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ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF THE EQUATION $[p(x)y']' - q(x)y = 0$ WITH COMPLEX-VALUED COEFFICIENTS

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In this paper conditions are derived under which the solutions of the equation $[p(x)\,y']' - q(x)\,y = 0$ with complex-valued coefficients can be approximated with the solutions of the equation $[p(x)\,z']' = 0$.

1. INTRODUCTION

Consider a differential equation

$$[p(x)\,y']' - q(x)\,y = 0 \tag{1}$$

where $p(x)$, $q(x)$ are continuous complex-valued functions on an x-interval $J = [a, \infty)$, $p(x) \neq 0$. '*Solution*' will mean '*complex-valued, non-trivial solution*' defined on J. The problem of the asymptotic integration of (1) is based on the finding of a differential equation of the same type with wellknown solutions approximating in a certain sense the solutions of (1) or, more generally, on the finding of a transformation of (1) on an equation which can be approximated in a suitable manner.

In our case, the object of interest will be asymptotic expansions of solutions of (1) when this equation can be considered as a perturbation of $[p(x)\,z']' = 0$ with the general solution

$$z(x) = \alpha + \beta \int_{\zeta}^{x} \frac{\mathrm{d}t}{p(t)}\,, \quad \zeta \in J. \tag{2}$$

If ξ, η are real numbers, $a \leqq \xi, \eta \leqq \infty$, then a function $y(x)$ is a solution of (1) satisfying conditions $y(\xi) = \eta$, $p(\eta)y'(\eta) = \beta$ if and only if $y(x)$ is a solution of the integral equation

$$y(x) - \int_{\xi}^{x}\int_{\eta}^{t} \frac{q(s)}{p(t)}\,y(s)\,\mathrm{d}s\,\mathrm{d}t = z(x). \tag{3}$$

Define

$$U^{\circ}z(x) = z(x), \quad U^{n}z(x) = \int\limits_{\xi}^{x}\int\limits_{\eta}^{t} \frac{q(s)}{p(t)}\, U^{n-1}\, z(s)\, ds\, dt.$$

It is easy to verify that the series

(4) $$y(x) = \sum_{0}^{\infty} U^{n}z(x)$$

is a formal solution of (1) with the formal derivative

(5) $$p(x)y'(x) = \beta + \sum_{1}^{\infty}\int_{\eta}^{x} q(t)\, U^{n-1}z(t)\, dt.$$

If the series (4) and (5) are uniformly convergent on some subinterval $I \subseteq J$, the first of them represents a solution of (1) with the derivative defined by (5) on I. We shall prove in Lemma 1 that both mentioned series converge uniformly on any finite subinterval $I \subseteq J$ choosing ξ, η, ζ to be in J. In Lemmas 2, 3 and 5 the uniform convergence of these series is considered on the whole interval J if

(6) $$\int\limits_{a}^{\infty}\int\limits_{a}^{t} \left| \frac{q(t)}{p(s)} \right| ds\, dt < \infty \quad \text{or} \quad \int\limits_{a}^{\infty}\int\limits_{a}^{t} \left| \frac{q(s)}{p(t)} \right| ds\, dt < \infty$$

holds. Both conditions are fulfilled if there is

(7) $$\int\limits_{a}^{\infty} \frac{dt}{|\,p(t)\,|} < \infty, \quad \int\limits_{a}^{\infty} |\,q(t)\,|\, dt < \infty\,;$$

but the former assumption in (6) is comparable with the following one

(8) $$\int\limits_{a}^{\infty} \frac{dt}{|\,p(t)\,|} = \infty\,,$$

and the latter with

(9) $$\int_{a}^{\infty} |q\,(t)\,|\, dt = \infty.$$

Under some combinations of conditions (6), (7), (8), (9) a detailed discussion of the asymptotic properties of solutions of (1) with real-valued coefficients can be found in a recent paper by U. Richard [1]. Unfortunately, in the case when one of the conditions (8), (9) holds, the uniform convergence of some expansions is proved only on a finite interval so that the k^{th} approximations

(10) $$z = \sum_{0}^{k} U^{n}z(x), \quad pz' = \beta + \sum_{n=1}^{k}\int_{\eta}^{x} q(t)\, U^{n-1}z(t)\, dt$$

do not expresse the asymptotic nature of solutions for $x \to \infty$. This suggested me an idea to consider the uniform convergence of (4) and (5) on the whole interval J

and for equations with complex-valued coefficients. In addition, it will be shown that combining this method with a transformation of (1), one can describe asymptotic nature of solutions of (1) if the absolute convergence of integrals in (6) is replaced by the conditional one.

Note that the first approximation of solutions of (1) forms the function (2) as it can be seen from (10) for $k = 0$.

The first approximations of solutions of (1) were derived by P. Hartman [2,] p. 375 under the assumptions (19) and in the case when these conditions were weakened as follows

$$\lim_{x\to\infty}\int_a^x \frac{dt}{p(t)} \text{ exists}, \quad \int_a^\infty |q(t)| \sup_{t\le s} \left|\int_s^\infty \frac{dr}{p(r)}\right| dt < \infty.$$

2. DEFINITIONS AND FUNDAMENTAL PROPERTIES OF THE OPERATORS $U^n_{\xi,\eta}$ $V^n_{\xi,\eta}$

Let p, q, w be continuous complex-valued functions on J, $p(x) \neq 0$. Define linear operators $U^n_{\xi,\eta}$, $V^n_{\xi,\eta}$ on the space $C^\circ(J)$ by putting

$$U^\circ_{\xi,\eta}\, w = w(x), \quad U^n_{\xi,\eta}\, w = \int_\xi^x \int_\eta^t \frac{q(s)}{p(t)}\, U^{n-1}_{\xi,\eta} w(s)\, ds\, dt,\ n = 1, 2, \ldots$$

and

$$V^\circ_{\xi,\eta}\, w = w(x), \quad V^n_{\xi,\eta}\, w = \int_\xi^x \int_\eta^t \frac{q(t)}{p(s)}\, V^{n-1}_{\xi,\eta}\, w(s)\, ds\, dt,\ n = 1, 2, \ldots.$$

The symbols $U^n_{\xi,\eta}(x)$, $V^n_{\xi,\eta}(x)$ will be used instead of $U^n_{\xi,\eta}\, w(x)$, $V^n_{\xi,\eta}\, w(x)$, respectively, for $w(x) \equiv 1$.

The following relations between the operators $U^n_{\xi,\eta}$, $V^n_{\xi,\eta}$ can be easily verified and will be often used in the following:

$$U^n_{\xi,\eta}(x) = U^{n-m}_{\xi,\eta}\, U^m_{\xi,\eta}(x), \qquad V^n_{\xi,\eta}(x) = V^{n-m}_{\xi,\eta}\, V^m_{\xi,\eta}(x),$$

$$U^n_{\xi,\eta}(x) = \int_\xi^x V^{n-1}_{\eta,\xi} \int_\eta^t q(s)\, ds \frac{dt}{p(t)}, \qquad V^n_{\xi,\eta}(x) = \int_\xi^x U^{n-1}_{\eta,\xi} \int_\eta^t \frac{ds}{p(s)} q(rt)\, dt,$$

$$U^m_{\xi,\eta} \int_\xi^x \frac{dt}{p(t)} = \int_\xi^x V^m_{\eta,\xi}(t) \frac{dt}{p(t)}, \qquad V^m_{\xi,\eta} \int_\eta^x q(t)\, dt = \int_\xi^x U^m_{\xi,\eta}(t)\, q(t)\, dt.$$

Here m, n are non-negative integers, $m < n$. Next, denote

(11) $$\sigma_{\xi,\eta}(x) = \left| \int_{\xi}^{x} \left| \int_{\eta}^{t} \left| \frac{q(s)}{p(t)} \right| \mathrm{d}s \right| \mathrm{d}t \right|, \quad \tau_{\xi,\eta}(x) = \left| \int_{\xi}^{x} \left| \int_{\eta}^{t} \left| \frac{q(t)}{p(s)} \right| \mathrm{d}s \right| \mathrm{d}t \right|$$

and if $w(x)$ is bounded on J, put

$$| w(x) |_J = \sup_{x \in J} | w(x) |, \quad | w(t) |_x = \sup_{t \geq x} | w(t) |.$$

On interchanging the order of integration in (11) we see that

(12) $$\sigma_{\infty,a}(a) = \tau_{\infty,\infty}(a), \quad \tau_{\infty,a}(a) = \sigma_{\infty,\infty}(a)$$

holds.

3. THE UNIFORM CONVERGENCE OF THE SERIES $\sum_0^\infty U^n_{\xi,\eta} w(x)$, $\sum_0^\infty V^n_{\xi,\eta} w(x)$

In the folloving Lemmas it will be supposed that p, q, w are continuous complex-valued functions on the investigated interval; further, let $p(x) \neq 0$ and $w(x)$ be bounded.

Lemma 1. *The series*

(13) $$\sum_0^\infty U^n_{a,a} w(x), \quad \sum_0^\infty V^n_{a,a} w(x)$$

converge uniformly on any finite interval $I = [x_0, X]$ *and it holds on* I

(14) $$\left| \sum_{n=k}^{\infty} U^n_{a,a} w(x) \right| \leq | w(x) |_I \frac{\sigma^k_{a,a}(x)}{k!} \exp\{ \sigma_{a,a}(x) \},$$
$$\left| \sum_{n=k}^{\infty} V^n_{a,a} w(x) \right| \leq | w(x) |_I \frac{\tau^k_{a,a}(x)}{k!} \exp\{ \tau_{a,a}(x) \}$$

for any k.

Proof. First, it will be proved by induction that

(15) $$| U^n_{a,a} w(x) | \leq | w(x) |_I \frac{\tau^n_{a,a}(x)}{n!}.$$

For $n = 0$ we have $| U^0_{a,a} w(x) | = | w(x) | \leq | w(x) |_I$. If (15) holds, it follows

$$| U^{n+1}_{a,a} w | = \left| \int_a^x \int_a^t \frac{q(s)}{p(t)} U^n_{a,a} w(s) \,\mathrm{d}s\,\mathrm{d}t \right| \leq | w(x) |_I \int_a^x \int_a^t \left| \frac{q(s)}{p(t)} \right| \frac{\sigma^n_{a,a}(s)}{n!} \,\mathrm{d}s\,\mathrm{d}t \leq$$

$$\leqq |w(x)|_I \int_a^x \frac{\tau^n_{a,a}(t)}{n!} \int_a^t \left| \frac{q(s)}{p(t)} \right| ds\, dt = |w(x)|_I \int_a^x \frac{\sigma^n_{a,a}(t)}{n!} \sigma'(t)\, dt = |w(x)|_I \frac{\sigma^{n+1}_{a,a}(x)}{(n+1)!}$$

and the induction is complete. Consequently,

$$\left| \sum_{n=k}^{\infty} U^n_{a,a} w(x) \right| \leqq \sum_{n=k}^{\infty} |U^n_{a,a} w(x)| \leqq |w(x)|_I \sum_{n=k}^{\infty} \frac{\sigma^n_{a,a}(x)}{n!} \leqq$$

$$\leqq |w(x)|_I \frac{\sigma^k_{a,a}(x)}{k!} \exp\{\sigma_{a,a}(x)\} \leqq |w(x)|_I \frac{\sigma^k_{a,a}(X)}{k!} \exp\{\sigma_{a,a}(X)\},$$

so that $\sum_0^{\infty} U^n_{a,a} w(x)$ is uniformly convergent on I and the first inequality in (14) holds.

In the same manner it is shown

$$|V^n_{a,a} w(x)| \leqq |w(x)|_I \frac{\tau^n_{a,a}(x)}{n!},$$

the uniform convergence of $\sum_0^{\infty} V^n_{a,a} w(x)$ and the latter estimate in (14).

The proof is complete.

Lemma 2. *Suppose that*

$$\int_a^{\infty} |q(t)|\, dt < \infty, \quad \int_a^{\infty} \int_t^{\infty} \left| \frac{q(s)}{p(t)} \right| ds\, dt < \infty. \tag{16}$$

Then the series $\sum_0^{\infty} U^n_{\infty,\infty} w(x)$ *converges uniformly on* J *and it holds for any* k

$$\left| \sum_{n=k}^{\infty} U^n_{\infty,\infty} w(x) \right| \leqq |w(t)|_X \frac{\sigma^k_{\infty,\infty}(x)}{k!} \exp\{\sigma_{\infty,\infty}(x)\}. \tag{17}$$

Proof. First of all it can be seen from (16) that in $\sigma_{\xi,\eta}(x)$ defined by (11) it can be taken $\xi, \eta = \infty$. It will be proved by induction that

$$|U^n_{\infty,\infty} w(x)| \leqq |w(t)|_X \frac{\sigma^n_{\infty,\infty}(x)}{n!}. \tag{18}$$

It is clear that (18) holds for $n = 0$. If (18) holds for an n, it follows

$$|U^{n+1}_{\infty,\infty} w(x)| = \left| \int_{\infty}^x \int_{\infty}^t \frac{q(s)}{p(t)} U^n_{\infty,\infty} w(s)\, ds\, dt \right| \leqq \int_x^{\infty} \int_t^{\infty} \left| \frac{q(s)}{p(t)} \right| |w(t)|_s \frac{\sigma^n_{\infty,\infty}(s)}{n!}\, ds\, dt \leqq$$

$$\leqq |w(t)|_x \int_x^\infty \int_t^\infty \left| \frac{q(s)}{p(t)} \right| \frac{\sigma^n_{\infty,\infty}(t)}{n!} \, ds \, dt = |w(t)|_x \int_x^\infty -\sigma_{\infty,\infty}(t) \frac{\sigma^n_{\infty,\infty}(t)}{n!} \, dt =$$

$$= |w(t)|_x \frac{\sigma^{n+1}_{\infty,\infty}(x)}{(n+1)!} .$$

Hence, the series $\sum U^n_{\infty\infty} w$ is uniformly convergent on J and it holds (17). This completes the proof.

In the same manner, it can be proved the following

Lemma 3. *Suppose that*

$$\int_a^\infty \frac{dt}{|p(t)|} < \infty, \int_a^\infty \int_t^\infty \left| \frac{q(t)}{p(s)} \right| ds \, dt < \infty . \tag{19}$$

Then the series $\sum_0^\infty V^n_{\infty,\infty} w(x)$ *converges uniformly on* J *and it holds for any* k

$$\left| \sum_{n=k}^\infty V^n_{\infty,\infty} w(x) \right| \leqq |w(t)|_x \frac{\tau^k_{\infty,\infty}(x)}{k!} \exp\{\tau_{\infty,\infty}(x)\} . \tag{20}$$

Note. If k is so large that $\frac{\sigma_{\infty,\infty}(a)}{k+1} < \frac{1}{2}$, $\left[\frac{\tau_{\infty,\infty}(a)}{k+1} < \frac{1}{2}\right]$ the estimate (17) [(20)] can be sharpened as follows

$$\left| \sum_{n=k}^\infty U^n_{\infty,\infty} w(x) \right| \leqq |w(t)|_x \frac{\sigma^k_{\infty,\infty}(x)}{k!} \left[1 + \frac{\sigma_{\infty,\infty}(x)}{k+1} + \frac{\sigma^2_{\infty,\infty}(x)}{(k+1)\,(k+2)} + \dots \right] \leqq$$

$$\leqq |w(t)|_x \frac{\sigma^k_{\infty,\infty}(x)}{k!} \sum_0^\infty \left[\frac{\sigma_{\infty,\infty}(x)}{k+1} \right]^n = |w(t)|_x \frac{\sigma^k_{\infty,\infty}(x)}{k!} \frac{1}{1 - \dfrac{\sigma_{\infty,\infty}(a)}{k+1}} \leqq$$

$$\leqq 2\, |w(t)|_x \frac{\sigma^k_{\infty,\infty}(x)}{k!} . \qquad \left[\left| \sum_{n=k}^\infty V^n_{\infty,\infty} w(x) \right| \leqq 2\, |w(t)|_x \frac{\tau^k_{\infty,\infty}(x)}{k!} . \right]$$

Lemma 4. *Suppose that*

$$\int_v^8 \int_a^t \left| \frac{q(t)}{p(s)} \right| ds \, dt < 1 . \tag{21}$$

Then the series

$$\sum_0^\infty U^n_{a,\infty} w(x) , \qquad \sum_0^\infty V^n_{\infty,a} w(x) \tag{22}$$

converge uniformly on J and for any k it holds

$$\left.\begin{array}{l} \left|\sum_{n=k}^{\infty} U^{n}_{a,\infty}\, w(x)\right| \\ \\ \left|\sum_{n-k}^{\infty} V^{n}_{\infty,a}\, w(x)\right| \end{array}\right\} \leqq |\, w(t)\,|_a \frac{\sigma^{k}_{\infty,\infty}(a)}{1-\sigma_{\infty,\infty}(a)} . \tag{23}$$

Proof. Suppose that we have for some n the estimates

$$|\, U^{n}_{a,\infty}\, w(x)\,| \leqq |\, w(t)\,|_a\, \sigma^{n}_{\infty,\infty}(a)\,,\quad |\, V^{n}_{\infty,a} w(x)\,| \leqq |\, w(t)\,|_a\, \sigma^{n}_{\infty,\infty}(a)\,. \tag{24}$$

These are certainly true for $n = 0$. Then we get

$$|\, U^{n+1}_{a,\infty}\, w(x)\,| = \left|\int_a^x \int_\infty^t \frac{q(s)}{p(t)}\, U^{n}_{a,\infty}\, w(s)\, ds\, dt\,\right| \leqq \int_a^x \int_t^\infty \left|\frac{q(s)}{p(t)}\right| |\, U^{n}_{a,\infty}\, w(s)\,|\, ds\, dt \leqq$$

$$\leqq |\, w(t)\,|_a\, \sigma^{n}_{\infty,\infty}(a) \int_a^x \int_t^\infty \left|\frac{q(s)}{p(t)}\right| ds\, dt \leqq |\, w(t)\,|_a\, \sigma^{n+1}_{\infty,\infty}(a)\,,$$

$$|\, V^{n+1}_{\infty,a}\, w(x)\,| = \left|\int_\infty^x \int_a^t \frac{q(t)}{p(s)}\, V^{n}_{\infty,a}\, w(s)\, ds\, dt\,\right| \leqq \int_x^\infty \int_a^t \left|\frac{q(t)}{p(s)}\right| |\, V^{n}_{\infty,a}\, w(s)\,|\, ds\, dt \leqq$$

$$\leqq |\, w(t)\,|_a\, \sigma^{n}_{\infty,\infty}(a) \int_x^\infty \int_a^t \left|\frac{q(t)}{p(s)}\right| ds\, dt = |\, w(t)\,|_a\, \sigma^{n}_{\infty,\infty}(a) \int_a^\infty \int_s^\infty \left|\frac{q(t)}{p(s)}\right| dt\, ds \leqq$$

$$\leqq |\, w(t)\,|_a\, \sigma^{n+1}_{\infty,\infty}(a)\,.$$

So the estimates are true for all n.

Consequently, the series $\sum_{n=0}^{\infty} |\, w(t)\,|_a\, \sigma^{n}_{\infty,\infty}(a) = \dfrac{|\, w(t)\,|_a}{1-\sigma_{\infty,\infty}(a)}$ is a majorant for (22), so that both series (22) are uniformly convergent on J and it holds (23). The Lemma is proved.

Lemma 5. *Suppose that*

$$\int_a^\infty \int_a^t \left|\frac{q(s)}{p(t)}\right| ds\, dt < 1\,.$$

Then the series $\sum_0^\infty U^{n}_{\infty,a}\, w(x)$, $\sum_0^\infty V^{n}_{a,\infty}\, w(x)$ *converge uniformly on J and for any k it holds*

$$\left.\begin{array}{l} \left|\sum\limits_{n=k}^{\infty} U^{n}_{\infty,a}\, w(x)\right| \\ \\ \left|\sum\limits_{n=k}^{\infty} V^{n}_{a,\infty}\, w(x)\right| \end{array}\right\} \leqq |\, w(t)\,|_a \frac{\tau^{k}_{\infty,\infty}(a)}{1-\tau_{\infty,\infty}(a)}\,.$$

The proof of this assertion is analogous to the proof of the preceding Lemma and will be omitted here.

Lemma 6. *If* $y(x) = \sum\limits_{0}^{\infty} U^{n}_{\xi,\eta}\left[\alpha + \beta \int\limits_{\xi}^{x} \frac{dt}{p(t)}\right]$, *then it is formaly* $p(x)y'(x) = \sum\limits_{0}^{\infty} V^{n}_{\eta,\xi}\,[\,\beta + \alpha \int\limits_{\eta}^{x} q(t)\,dt\,]$.

Proof. It is $y = \alpha + \alpha \sum\limits_{0}^{\infty} U^{n+1}_{\xi,\eta}(x) + \beta \sum\limits_{0}^{\infty} U^{n}_{\xi,\eta} \int\limits_{\xi}^{x} \frac{dt}{p(t)}$, so that

$$y' = \alpha \sum_{0}^{\infty} \frac{d}{dx} U^{n+1}_{\xi,\eta}(x) + \beta \sum_{0}^{\infty} \frac{d}{dx} U^{n}_{\xi,\eta} \int\limits_{\xi}^{x} \frac{dt}{p(t)},\quad py' = \alpha \sum_{0}^{\infty} V^{n}_{\eta,\xi} \int\limits_{\eta}^{x} q(t)\,dt +$$

$$+ \beta \sum_{0}^{\infty} V^{n}_{\eta,\xi}(x) = \sum_{0}^{\infty} V^{n}_{\eta,\xi}\,[\beta + \alpha \int\limits_{\eta}^{x} q(t)\,dt]$$ as was to be proved.

4. ASYMPTOTIC EXPANSIONS

In this paragraph conditions are derived guaranteeing the uniform convergence of the series (4) and (5) on the whole interval J.

Theorem 1. *Suppose that* $p(x)$, $q(x) \in C^{\circ}(J)$, $p(x) \neq 0$ *and*

$$\int\limits_{a}^{\infty} |\,q(t)\,|\,dt < \infty\,, \qquad \int\limits_{a}^{\infty} \frac{dt}{|\,p(t)\,|} < \infty\,. \tag{25}$$

Then the general solution of (1) *and its derivative can be expressed as uniformly convergent series on J*

$$y(x) = \sum_{0}^{\infty} U^{n}_{\infty,\infty}\left[\alpha + \beta \int\limits_{\infty}^{x} \frac{dt}{p(t)}\right],\quad p(x)y'(x) = \sum_{0}^{\infty} V^{n}_{\infty,\infty}\,[\,\beta + \alpha \int\limits_{\infty}^{x} q(t)\,dt\,] \tag{26}$$

with the following estimates for the k^{th} *approximations*

$$\left|\, y(x) - \sum_{0}^{k-1} U^{n}_{\infty,\infty}\left[\alpha + \beta \int\limits_{\infty}^{x} \frac{dt}{p(t)}\right]\right| \leqq \left|\, \alpha + \beta \int\limits_{\infty}^{t} \frac{ds}{p(s)}\right|_{x} \frac{\sigma^{k}_{\infty,\infty}(x)}{k!} \exp\{\,\sigma_{\infty,\infty}(x)\,\}, \tag{27}$$

(28)

$$|\, p(x)y'(x) - \sum_0^{k-1} V^n_{\infty,\infty}\,[\,\beta + \alpha \int_\infty^x q(t)\,dt\,]\,| \leqq |\,\beta + \alpha \int_\infty^t q(s)\,ds\,|_x \frac{\tau^k_{\infty,\infty}(x)}{k!} \exp\{\, \tau_{\infty,\infty}(x)\,\}.$$

Proof. Assumptions (25) imply (16) and (19). The statement follows immediately from Lemma 2 and 3 choosing $w = \alpha + \beta \int_\infty^x \frac{dt}{p(t)}$, $w = \beta + \alpha \int_\infty^x q(t)\,dt$ respectively, and Lemma 6.

Theorem 2. *Suppose that* $p(x)$, $q(x) \in C^\circ(J)$, $p(x) \neq 0$ *and*

$$\int_a^\infty \frac{dt}{|\,p(t)\,|} = \infty, \quad \int_a^\infty \int_a^t \left| \frac{q(t)}{p(s)} \right| ds\, dt < \infty\,. \tag{29}$$

Then the equation (1) *has a solution* y_1 *of the form*

$$y_1 = \sum_0^\infty U^n_{\infty,\infty}(x)\,, \tag{30}$$

$$py'_1 = \sum_0^\infty V^n_{\infty,\infty} \int_\infty^x q(t)\,dt \tag{31}$$

with the following estimates for the k^{th} *approximations*

$$|\, y_1(x) - \sum_0^{k-1} U^n_{\infty,\infty}(x)\,| \leqq \frac{\sigma^k_{\infty,\infty}(x)}{k!} \exp\{\, \sigma_{\infty,\infty}(x)\,\}\,, \tag{32}$$

$$|\, p(x)\, y'_1(x) - \sum_0^{k-1} V^n_{\infty,\infty} \int_\infty^x q(t)\,dt\,| \leqq \int_x^\infty |\,q(t)\,|\,dt\, \frac{\sigma^k_{\infty,\infty}(x)}{k!} \exp\{\, \sigma_{\infty,\infty}(x)\,\}\,. \tag{33}$$

Doing further assumption

$$\int_a^\infty \frac{\tau_{\infty,a}(t)}{|\,p(t)\,|}\,dt < \infty \tag{34}$$

then there exists another solution y_2 *of the form*

$$y_2(x) = \sum_0^\infty U^n_{\infty,\infty} \int_a^x \frac{dt}{p(t)}\,, \tag{35}$$

$$p(x)\, y'_2(x) = 1 + \sum_0^\infty V^n_{\infty,\infty}\, V^1_{\infty,a}(x) \tag{36}$$

and it holds

$$(37)\qquad \left| y_2(x) - \sum_0^k U^n_{\infty,\infty} \int_a^x \frac{dt}{p(t)} \right| \leqq \int_x^\infty \tau_{\infty,a}(t) \frac{dt}{|p(t)|} \frac{\sigma^k_{\infty,\infty}(x)}{k!} \exp\{\sigma_{\infty,\infty}(x)\},$$

$$(38)\qquad | p(x)y_2'(x) - 1 - \sum_0^{k-1} V^n_{\infty,\infty} V^1_{\infty,a}(x) | \leqq \int_x^\infty | q(t) | \,dt \int_x^\infty \tau_{\infty,a}(t) \frac{dt}{|p(t)|} \cdot$$

$$\cdot \frac{\sigma^{k-1}_{\infty,\infty}(x)}{(k-1)!} \exp\{\sigma_{\infty,\infty}(x)\}.$$

If the assumption (34) *is replaced by some of the conditions*

$$(39)\qquad \int_a^x \frac{dt}{p(t)} \text{ is bounded on } J,$$

$$(40)\qquad \left| \int_a^\infty V^1_{\infty,a}(t) \frac{dt}{p(t)} \right| < \infty,$$

then, in the estimates (37), (38), *the right hand sides have to be replaced by*

$$(41)\qquad \left| \int_a^x \frac{dt}{p(t)} \right|_J \frac{\sigma^{k+1}_{\infty,\infty}(x)}{(k+1)!} \exp\{\sigma_{\infty,\infty}(x)\},$$

$$(42)\qquad \int_x^\infty | q(t) | \,dt \left| \int_a^x \frac{dt}{p(t)} \right|_J \frac{\sigma^k_{\infty,\infty}(x)}{k!} \exp\{\sigma_{\infty,\infty}(x)\}, \text{ respectively, in the case (39)}$$

and

$$(43)\qquad \left| U^1_{\infty,\infty} \int_a^x \frac{dt}{p(t)} \right|_J \frac{\sigma^k_{\infty,\infty}(x)}{k!} \exp\{\sigma_{\infty,\infty}(x)\},$$

$$(44)\qquad \int_x^\infty | q(t) | \,dt \left| U^1_{\infty,\infty} \int_a^x \frac{dt}{p(t)} \right|_J \frac{\sigma^{k-1}_{\infty,\infty}(x)}{(k-1)!} \exp\{\sigma_{\infty,\infty}(x)\}, \text{ respectively, in the}$$

case (40).

If the assumption (34) *is replaced by the following one*

$$(45)\qquad \int^\infty V^k_{\infty,a}(t) \frac{dt}{p(t)} = \infty, \text{ for } k = 1, 2, \ldots n$$

$$\left|\int_a^\infty V^{n+1}_{\infty,a}(t)\frac{dt}{p(t)}\right| < \infty, \quad \sigma_{\infty,\infty}(a) < 1,$$

then, there exists a solution y_2 of the form

(46)
$$y_2(x) = \sum_0^\infty U^n_{a,\infty}\int_a^x \frac{dt}{p(t)},$$

(47)
$$p(x)y_2'(x) = \sum_0^\infty V^n_{\infty,a}(x)$$

and it holds

(48)
$$\left|y_2(x) - \sum_{l=0}^{n+k} U^l_{a,\infty}\int_a^x \frac{dt}{p(t)}\right| \leq \left|\int_a^x V^{n+1}_{\infty,a}(t)\frac{dt}{p(t)}\right| \frac{\sigma^k_{\infty,\infty}(a)}{1-\sigma_{\infty,\infty}(a)},$$

(49)
$$\left|p(x)y_2'(x) - \sum_0^k V^n_{\infty,a}(x)\right| \leq \frac{\sigma^{k+1}_{\infty,\infty}(a)}{1-\sigma_{\infty,\infty}(a)}.$$

Proof. It follows from Lemma 6 that the series (31), (36) and (47) are formal derivatives of the series (30), (35) and (46) respectively, so that it is sufficient to prove the uniform convergence of the mentioned series.

First of all the conditions (29) imply (16) and the uniform convergence of (30) follows from Lemma 2. Further, we have

$$\left|V^n_{\infty,\infty}\int_\infty^x q(t)\,dt\right| = \left|\int_x^\infty q(t)\,U^n_{\infty,\infty}(t)\,dt\right| \leq \int_x^\infty |q(t)|\,dt\,\frac{\sigma^n_{\infty,\infty}(x)}{n!}.$$

Hence the series (31) is uniformly convergent and (33) holds. Under the assumption (34) we have

$$\left|U^n_{\infty,\infty}\int_a^x \frac{dt}{p(t)}\right| = \left|U^{n-1}_{\infty,\infty}U^1_{\infty,\infty}\int_a^x \frac{dt}{p(t)}\right| = \left|U^{n-1}_{\infty,\infty}\int_x^\infty V^1_{\infty,\infty}(t)\frac{dt}{p(t)}\right| \leq$$

$$\leq \int_x \tau_{\infty,a}(t)\frac{dt}{|p(t)|}\,\frac{\sigma^{n-1}_{\infty,\infty}(x)}{(n-1)!}$$

and this implies the uniform convergence of (35) and the estimate (37).

The uniform convergence of (36) and the estimate (38) follows from

$$\left|V^n_{\infty,\infty}V^1_{\infty,a}(x)\right| = \left|\int_\infty^x q(t)\,U^n_{\infty,\infty}\int_a^t \frac{ds}{p(s)}\right| = \left|\int_\infty^x q(t)\,U^{n-0}_{\infty,\infty}U^1_{\infty,\infty}\int_a^t \frac{ds}{p(s)}\right| \leq$$

$$\geqq \left| \int_{\infty}^{x} q(t)\, U^{n-1}_{\infty,\infty} \int_{\infty} V^{1}_{\infty,a}(s) \frac{ds}{p(s)} \right| \leqq \int_{x}^{\infty} |q(t)|\, dt \int_{x}^{\infty} \tau_{\infty,a}(t) \frac{dt}{|p(t)|} \; \frac{\sigma^{n-1}_{\infty,\infty}(x)}{(n-1)!} .$$

If we replace the assumption (34) by (39) we have

$$\left| U^{n}_{\infty,\infty} \int_{a}^{x} \frac{dt}{p(t)} \right| \leqq \left| \int_{a}^{x} \frac{dt}{p(t)} \right|_{J} \frac{\sigma^{n}_{\infty,\infty}(x)}{n!} ,$$

and under the assumption (40)

$$\left| U^{n}_{\infty,\infty} \int_{a}^{x} \frac{dt}{p(t)} \right| = \left| U^{n-1}_{\infty,\infty} U^{1}_{\infty,\infty} \int_{a}^{x} \frac{dt}{p(t)} \right| \leqq \left| U^{1}_{\infty,\infty} \int_{a}^{x} \frac{dt}{p(t)} \right|_{J} \frac{\sigma^{n-1}_{\infty,\infty}(x)}{(n-1)!} ,$$

$$\left| V^{n}_{\infty,\infty} V^{1}_{\infty,a}(x) \right| \leqq \left| \int_{a}^{x} q(t)\, U^{n}_{\infty,\infty} \int_{a}^{t} \frac{ds}{p(s)} \right| \leqq \int_{x}^{\infty} |q(t)|\, dt \left| U^{1}_{\infty,\infty} \int_{a}^{x} \frac{dt}{p(t)} \right|_{J} \frac{\sigma^{n-1}_{\infty,\infty}(x)}{(n-1)!} .$$

This implies the uniform convergence of (35) and (36) and the estimates (43) and (44).

Finally, the assertion concerning the series (46), (47) follows immediately from Lemma 4 letting there

$$w(x) = U^{n+1}_{a,\infty} \int_{a}^{x} \frac{dt}{p(t)} ,\; w(x) \equiv 1, \text{ respectively. Theorem is proved.}$$

Theorem 3. *Suppose that $p(x), q(x) \in C^{\circ}(J)$, $p(x) \neq 0$ and*

$$\int_{a}^{\infty} |q(t)|\, dt = \infty , \quad \int_{a}^{\infty} \int_{a}^{t} \left| \frac{q(s)}{p(t)} \right| ds\, dt < \infty .$$

Then the equation (1) *has a solution y_1 of the form*

$$y_1(x) = \sum_{0}^{\infty} U^{n}_{\infty,\infty} \int_{x}^{\infty} \frac{dt}{p(t)} , \qquad p(x) y_1'(x) = \sum_{0}^{\infty} V^{n}_{\infty,\infty}(x)$$

with the following estimates for the k^{th} approximations

$$\left| y_1(x) - \sum_{0}^{k-1} U^{n}_{\infty,\infty} \int_{x}^{\infty} \frac{dt}{p(t)} \right| \leqq \int_{x}^{\infty} \frac{dt}{|p(t)|} \; \frac{\tau^{k}_{\infty,\infty}(x)}{k!} \exp\{ \tau_{\infty,\infty}(x) \} ,$$

$$\left| p(x) y_1'(x) - \sum_{0}^{k-1} V^{n}_{\infty,\infty}(x) \right| \leqq \frac{\tau^{k}_{\infty,\infty}(x)}{k!} \exp\{ \tau_{\infty,\infty}(x) \} .$$

Doing further assumption

$$\int_a^\infty |q(t)|\,\sigma_{\infty,a}(t)\,dt < \infty , \tag{50}$$

then there exists another solution y_2 *of the form*

$$y_2(x) = 1 + \sum_0^\infty U^n_{\infty,\infty}\, U_{\infty,a}(x) , \qquad p(x)\, y_2'(x) = \sum_0^\infty V^n_{\infty,\infty} \int_a^x q(t)\,dt \tag{51}$$

and it holds

$$| y_2(x) - 1 - \sum_0^{k-1} U^n_{\infty,\infty}\, U_{\infty,a}(x)| \leqq \int_x^\infty \frac{dt}{|p(t)|} \int_x^\infty |q(t)|\,\sigma_{\infty,a}(t)\,dt\, \frac{\tau^{k-1}_{\infty,\infty}(x)}{(k-1)\,!} \exp\{\tau_{\infty,\infty}(x)\}, \tag{52}$$

$$| p(x)\, y_2'(x) - \sum_0^k V^n_{\infty,\infty} \int_a^x q(t)\,dt \,| \leqq \int_x^\infty |q(t)|\,\sigma_{\infty,a}(t)\,dt\, \frac{\tau^k_{\infty,\infty}(x)}{k!} \exp\{\tau_{\infty,\infty}(x)\} . \tag{53}$$

If the assumption (50) *is replaced by some of the conditions*

$$\int_a^\infty q(t)\,dt \text{ is bounded on } J , \tag{54}$$

$$\left| \int_a^\infty q(t)\, U_{\infty,a}(t)\,dt \right| < \infty , \tag{55}$$

then in the estimates (52), (53) *the right hand sides have to be replaced by*

$$\int_x^\infty \frac{dt}{|p(t)|} \left| \int_a^x q(t)\,dt \right|_J \frac{\tau^k_{\infty,\infty}(x)}{k!} \exp\{\tau_{\infty,\infty}(x)\} ,$$

$$\left| \int_a^x q(t)\,dt \right|_J \frac{\tau^k_{\infty,\infty}(x)}{k!} \exp\{\tau_{\infty,\infty}(x)\}, \text{ respectively, in the case (54)}$$

and by

$$\int_x^\infty \frac{dt}{|p(t)|} \left| \int_x^\infty q(t)\, U^1_{\infty,a}(t)\,dt \right|_J \frac{\tau^{k-1}_{\infty,\infty}(x)}{(k-1)\,!} \exp\{\tau_{\infty,\infty}(x)\} ,$$

$$\left| \int_x^\infty q(t)\, U^1_{\infty,a}(t)\,dt \right|_J \frac{\tau^k_{\infty,\infty}(x)}{k!} \exp\{\tau_{\infty,\infty}(x)\}, \text{ respectively, in the case (55).}$$

If the assumption (50) *is replaced by the following one*

$$\int_a^\infty q(t)\, U^k_{\infty,a}(t)\, dt = \infty \qquad for \qquad k = 0, 1, \ldots, n\,,$$

$$\left| \int_a^\infty q(t)\, U^{n+1}(t)\, dt \right| < \infty\,, \qquad \tau_{\infty,\infty}(a) < 1\,,$$

then, there exists a solution y_2 of the form

$$y_2(x) = \sum_0^\infty U^n_{\infty,\infty}(x)\,, \qquad p(x) y_2'(x) = \sum_0^\infty V^n_{a,\infty} \int_a^x q(t)\, dt$$

and it holds

$$\left| y_2(x) - \sum_0^{k-1} U^n_{\infty,a}(x) \right| \leq \frac{\tau^k_{\infty,\infty}(a)}{1 - \tau_{\infty,\infty}(a)}\,,$$

$$\left| p(x)\, y_2'(_2x) - \sum_{l=0}^{n+k} V^l_{a,\infty} \int_a^x q(t)\, dt \right| \leq \left| \int_a^x q(t)\, U^{n+1}_{\infty,a}(t)\, dt \right|_J \frac{\tau^k_{\infty,\infty}(a)}{1 - \tau_{\infty,\infty}(a)}\,.$$

The proof of this Theorem is similar to the proof of the preceding statement and will be omitted here.

Appendix. Using the preceding notation let us consider the following transformation of the equation (1)

$$y = f(x)\, z, \qquad f(x) = \sum_{k=0}^{n} U^k_{\xi,\eta}\, w(x)\,,$$

whose special form $\left[w(x) = \int_a^x \frac{dt}{p(t)} \right]$ in the real case appeared in the paper [1] by U. Richard. We get

$$[P(x)\, z']' - Q(x)\, z = 0,$$

$$P(x) = p(x) f^2(x), \qquad Q(x) = q(x) f(x)\, U^n_{\xi,\eta}\, w(x)\,.$$

This equation is of the same type like (1) and above theorems can be applied.

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