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TWO THEOREMS CONCERNING COMMON FIXED POINT OF COMMUTATIVE MAPPINGS

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We use the following notation: if F is a system of mappings from the set X into itself, then, for any $Y \subset X$, $F(Y)$ is the set of all $f(y)$, $f \in F$, $y \in Y$; instead of $F((y))$, $F(y)$ is written. If $Y \subset X$, $F(Y) \subset Y$, then $F|Y$ denotes the set of all $f \in F$ restricted to Y.

The operation in all semi-groups throughout this remark is the composition of mappings.

Let F be a commutative semi-group of mappings from the set X into itself. F is said to be a maximal commutative semi-group of mappings, if there exists no mapping from X into X which commutes with all mappings from F and does not belong to F.

Let F be a system of mappings from a set X into itself. By $r(F)$ we denote the set of all $f \in F$ such that for each $f_1 \in F$ there exists $f_2 \in F$ and $f = f_1 \circ f_2$ holds. By $f_1 \circ f_2$ we denote, as usual, the composition of mappings f_1 and f_2, that is, $f_1 \circ f_2(x) = f_1[f_2(x)]$ for every $x \in X$.

We now examine the situation in which all mappings from a system F commute and each of them has a fixed point.

In order to illustrate, let us consider the extremely simple system of mappings. Let X consist of six points, $1,2,\dots,6$, and F consist of four mappings, f_1, f_2, f_3, f_4, from the set X into itself defined as follows:

	1	2	3	4	5	6
f_1:	1	2	3	4	5	6
f_2:	2	1	4	3	5	6
f_3:	2	1	3	4	6	5
f_4:	1	2	4	3	6	5

Obviously, F is a commutative semi-group of mappings. Each mapping from F has a fixed point, but there exists no common fixed point of all mappings from F. Therefore it is not true that every commutative semi-group of mappings from a finite set into itself has common fixed point provided that each mapping from the semi-group has a fixed point. But this assertion is true under assumption that F is a maximal commutative semi-group. We prove:

Theorem 1. Let F be a maximal commutative semi-group of mappings from a set X into itself, $r(F) \neq \emptyset$. If each $f \in F$ has a fixed point, then all mappings from F have precisely one common fixed point.

If X is a finite set, then also F is finite and the composition of all mappings from F belongs to $r(F)$, and therefore $r(F) \neq \emptyset$. We obtain immediately from Theorem 1 :

Corrolary: Let F be a maximal commutative semi-group of mappings from a finite set X into itself. If each $f \in F$ has a fixed point, then all mappings from F have precisely one common fixed point.

Proof of Theorem 1 :

Let $f' \in r(F)$. Define a mapping u from the set X into $\exp X$ as follows:

$$u(x) = F\,[f'(x)] \quad .$$

Assuredly, $F[u(x)] \subset u(x)$.

Let $y \in u(x)$. Then $y = f \circ f'(x)$ for some $f \in F$. Therefore $F(y) \subset u(x)$. As $f' \in r(F)$, $f' \circ f \in F$, there exists $g \in F$ such that

$$f' = f' \circ f \circ g ,$$

and hence $f'(x) = f' \circ f \circ g(x) = g(y)$.

This implies $u(x) \subset F(x)$, and finally $u(x) = F(y)$.

If $x_1, x_2 \in X$, then either $u(x_1) = u(x_2)$ or $u(x_1) \cap u(x_2) = \emptyset$. Indeed, if $x \in u(x_1) \cap u(x_2)$, then $x = f_1 \circ f'(x_1) = f_2 \circ f'(x_2)$, where $f_1 \in F$, $f_2 \in F$, and $F(x) = u(x_1) = u(x_2)$.

Therefore we can choose x_a , $a \in D$ such that

$$\bigcup_{a \in D} u(x_a) = \bigcup_{x \in X} u(x), \text{ and } u(x_{a_1}) \cap u(x_{a_2}) = \emptyset \text{ for } a_1 \neq a_2 .$$

For each $x \in X$ and $f \in F$ we have

$$u[f(x)] \subset F[f'(x)] ,$$

and hence

$$u[f(x)] = u(x) .$$

This implies the image of $\overset{-1}{u}[u(x)]$ under F is contained in $\overset{-1}{u}[u(x)]$. The sets $\overset{-1}{u}[u(x_a)]$, $a \in D$, cover X and are disjoint.

If any of the sets $u(x_a)$ contains only one point, then this point is a common fixed point of all mappings from F .

Let $u(x_a)$ contain at least two points. We obtain a contradiction.

Denote $F_a = F \mid u(x_a)$. F_a is a group of mappings from the set $u(x_a)$ into itself, for each $a \in D$, as $F_a(x) = u(x_a)$ for each $x \in u(x_a)$. (See lemma in [1]). Hence there must exist, for each $a \in D$, a mapping $f_a \in F$ such that $f_a | u(x_a) \neq i | u(x_a)$, where by i we denote the identical mapping from X into itself. We introduce an auxiliary mapping g from X into itself as follows:

$$g \mid \overset{-1}{u} [u(x_a)] = f' \mid \overset{-1}{u} [u(x_a)] \quad \text{if} \quad f' \mid \overset{-1}{u} [u(x_a)] \neq$$
$$\neq i \mid \overset{-1}{u} [u(x_a)] \quad ,$$

and

$$g \mid \overset{-1}{u} [u(x_a)] = f_a \circ f' \mid \overset{-1}{u} [u(x_a)] \quad \text{otherwise.}$$

As the sets $\overset{-1}{u} [u(x_a)]$ cover X and are disjoint, g is a mapping from X into X. Certainly, g commutes with each $f \in F$. As F is maximal commutative semi-group, we obtain $g \in F$.

But g has no fixed point on X, as for each $x \in X$ $g(x) \in u(x_a)$ for some $a \in D$. $g | u(x_a) \in F_a$ and $g | u(x_a)$ is not identical mapping from $u(x_a)$ into itself. As F_a is a group, g has no fixed point on $u(x_a)$ (See lemma 1 in [1]). This is a contradiction. All mappings from F have at least one common fixed point.

Let x_1, x_2 be common fixed points of all mappings from F. Then the mapping $f(x) = x_1$ for every x X commutes with each mapping from F and therefore $f \in F$. $f(x_2) = x_1$ and therefore $x_1 = x_2$. The theorem is proved.

Theorem 2. Let f and g be mappings from an arbitrary set X into itself, $f \circ g = g \circ f$. Let f have precisely n fixed points, n natural number. Then, there exists a natural number k , $1 \leq k \leq n$, such that f and $g^k = \underbrace{g \circ g \circ \dots \circ g}_{k\text{-times}}$ have a common fixed point.

Proof. Let us denote the set of all fixed points of f by Y . Obviously, $g(Y) \subset Y$. Hence $g|Y$ is a mapping from a set Y , which has n points, into itself. There must exist a k , $1 \leq k \leq n$, such that $\underbrace{g|Y \circ g|Y \circ \dots \circ g|Y}_{k\text{-times}}$ has a fixed point in Y , and this is the assertion of the theorem.

Reference

[1] Z. HEDRLÍN: On common fixed points of commutative mappings, Commentationes Mathematicae Universitatis Carolinae, 2,4 (1961).