Commentationes Mathematicae Universitatis Carolinae

Bohuslav Diviš

On lattice points in high dimensional ellipsoids (Preliminary communication)

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 9 (1968), No. 2, 199--205

Persistent URL: `http://dml.cz/dmlcz/105172`

Commentationes Mathematicae Universitatis Carolinae
9,2 (1968)

# ON LATTICE POINTS IN HIGH-DIMENSIONAL ELLIPSOIDS

( Preliminary communication )

Bohuslav DIVIŠ, Praha

Let $n$ be an integer, $n \geq 5$ and

(1) $Q(u) = \sum_{j=1}^{n} \alpha_j u_j^2 , \quad \alpha_j > 0 \quad (j = 1, 2, \dots, n) .$

For $x > 0$ let $A_Q(x)$ be a number of lattice points in a closed ellipsoid $Q(u) \leq x$ , the volume of which is expressed by

$$V_Q(x) = \frac{\pi^{\frac{n}{2}} x^{\frac{n}{2}}}{\sqrt{\alpha_1 \alpha_2 \dots \alpha_n}\, \Gamma(\frac{n}{2}+1)} ;$$

We shall put

$$P_Q(x) = A_Q(x) - V_Q(x) .$$

The quadratic form $Q(u)$ is said to be "rational" if there is such a real number $\alpha$ that all $\alpha_j$ $(j = 1, 2, \dots, n)$ are integer multiples of $\alpha$ ; otherwise we say that $Q(u)$ is "irrational".

The following results are well-known:

I. $P_Q(x) = O(x^{\frac{n}{2}-1})$ when $Q$ is rational (see [6]) and this estimate is definitive; namely the following result is true:

II. $P_Q(x) = \Omega(x^{\frac{n}{2}-1})$ for $Q$ rational (see [7]).

On the other hand:

III. $P_Q(x) = o(x^{\frac{r}{2}-1})$ for $Q$ irrational (for $r \geqslant$ $\geqslant 6$ see [1], for $r = 5$ see [5]).

Furthermore we know that it is:

IV. $P_Q(x) = \Omega(x^{\frac{r-1}{4}})$ (see [8]).

V. For almost all systems $\alpha_1, \alpha_2, \dots, \alpha_r$ of positive real numbers (in the sense of the Lebesgue measure in the r-dimensional Euclidean space $E_r$ ) even the following is true: $P_Q(x) = O(x^{\frac{r}{4}+\varepsilon})$ for every $\varepsilon > 0$ (see [2]).

It is unknown if the estimates IV or V can be improved but we know that III cannot be, in general, improved, as it can be seen from the assertion:

VI. If $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) \to 0$ for $x \to +\infty$, then for arbitrary $r \geqslant 5$ there exists an irrational form $Q$ of the type (1) such that

$$P_Q(x) = \Omega(x^{\frac{r}{2}-1}\varphi(x)) \qquad \text{(see [9]).}$$

For a deeper and more detailed study of the function $P_Q(x)$ further specialization of the form $Q$ turns out advantageous. Let $\sigma$ and $r_j$ be integers, $\sigma \geqslant 2$; $r_j \geqslant 4$ $(j = 1, 2, \dots, \sigma)$; $r = r_1 + r_2 + \dots + r_\sigma$, let $\alpha_j > 0$ $(j = 1, 2, \dots, \sigma)$. We shall consider the following forms:

$$(2)\quad Q(u) = \sum_{j=1}^{\sigma} \alpha_j (u_{1,j}^2 + u_{2,j}^2 + \dots + u_{r_j,j}^2), \; \alpha_j > 0 \; (j = 1, 2, \dots, \sigma).$$

The assertions I - IV, of course, remain true also for the forms of the type (2); moreover, IV can be essentially strengthened due to the special choice (2) of the forms $Q$ :

VII. Let $\sigma$ and $\kappa_j$ $(j = 1, 2, \dots, \sigma)$ be integers, $\kappa_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, $\sigma \geq 2$, $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$. The following estimate holds for the forms of the type (2):

$$P_Q(x) = \Omega(x^{\frac{\kappa}{2} - \sigma}) \qquad \text{(see [2]).}$$

We have now:

VIII. Let $\sigma$ and $\kappa_j$ be integers, $\sigma \geq 2$, $\kappa_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$. Then for almost all systems of positive numbers $\alpha_1, \alpha_2, \dots, \alpha_\sigma$ (in the sense of the Lebesgue measure in $E_\sigma$) the estimate

$$P_Q(x) = O(x^{\frac{\kappa}{2} - \sigma + \varepsilon})$$

for every $\varepsilon > 0$ holds for the forms (2). (See [2].) For any form $Q$ of the type (1) let $f = f(Q)$ be the infimum of those real numbers $\omega$ for which

$$P_Q(x) = O(x^{\omega}),$$

i.e.,for every $\varepsilon > 0$

$$P_Q(x) = O(x^{f+\varepsilon}), \quad P_Q(x) = \Omega(x^{f-\varepsilon}).$$

Then, according to I,III,VII there is

$$\frac{\kappa}{2} - \sigma \leq f(Q) \leq \frac{\kappa}{2} - 1$$

for the forms (2); $\sigma$, $\kappa_j$ being integers, $\sigma \geq 2$, $\kappa_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$.

It is obvious that for $Q$ rational there is $f(Q) = \frac{\kappa}{2} - 1$ owing to II and by VIII it follows that for "almost all" $Q$ there is $f(Q) = \frac{\kappa}{2} - \sigma$.

Let us denote by $\beta = \beta(\alpha)$ the supremum of those real $\omega$ for which the inequality

$$\left| \frac{p}{q} - \alpha \right| < \frac{1}{q^{\omega}}$$

is satisfied for infinitely many pairs of integers $\{p_n;\ q_n\}_{n=1}^{\infty}$, $q_n \geq 1$, $q_n \to +\infty$. Notice that $\beta(\alpha) = \beta(\frac{1}{\alpha})$ and that always $2 \leq \beta(\alpha) \leq +\infty$; moreover, for almost all $\alpha$ (in the sense of the Lebesgue measure in $E_1$) we have $\beta(\alpha) = 2$.

Now, the following statement is true:

IX. Let $k_j$ be integers, $k_j \geq 4$ $(j = 1, 2)$, $k = k_1 + k_2$, $\alpha_j > 0$ $(j = 1, 2)$; let $\beta = \beta(\frac{\alpha_1}{\alpha_2})$,

$$Q(u) = \alpha_1 (u_{1,1}^2 + \dots + u_{k_1,1}^2) + \alpha_2 (u_{1,2}^2 + \dots + u_{k_2,2}^2).$$

Then

$$f(Q) = \frac{k}{2} - 1 - \frac{1}{\beta - 1} \qquad \text{(see [3])}$$

where for $\beta = +\infty$ we put $\frac{1}{\beta - 1} = 0$. Notice that

$\beta(\frac{\alpha_1}{\alpha_2}) = \beta(\frac{\alpha_2}{\alpha_1})$. Thus, the assertion IX solves the question of finding $f(Q)$ for the forms (2) in the case $\sigma = 2$. For $\sigma > 2$ the following result is known:

X. Let $\sigma$, $k_j$ be integers, $\sigma > 2$, $k_j \geq 4$ $(j = 1, 2, \dots, \sigma)$, let $k = k_1 + k_2 + \dots + k_\sigma$, $\frac{k}{2} - \sigma \leq f \leq \frac{k}{2} - 1$. Then there

exists a form (2) such that $f(Q) = f$ (see [4]).

Now, let us denote by $\beta = \beta(\alpha_1, \dots, \alpha_k)$ the supremum of those real numbers $\omega$ for which the system of inequalities

$$\left| \frac{p_j}{q} - \alpha_j \right| < \frac{1}{q^{\omega}} \qquad (j = 1, 2, \dots, k)$$

is satisfied for infinitely many (k+1)-tuples $\{p_{1n}, \dots, p_{kn}; q_n\}_{n=1}^{\infty}$ of integers, $q_n \geq 1$, $q_n \to +\infty$. Notice that $\frac{k+1}{k} \leq \beta(\alpha_1, \dots, \alpha_k) \leq +\infty$ and that for almost all systems $\alpha_1, \alpha_2, \dots, \alpha_k$ (in the sense of the Lebesgue measure in $E_k$) it is $\beta(\alpha_1, \dots, \alpha_k) = \frac{k+1}{k}$.

Our contribution is the following

Theorem 1. Let $\sigma$ be an integer, $\sigma \geq 2$, let $\alpha_j > 0$ $(j = 1, 2, \dots, \sigma)$, let $\beta = \beta\left(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1}, \dots, \frac{\alpha_\sigma}{\alpha_1}\right)$;

let $\kappa_j \geq \frac{2\beta}{\beta - 1}$, $\kappa_j$ integers $(j = 1, 2, \dots, \sigma)$; $\kappa = \kappa_1 + \kappa_2 + \dots + \kappa_\sigma$.

Then for the forms (2) we have

$$f(Q) = \frac{\kappa}{2} - 1 - \frac{1}{\beta - 1}.$$

We put $\frac{1}{\beta - 1} = 0$, $\frac{2\beta}{\beta - 1} = 2$ for $\beta = +\infty$.

Assymmetry of the assumptions of Theorem 1 is only seeming because $\beta\left(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1}, \dots, \frac{\alpha_\sigma}{\alpha_1}\right) = \beta\left(\frac{\alpha_1}{\alpha_2}, \frac{\alpha_3}{\alpha_2}, \dots, \frac{\alpha_\sigma}{\alpha_2}\right) = \dots = \beta\left(\frac{\alpha_1}{\alpha_\sigma}, \frac{\alpha_2}{\alpha_\sigma}, \dots, \frac{\alpha_{\sigma-1}}{\alpha_\sigma}\right)$.

It is $\beta \geq \frac{\sigma}{\sigma - 1}$ so that $\frac{2\beta}{\beta - 1} \leq 2\sigma$. According to this fact we see that the assumption $\kappa_j \geq \frac{2\beta}{\beta - 1}$ is automatically satisfied as soon as $\kappa_j \geq 2\sigma$. If $\beta$

passes through the interval $\langle \frac{\sigma}{\sigma-1}, +\infty \rangle$, $f(Q)$ passes through the interval $\langle \frac{r}{2} - \sigma, \frac{r}{2} - 1 \rangle$.

Thus, the Theorem 1 generalizes the assertion IX to the general case $\sigma \geqslant 2$ for which only the existence statement X was known up to now. The Theorem 1 solves the question of finding of $f(Q)$ for sufficiently large $r_j$ for every form (2). Jarník's method was used for the proof (see V. Jarník [3]).

## R e f e r e n c e s


[1] V. JARNÍK: Über Gitterpunkte in mehrdimensionalen Ellipsoiden (zweite Abhandlung), Math. Annalen 101 (1929), 136-146.

[2] Über Gitterpunkte in mehrdimensionalen Ellipsoiden, Math. Annalen 100(1928), 699-721.

[3] Über Gitterpunkte in mehrdimensionalen Ellipsoiden, The Tôhoku Math. Journal 30(1929), 354-371.

[4] Über Gitterpunkte in mehrdimensionalen Ellipsoiden: eine Anwendung des Hausdorffschen Massbegriffes, Math. Zeitschrift 38(1934), 217-256.

[5] V. JARNÍK, A. WALFISZ: Über Gitterpunkte in mehrdimensionalen Ellipsoiden, Math. Zeitschrift 32 (1930), 152-160.

[6] E. LANDAU: Über Gitterpunkte in mehrdimensionalen Ellipsoiden, Math. Zeitschrift 21(1924), 126-132.

[7] Über Gitterpunkte in mehrdimensionalen Ellipsoiden(zweite Abhandlung), Math. Zeitschrift 24 (1926), 299-310.

[8] E. LANDAU: Über die Anzahl der Gitterpunkte in gewissen Bereichen(vierte Abhandlung),Göttinger Nachrichten 1924,137-150.

[9] A. WALFISZ: Über Gitterpunkte in mehrdimensionalen Ellipsoiden(dritte Abhandlung),Math.Zeitschrift 27(1927),245-268.

(Received January 22, 1968)