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BANACH SPACES WITH THE DIFFERENTIABLE NORMS

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§ 1. Notations. The word space X denotes the Banach space X , $x_n \to x$, resp. $x_n \xrightarrow{w} x$ strong, resp. weak convergence in the space X , X* is dual of X .
Further we use the following notations:

$$S_k = \{x \in X;\ \|x\| = k\},\ K_k = \{x \in X;\ \|x\| \le k\},$$
$$S_k^* = \{f \in X^*;\ \|f\| = k\},\ K_k^* = \{f \in X^*;\ \|f\| \le k\}.$$

w^* -topology in X* is the topology of pointwise convergence in X . The isomorphism of X,Y is taken as the linear isomorphism of X,Y . P denotes the set of all real numbers. (X→Y) denotes the space of all continuous linear mappings of X into Y .

§ 2. Fundamental definitions.

Definition 1. We say that $x \in S_1 \subset X$ is a point of the weak or Gâteaux smoothness if the norm of X is Gâteaux differentiable at $x \in S_1$, i.e. if the limit

$$(1)\qquad \lim_{t \to 0} \frac{\|x+th\| - \|x\|}{t} = D\|\cdot\|(x, h)$$

exists for every $h \in X$.
We say that X is the weak or Gâteaux smooth (G) if every

point of S_1 is the point of the weak smoothness of S_1 .

Remark 1. The Gâteaux differential of every norm is linear in h (see for example [65]) and continuous in h (see for example [38]). The condition of Gâteaux differentiability of $\|x\|$ at x is equivalent to the fact that there exists only one support hyperplane at $x \in S_1$ ([78]).

Definition 2. A space X is called uniformly Gâteaux smooth or uniformly weak smooth (UG) if the norm of X is uniformly Gâteaux differentiable for $x \in S_1$, i.e. the limit (1) is uniform with respect to $x \in S_1$.

Definition 3. A point $x \in S_1$ is called a point of the strong or Fréchet smoothness of S_1 if the limit (1) is uniform in $h \in S_1$.

A space X is said to be Fréchet or strong smooth (F) if every point of S_1 is a point of the strong smoothness of S_1 .

Definition 4. A space X is said to be uniformly strong smooth or uniformly Fréchet smooth (UF) if the limit (1) is uniform in x , $h \in S_1$.

Definition 5. A space X is called uniformly rotund (UR) if the following implication is true:

$$(x_n, y_n \in S_1,\ \|\frac{x_n + y_n}{2}\| \longrightarrow 1) \Longrightarrow x_n - y_n \rightarrow 0 .$$

Remark 2. S. Kakutani ([52]) has proved that the condition (UR) is equivalent to the following one:

$$(x_n, y_n \in K_1,\ \|\frac{x_n + y_n}{2}\| \longrightarrow 1) \Longrightarrow x_n - y_n \rightarrow 0 .$$

Definition 6. A space X is called locally uniformly rotund (LUR) if the following implication is true:

$$(x_n, x_0 \in S_1, \|\frac{x_n + x_0}{2}\| \to 1) \Rightarrow x_n - x_0 \to 0 \; .$$

Definition 7. A space X is said to be weakly uniformly rotund (WUR) if the following implication is valid:

$$(x_n, y_n \in S_1, \|\frac{x_n + y_n}{2}\| \to 1) \Rightarrow x_n - y_n \xrightarrow{w} 0 .$$

Definition 8. A space X^* is said to be weakly* uniformly rotund (W*UR) if the following implication is true:

$$(f_n, g_n \in S_1^*, \|\frac{f_n + g_n}{2}\| \to 1) \Longrightarrow f_n - g_n \xrightarrow{w^*} 0 .$$

Definition 9. A point $x \in S_1$ is called an extremal point of S_1 if x is not an interior point of any segment in S_1. The set of all extremal points of S_1 is denoted by $ext\, S_1$.

Definition 10. A point $x \in S_1$ is called an exposed point of S_1 if there exists $f \in S_1^*$ such that $1 = f(x) > f(y)$ for each $y \in S_1$, $y \neq x$.

Definition 11. A space X is called rotund or strictly convex (R) if every point of S_1 is extremal point of S_1 .

Remark 3. The following well-known theorem is due to M.G. Krejn ([1]):

X is (R) iff every $f \in X^*$ attains its supremum at most at one point of S_1 .

From this theorem it follows immediately that each point of S_1 is exposed if X is (R).

Definition 12. We say that X has a property (p) when X is isomorphic to a space Y with the property (P) ((P) is (R),(G) and so on). X is said to have a property $(P_1 P_2)$ if X has the properties (P_1) and (P_2) jointly.

Remark 4. It is well-known that the condition (R) is equivalent to the following one:

$$(\|x+y\| = \|x\| + \|y\|,\ x \neq 0,\ y \neq 0\ ) \rightarrow x = ty,\ t > 0.$$

§ 3. List of known results.

S. Banach has proved that the norm of $C\langle 0, 1\rangle$ is Fréchet differentiable at $x_0 \in C\langle 0, 1\rangle$ iff x_0 attains its supremum at only one point of $\langle 0, 1\rangle$ ([3]). S. Mazur ([78]) has proved that similar condition is true for the space of bounded functions and that L_p, $p > 1$ is (F). He has proved that the set of all weak smooth points of S_1 in separable space contains a set which is G_δ and dense in S_1. V.L. Šmuljan ([87], [89], [90]) has observed that necessary and sufficient condition for the fact that the norm of X^* is Fréchet (resp.Gateaux) differentiable in $f \in S_1^*$ is that the following implication is valid:

$(x_n \in S_1,\ f(x_n) \to 1) \Longrightarrow \{x_n\}$ is strongly (resp.weakly) Cauchy-sequence. He has shown ([89]) that X is (R) (resp. (G)) when X^* is (G)(resp.(R)). Moreover he has established the following theorems ([87], [90]):

X^* is (UF) iff for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\|x - y\| \leq \varepsilon$ if $f(x) > 1 - \delta_\varepsilon$, $f(y) > 1 - \delta_\varepsilon$ for some $f \in S_1^*$ and $x, y \in S_1$.

X^* is (UG) iff for every $\varepsilon > 0$ and $g \in X^*$ there exists $\delta_{\varepsilon,g} > 0$ that $|g(x - y)| \leq \varepsilon$ whenever $f(x) > 1 - \delta_{\varepsilon,g}$, $f(y) > 1 - \delta_{\varepsilon,g}$ for some $f \in S_1^*$

and $x, y \in S_1$.

It means in fact in the terminology of [17 - 18] that X^* is (UF) (resp.(UG)) iff X is (UR) (resp. (WUR)).

Analogically: X is (UF) iff X^* is UR; further: X is (UG) iff X^* is (W^*UR).

The spaces (UR) have been intoduced by J. Clarkson in [16], the spaces (WUR),(W^*UR) by D.Cudia in [17 - 18].

V.L. Klee and M.M. Day have proved many fundamental theorems in these topics ([54 - 59],[21 - 26]).

The questions concerning the Čebyšev-subsets of Banach spaces are studied for example in the papers of V.L. Klee (see for example [59]), L.P. Vlasov ([94 - 97]), N.V. Jefimov and S.B. Stečkin ([44a - c]). J. Clarkson ([16]) has established that every separable Banach space is (r) . M.M. Day has proved ([25]) that every separable Banach space is (rg). V.L. Klee ([58]) has shown that every separable Banach space X is isomorphic to a space Y which norm is (GR) and its dual is (R).

Other types of rotundity have been also studied by R.C.James ([43]), D.P. Giesy ([33]), A. Beck ([4]). M.I.Kadec ([50]) has established for example that every separable space is isomorphic to a space (LUR) and that all separable spaces are homeomorphic in nonlinear sense ([48]). V.L. Klee ([57]) has shown that the following theorem is valid: Suppose X^* is separable. Then there exists an equivalent norm in X that its dual norm in X^* is (R) and the relations

$f_\alpha \in X^*$, $f_\alpha \xrightarrow{w^*} f$, $\|f_\alpha\| \to \|f\|$ implies $\|f_\alpha - f\| \to 0$.

J. Lindenstrauss ([72]) has proved that every reflexive Banach space is isomorphic to a space (R) and then it is also isomorphic to a space (G). He has also established ([72]) for example that the set of all points of Gâteaux-smoothness of S_1 is dense in S_1 in every reflexive Banach space and that the same result is valid concerning the points of Fréchet differentiability in the reflexive separable space.

E. Asplund has shown a general method of the construction of some special norms which gives for example:

Every reflexive space is (r g);

every reflexive separable space is (f lur).

E. Bishop and R.R. Phelps ([5]) have proved that the set of all $f \in X^*$ which attain their norms is norm-dense in X^*.

J. Lindenstrauss ([70]) has shown that for example the set of all linear continuous operations of X into Y which attain their norm is norm-dense in the space of all linear continuous operators of X into Y, where X is a reflexive space, Y is an arbitrary space.

J. Kurzweil ([66]) has studied the differentiability of higher order of the norm of L_p, $p > 1$ and the properties of analytic operators in real spaces ([67]).

K. Sundaresan ([86]) has studied a twice-differentiable norm.

The properties of the modulus of rotundity are studied in [69],[36- 37],[34] for example.
Using the geometrical properties of the space C(K) J.Wada ([98]) has established some topological characterizations of the space K .
Some other types of rotundity have been introduced by K. Fan and J. Glicksberg ([29],[30]).

Lists of the papers in these topics are contained, for example, in [18],[26],[27] and in this paper. But the latter one is not complete. It does not contain even many fundamental and important articles.

§ 4. Summary. This paper concerns the questions of the duality mappings, the isomorphisms of separable spaces, one fixed point theorem and a modification of one corollary of one well-known excellent **theorem of J. Rainwater**.
I wish to thank J.Kolomý for calling my attention to these problems.

§ 5. The duality mappings

Theorem 1. X is (WUR) iff the following implication is true:

$$(x_n, y_n \in K_1, \ \left\|\frac{x_n+y_n}{2}\right\| \to 1) \Longrightarrow x_n - y_n \xrightarrow{w} 0 .$$

Proof (is analogical to that of S. Kakutani for (UR)([52])):
Let X be (WUR). It is evident that it is sufficient to prove the following implication: For every $g \in S_1^*$ and for every $\varepsilon > 0$ there exists $\delta_{\varepsilon,g} > 0$ that

$$|g(x-y)| \geq \varepsilon \cdot \max(\|x\|, \|y\|) \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq (1-\delta_{\varepsilon,g}) \cdot \max(\|x\|, \|y\|).$$

Using the symmetry of x, y we suppose that $\|x\| \geq \|y\|$. It is sufficient to assume that $\|x\| = 1 \geq \|y\|$ because the general case can be proved from this one by changing x into $\frac{x}{\|x\|}$, y into $\frac{y}{\|x\|}$.

Let $|g(x-y)| \geq \varepsilon$, $\varepsilon \in (0,1)$, $1 = \|x\| \geq \|y\|$, $\eta \in (0, \varepsilon)$.

First of all let $\|y\| \leq 1$, $\|y\| \geq 1 - \eta$. Define $z = \frac{y}{\|y\|}$. Then

$$g(x-z) = g(x-y) + g(y-z).$$

Thus

$$|g(x-z)| \geq \varepsilon - |g(y-z)|.$$

Using the inequalities we obtain:

$$|g(x-z)| \geq \varepsilon - \eta .$$

Thus

$$\left\| \frac{x+z}{2} \right\| \leq 1 - \delta_{\varepsilon-\eta,g} .$$

Then we have

$$\left\| \frac{x+y}{2} \right\| \leq \left\| \frac{x+z}{2} \right\| + \left\| \frac{y-z}{2} \right\| \leq 1 - \delta_{\varepsilon-\eta,g} + \frac{\eta}{2} .$$

Suppose that $\|y\| \leq 1 - \eta$; then $\left\| \frac{x+y}{2} \right\| \leq 1 - \frac{\eta}{2}$.

Now generally:

$$\left\| \frac{x+y}{2} \right\| \leq \max \left(1 - \delta_{\varepsilon-\eta,g} + \frac{\eta}{2} , 1 - \frac{\eta}{2}\right) .$$

It is easy to see that the right hand of this inequality can be made less than one only by changing of η.

<u>Definition 13</u>. The mapping $\varphi : E \to E^*$ is called w^*-demicontinuous on $S_1 \subset E$ if

$$x_n, x \in S_1, \ x_n \to x \quad \text{implies } \varphi(x_n) \xrightarrow{w^*} \varphi(x).$$

<u>Definition 14</u>. Let X be (G)-space. The duality mapping on $S_1 \subset X$ is the mapping $\mathcal{J}$ defined by following: For $x \in S_1$ $\mathcal{J}(x) = f \in S_1^*$, where $f(x) = 1$.

Remark 5. For the simplification of notations we define the duality mapping $\mathcal{J}$ only on S_1.

Theorem 2 (V.L. Šmuljan ([89, 91]). Let X be a (G)-space. Then the duality mapping $\mathcal{J}$ is w^*-demicontinuous mapping of S_1 into S_1^*. $\mathcal{J}$ is continuous if X is a (F)-space.

Definition 15. The mapping $\varphi : X \to X^*$ is said to be uniformly w^*-demicontinuous if for every $\varepsilon > 0$ and $x \in X$ there exists $\delta_{\varepsilon,x} > 0$ such that $|(\varphi(x) - \varphi(y))(x)| \leq \varepsilon$ whenever $\|x-y\| \leq \delta_{\varepsilon,x}$.

Proposition 1. Let X be a (UG)-space. Then the duality mapping $\mathcal{J}$ is uniformly w^*-demicontinuous from S_1 into S_1^*.

Proof. Let $\varepsilon > 0$, $x \in X$ be arbitrary elements. Let $\delta = \delta_{\varepsilon,x} > 0$ be the number from the definition of the (UG)-space X. Let $x, y \in S_1$, $\|x-y\| \leq \delta$. Let $\mathcal{J}x = f$, $\mathcal{J}y = g$. Then $f(x) = 1$, $g(x) = g(y) + g(x-y) \geq 1-\delta$. Thus $|(f-g)(x)| \leq \varepsilon$ by the well-known Šmuljan characterization of the (UG)-property of X.

One may prove the following proposition analogously:

Proposition 2 (see also [52a]). Let X be a (UF)-space. Then $\mathcal{J}$ is uniformly continuous mapping of S_1 into S_1^*.

Proposition 3. Let X be a (G)-space. Assume that the differential $D\|\cdot\|(x,h)$ of the norm of X is uniformly w^*-demicontinuous mapping. Then X is the (UG)-space.

Proof. Suppose that the norm of X is not uniformly Gâteaux differentiable. Let us write for $x_n, x \in S_1$, $t_n \in P$

$$\|x_n+t_n h\|-\|x_n\| = D\|\cdot\|(x_n,t_n h)+\omega(x_n,t_n h)$$

$$\|x+t_n h\| - \|x\| = D\|\cdot\|(x,t_n h)+\omega(x,t_n h).$$

Then there exist a $\varepsilon_0 > 0$ and $|t_n| \in (0, \frac{1}{n})$, $x_n \in$ $\in S_1$ such that

$$\left|\frac{\omega(x_n,t_n h)}{t_n}\right| \geq \varepsilon_0 .$$

We have

$$\left|\frac{\omega(x_n,t_n h)}{t_n} - \frac{\omega(x,t_n h)}{t_n}\right| = \left|\frac{\|x_n+t_n h\|-\|x_n\|}{t_n} - \right.$$
$$- \frac{\|x+t_n h\| - \|x\|}{t_n} +$$
$$\left. + D\|\cdot\|(x,h) - D\|\cdot\|(x_n,h)\right| .$$

By the mean-value theorem there exist τ_n, $\tau_n' \in (0,1)$ such that

$$\left|\frac{\omega(x_n,t_n h)}{t_n} - \frac{\omega(x,t_n h)}{t_n}\right| = \left| D\|\cdot\|(x_n+\tau_n t_n h, h) - \right.$$
$$- D\|\cdot\|(x_n,h) + D\|\cdot\|(x,h) -$$
$$\left. - D\|\cdot\|(x+\tau_n' t_n h, h)\right| .$$

The right side of this equality converges to zero as $n \to$ $\to \infty$. By our hypothesis we have a contradiction because

$\frac{\omega(x,t_n h)}{t_n} \longrightarrow 0$ whenever $n \to \infty$ because X is a (G)-space.

One may prove the following assertion similarly (see also Th.4.3 [93]):

Proposition 4. Let X be a (F)-space. Suppose the differential of the norm of X is uniformly continuous in sense of the space $(X \to P)$. Then X is a (UF)-space.

Proposition 5. Let X be a reflexive (UG)-space. Suppose the differential $D\|\cdot\|(x,h)$ is uniformly continuous in x and weakly continuous in h jointly. Then X is a (UF)-space.

Proof. Assume X is not a (UF)-space. Then there exist $\varepsilon_0 > 0$ and $\{x_n\} \subset S_1$, $\{h_n\} \subset S_1$ such that

$$\left|\frac{\omega(x_n, t_n h_n)}{t_n}\right| \geq \varepsilon_0 ,$$

where ω is defined as in Proposition 3.

Let $h_n \xrightarrow{w} h$. We have

$$\|x_n + t_n h_n\| - \|x_n\| = D\|\cdot\|(x_n, t_n h_n) + \omega(x_n, t_n h_n)$$

$$\|x_n + t_n h\| - \|x_n\| = D\|\cdot\|(x_n, t_n h) + \omega(x_n, t_n h).$$

Then

$$\left|\frac{\omega(x_n, t_n h_n)}{t_n} - \frac{\omega(x_n, t_n h)}{t_n}\right| = \left|\frac{\|x_n + t_n h_n\| - \|x_n\|}{t_n} - \frac{\|x_n + t_n h\| - \|x_n\|}{t_n} + D\|\cdot\|(x_n, h) - D\|\cdot\|(x_n, h_n)\right|.$$

Then there exist $\tau_n, \tau_n' \in (0, 1)$ such that

$$\left|\frac{\omega(x_n, t_n h_n)}{t_n} - \frac{\omega(x_n, t_n h)}{t_n}\right| = \left|D\|\cdot\|(x_n + \tau_n t_n h_n, h_n) - D\|\cdot\|(x_n + \tau_n' t_n h, h) + D\|\cdot\|(x_n, h) - D\|\cdot\|(x_n, h_n)\right|.$$

It is easy to see that the right side of this equality converges to zero as $n \to \infty$. But $\frac{\omega(x_n, t_n h)}{t_n} \to 0$ whenever $n \to \infty$ because X is a (UG)-space. This contradiction concludes the proof.

Proposition 6. Let X be a (UG)-space or a (F)-space. Suppose that

$$D\|\cdot\|(x_n+t_n h_n, h_n) - D\|\cdot\|(x_n, h_n) \to 0$$

whenever $h_n, x_n \in S_1$, $t_n \to 0$.

Then X is a (UF)-space.

Proof. Let us prove the part of our Proposition for the case of (F). Using the notations of the preceding propositions we have

$$\left|\frac{\omega(x_n, t_n h_n)}{t_n} - \frac{\omega(x, t_n h_n)}{t_n}\right| = |D\|\cdot\|(x_n+\tau_n t_n h_n, h_n) - D\|\cdot\|(x_n, h_n) + D\|\cdot\|(x, h_n) - D\|\cdot\|(x+\tau_n' t_n h_n, h_n)|.$$

Now we proceed analogously as in Proposition 5.

Using the criterium of reflexivity by R.C. James ([40], [41]) and Šmuljan's theorem D. Cudia has proved:

Theorem 3 (D. Cudia [17]). Let X be a weakly sequentially complete space, X^* a (G)-space. Then X is a reflexive space.

Theorem 4 (D. Cudia [17]). A space X is reflexive if X^* is a (F)-space.

Remark 6. It is known that the duality mapping $\mathcal{J}$ is weakly continuous in spaces l_p, $p > 1$ and that $\mathcal{J}$ is not weakly continuous in spaces L_p, $p > 1$ (for the references see for example [31]).

Remark 7. Using the well-known fact that rotundity of the space X implies its uniform rotundity in the case of finite dimension we can prove at once that for the finite-dimensional case (UG) implies (UF). From theorem of N.A. Ivanov it follows that $(G) \Longrightarrow (F)$ (this theorem asserts: the Lipschitz - mapping F has the Fréchet derivative $F'(x_0)$ at x_0 whenever it possesses the linear Gateaux differential at x_0).

§ 6. Isomorphisms and reflexivity of the smooth spaces

Using the modification of one method of M.I.Kadec ([50]) we have the following

Theorem 5. Let X^* be separable space. Then X is (w u r)-space.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a countable dense subset of S_1^*. Let us define the functional $I(x)$ on X by

$$I(x) = \sqrt{\sum_{k=1}^{\infty} \frac{1}{2^k} \cdot f_k^2(x)} .$$

Let $\|x\|$ denote the norm of X . Then it is easy to see that the functional

$$|||x||| = \sqrt{\|x\|^2 + I^2(x)}$$

is the equivalent norm to $\|x\|$.

We shall show that this norm $|||x|||$ is (WUR). Let $|||x_n||| = |||y_n||| = 1, \quad |||\frac{x_n+y_n}{2}||| \to 1 .$

We have

$$I^2(x_n+y_n) + I^2(x_n-y_n) = 2\cdot(I^2(x_n) + I^2(y_n)) .$$

It is easy to see that

$$\| x_n + y_n \|^2 \leqq 2 \cdot (\| x_n \|^2 + \| y_n \|^2) \ .$$

From these facts it follows (by addition) that

$$|\!|\!| x_n + y_n |\!|\!|^2 + I^2 (x_n - y_n) \leqq 2 \cdot (|\!|\!| x_n |\!|\!|^2 + |\!|\!| y_n |\!|\!|^2) \ .$$

The right side of this inequality is equal to 4, $|\!|\!| x_n + y_n |\!|\!|^2 \to 4$. Then $I^2(x_n - y_n) \to 0$. Thus we have

$$f_k(x_n - y_n) \to \text{ whenever } \quad n \to \infty$$

for every $k \in N$.

The sequence $\{x_n - y_n\}$ is bounded in X and $f_k(x_n - y_n) \to 0$ as $n \to \infty$ and $k = 1, 2, \ldots$. Hence by the well-known theorem $x_n - y_n \xrightarrow{w} 0$. This completes the proof.

Corollary 1. Let X^* be separable. Then X^* is (ug).

Corollary 2. Let X be a reflexive separable space. Then X is (w u r) and X is (ug).

Remark 9 (Construction of a space (WUR) which is not (ur): In the paper [21] M.M. Day has constructed a separable reflexive (R)-space X_0 which is not (ur). If we introduce in this space the norm as in the proof of Proposition 10 we obtain the example of the space (WUR) which is not (ur). By duality we obtain the example of the space (UG) which is not (uf).

Remark 10. From these facts it follows that not every (WUR)-space is reflexive.

Theorem 6. Let X be a separable space. Then X^* is (w^*ur) .

Proof. Let $\{x_k\}$ be a countable dense subset of S_1

and define the functional I on X^* by

$$(2) \qquad I(f) = \sqrt{\sum_{k=1}^{\infty} \frac{1}{2^k} \cdot f^2(x_k)} \; .$$

Further we proceed analogously as in Proposition 10.

Lemma 1 (see [51]). Let X be a separable space. Let the new norm $|||f|||$ in X^* be defined by (2) from the proof of Proposition 11. Then $|||f|||$ is w^*-lower-semicontinuous on X^*.

Proof. It follows immediately from well-known Fatou-lemma and from Theorem of resonance ([99], chapt. 3).

Theorem 7 (see also V.L. Klee [55], M.M. Day [25], I. Singer [85]). Let X be a space. Assume that the new norm $|||f|||$ in X^* is equivalent to the obvious supremum - norm of X^*. Suppose $|||f|||$ is w^*-lower-semicontinuous. Then $|||f|||$ is a dual norm of some norm $|||x|||$ in X which is equivalent to $\|x\|$.

Proof. Denote the unit closed ball in the norm $|||f|||$ by M. Then M is w^*-closed and thus $({}^{\circ}M)^{\circ} = M$ where ${}^{\circ}M$ denotes the polar set of M in X and $({}^{\circ}M)^{\circ}$ denotes the polar set of ${}^{\circ}M$ in X^*. Define $|||x|||$ in X by the set ${}^{\circ}M$ as its unit closed ball. Then $|||x|||$ satisfies all conditions in our theorem.

Theorem 8. Let X be a separable space. Then X is (ug).

Proof. It follows immediately from Theorems 6, 7 and Lemma 1.

Proposition 7. Let X be a weakly sequentially complete space, X^* be isomorphic to a (G)-space Y. Suppose that the unit ball of Y is w^*-closed. Then X is reflexive.

Proof. It follows immediately from the considerations of

Theorems 3.7.

Proposition 8. Suppose X^* be isomorphic to a (F)-space Y. Assume the unit ball of Y is w^*-closed. Then X is reflexive.

Proof. It follows from Theorems 4,7.

Now we shall prove the modification of the results of M.M. Day for the strong case.

Theorem 9. Let X be a (F)-space, Y be a (FR)-space. Let there exist a linear one-to-one continuous mapping L of X into Y. Then X is a (fr)-space.

Proof. Let $\|x\|$ denote the norm of X, $|||y|||$ denote the norm of Y. Define a new norm of X by

$$\left\|\!\left| x \right|\!\right\| = \|x\| + |||Lx||| .$$

This norm is evidently strictly convex and Fréchet differentiable (see [28]).

Corollary 3. Let X be a (f)-space and suppose there exists a linear one-to-one continuous mapping of X into a (fr)-space Y. Then X is a (fr)-space.

The following lemma is well-known.

Lemma 2. Let X be a separable space. Then there exists a linear one-to-one continuous mapping of X into $L_2 < 0,1 >$.

Using the fact that $L_2 < 0, 1 >$ is (UR UF) we have

Theorem 10. Let X^* be separable. Then X is a (fr)-space.

Proof. G. Restrepo ([84]) has proved that a separable space is (f) iff X^* is separable. Then X is (f). This fact, Lemma 3 and Theorem 9 imply this assertion.

Remark 8. The part "if" of Restrepo's theorem from the proof of Theorem 10 has been also established by M.I.Kadec([51]).

§ 7. A theorem concerning the fixed point of nonexpansive mapping

Definition 16. Let C be a subset of X . A mapping T : $: C \to C$ is said to be nonexpansive on C if $\|Tx - Ty\| \leq \|x - y\|$ whenever $x, y \in C$.

Definition 17. Let C be a bounded subset of the space X , $\sigma(C)$ denote its diameter. The point $x \in C$ is said to be a diametral point of C if $\sup_{y \in C} \|x - y\| = \sigma(C)$.

Definition 18. ([12]). A convex subset $C \subset X$ is said to have normal structure if every bounded convex subset $C_1 \subset C$ which contains more than one point contains a point which is not diametral of C_1 .

It is well-known ([13],[35]) that every nonexpansive mapping of a convex bounded closed subset C of a (UR)-space into C has a fixed point, i.e. there exists a point $x \in C$ such that $Tx = x$.

W.A. Kirk ([53]) has proved the following:

Theorem 11 (W.A. Kirk). Let X be a reflexive space, C be a bounded closed convex subset of X which has normal structure. Then every nonexpansive mapping T of C into itself has a fixed point.

Theorem 12 (D.G. Figueiredo [31] (= for example)).Let X be a (UR)-space. Then every bounded closed convex subset of X has normal structure.

Theorem 13. Let X be a (WUR)-space. Then every bounded closed convex subset of X has normal structure.

Proof (Analogical to that for the case (UR)). It is sufficient to prove that every bounded convex subset C of X which contains more than one point has a point which is not diametral.

Let $x, y \in C$, $\|x - y\| \geq \frac{1}{2}\,\delta(C)$. Let us take $u = \frac{x+y}{2}$. This point is not diametral. Suppose u is diametral. Then there exists a sequence $\{v_n\} \subset C$ such that $\|u - v_n\| \longrightarrow \delta(C)$. We have $\|x - v_n\| \leqq \delta(C)$ and $\|y - v_n\| \leqq \delta(C)$.

Let K denote the closed ball with the center 0 and the radius $\delta(C)$. Then we have

$$x - v_n \in K,\ y - v_n \in K,\ \left\|\frac{x - v_n + y - v_n}{2}\right\| = \|u - v_n\| \to \delta(C).$$

Since then $x - v_n - (y - v_n) = x - y \xrightarrow{w} 0$, it is x = y. This contradiction concludes the proof.

The following assertion follows at once from Kirk's Theorem and Theorem 13.

Theorem 14. Let X be a reflexive (WUR)-space, C a bounded closed convex subset of X, T a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorems 5,14 imply

Theorem 15. Let X be a separable, reflexive space. Then X is isomorphic to a space Y with the following property:

Every nonexpansive mapping T of a closed convex bounded subset C into itself has a fixed point.

§ 8. Appendix.

1. It is easy to see that the following theorem is true:

Theorem 16. X^* is $(W^* \mathcal{U} R)$ iff the following implication is valid:

$$(f_n, g_n \in K_1^*,\ \| \frac{f_n + g_n}{2} \| \to 1) \Rightarrow f_n - g_n \xrightarrow{w^*} 0 .$$

Proof. It is analogical to that of Theorem 1.

2. The following assertion is analogical to Theorem 13.

Theorem 17. Let X^* be a $(W^* \mathcal{U} R)$ -space. Then every bounded closed convex subset of X^* has normal structure.

Proof. It is a modification of the proof of Theorem 13.

3. J. Rainwater ([82]) has proved the following very important theorem:

Theorem 18 (J. Rainwater). Let X be a Banach space, $\{x_n\}$ a bounded sequence in X, $x \in X$. Then $x_n \xrightarrow{w} x$ if $f(x_n) \to f(x)$ for each $f \in ext\, S_1^*$.

It follows immediately from this assertion that the following generalization of Theorem 5 is valid.

Theorem 19. Suppose there exists a countable subset $M \subset X^*$ such that $\overline{M} \supset ext\ S_1^*$. Then X is (wur)-space.

Proof. Let $M = \{f_k\}_{k=1}^{\infty}$. Define the functional $I(x)$ on X by

$$I(x) = \sqrt{\sum_{k=1}^{\infty} \frac{1}{2^k} \cdot f_k^2(x)} .$$

Let $\| x \|$ denote the norm of X. Define the new equivalent norm to $\| x \|$ by

$$||| x ||| = \sqrt{\| x \|^2 + I^2(x)} .$$

Suppose $|||x_n||| = |||y_n||| = 1$,

$$||| \frac{x_n + y_n}{2} ||| \to 1 .$$

As in proof of Theorem 5 we obtain that $f_k(x_n - y_n) \to_{n\to\infty} 0$ for every k. $\{x_n - y_n\}$ is a bounded sequence in X. Thus $g(x_n - y_n) \longrightarrow 0$ for each $g \in ext\, S_1^*$. Using the Theorem of J. Rainwater we have that $x_n - y_n \xrightarrow{w} 0$. Theorem is proved.

4. Theorem 10 is the consequence of the general method of E. Asplund ([1a])

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(The notations of the journals: BAMS = Bull. of the Amer. Math. Soc., BPA = Bull. de L.Acad.Pol., CMUC = Comment. Math.Univ.Carol., DAN = Dokl.Ak.Nauk SSSR, IJM = Israel Journal of Mathem., PAMS = Proc.of the Amer.Math.Soc., PNA = Proc. of the Nat.Ac.Sc.USA, TAMS = Trans.of the Amer. Math.Soc., UMN = Uspechi mat.nauk SSSR.)

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