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APPROXIMATION BY HILL FUNCTIONS x)

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Introduction

The finite element method has become a very effective method for numerical solution of partial differential equations. See e.g.[1],[2],[3] and many others that deal with the engineering or mathematical aspects. In a series of papers we shall build up one variant of this method for boundary value problems of partial differential equations especially of elliptic type. See e.g. [4] - [15] . The problem of approximation in the fractional Sobolev spaces $W_2^{\alpha}(R_m)$ is of special importance for this approach. The problem is the following. To study functions $\omega_h(x)$ with compact support such that for every $f \in W_2^{\alpha}(R_m)$ and $1 > h > 0$ there exist $C_h(\underline{k}, j)$, $\underline{k} = (k_1, \dots, k_m)$, k_i integral $j = 1, \dots, \kappa$ such that

$$\| e(f) \|_{W_2^{\beta}(R_m)} = \| f(x) - \sum_{j=1}^{\kappa} \sum_{\underline{k}} C_h(\underline{k}, j)\, \omega_j\left(\frac{\underline{x} - hA\underline{k}}{h}\right) \|_{W_2^{\beta}(R_m)}$$

$$\leq C \| f \|_{W_2^{\alpha}(R_m)} h^{\mu}$$

provided $0 \leq \beta \leq k' \leq k$, $\alpha \geq \beta$, with

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$\mu = \min(k-\beta, \alpha-\beta)$ and C is not dependent on f and h and A is a non singular matrix and that the support of $\omega(f)$ lies in an Lh neighborhood of the support of f with L independent of f and h. An approximation property of this type will play a very basic and important role in further papers (see e.g. [4] - [10]).

In this paper we analyze some necessary and sufficient conditions on $\omega(x)$ 1) for the above approximation property.

The name "hill functions" describes the fact that the support of the functions $\omega(\frac{x}{h})$ is small (of order h). The special kinds of these "hill functions" have been studied by different authors and called by different names.

1. Some results of the theory of the Fourier Transform

We shall quote here some known results of the theory of Fourier Transform of generalized functions without proofs. For the proofs see e.g. K. Yosida [16] or Gelfand [17].

We denote R_m the m-dimensional Euclid space: $\underline{x} \equiv (x_1, \dots, x_m)$, $\|\underline{x}\|^2 = \sum_{i=1}^{m} (x_i)^2$. Let $S(R_m)$ 2) be the totality of all rapidly decreasing functions (at ∞) with the usual topology (see K. Yosida

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1) After finishing this paper I received information that other authors received result very closed to that in this paper, esp. Fix, Strang, De Guglielmo, see [18] - [21].

2) We shall very often write simply S instead of $S(R_m)$ in this and analogous cases.

[16], p.146).

The space of generalized functions over $S(R_n)$ will be denoted $S'(R_n)$ [1]. For any $\phi \in S(R_n)$ we define Fourier transform: $F(\phi)(\underline{\sigma})$.

(1.1) $$F(\phi)(\underline{\sigma}) = \tilde{\phi}(\underline{\sigma}) = \int_{R_n} e^{i\langle \underline{x}, \underline{\sigma}\rangle} \phi(\underline{x})\, d\underline{x} \ ,$$

$$\text{with } \langle \underline{x}, \underline{\sigma} \rangle = \sum_{j=1}^{n} x_j \sigma_j \ ,$$

and the inverse transform

(1.2) $$F^{-1}(\phi)(\underline{\sigma}) = (2\pi)^{-n} \int e^{-i\langle \underline{x}, \underline{\sigma}\rangle} \phi(\underline{x})\, d\underline{x} \ .$$

It is well known that the Fourier transform is a continuous mapping of S on S (See e.g. [17], vol.2,III.§ 1.1). Let $\phi \in S$, then

(1.3) $$F[F(\phi)] = (2\pi)^n \phi(-x) \ .$$

Let $f \in S'$. The Fourier transform of f, i.e. $F(f)$ will be defined by the equation

(1.4) $$(F(f), F(\phi)) = (2\pi)^n (f, \phi) \ .$$

Let $f \in L_2 \subset S'$ with L_2 the space of all square integrable functions on R_n, then $F(f) \in L_2$ and

(1.5) $$\| F(f) \|_{L_2}^2 = (2\pi)^n \| f \|_{L_2}^2 \ .$$

Let $\underline{A}$ now denote a linear mapping R_n on R_n - let this mapping be given by the matrix A of order n (which is necessary nonsingular) [2]. Let A^{-1} be the inverse mapping.

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1) If $f \in L_1$ then $(f, \phi) = \int_{R_n} \bar{f} \phi \, d\underline{x}$, $\phi \in S$.

2) We shall denote the matrix and the mapping by the same symbol.

Let $f \in L_2$; and let us denote $(Af)(x) = f(A^{-1}x) \in L_2$, $(A^{-1}f)(x) = f(Ax)$. Now let $f \in S'$, then $Af \in S'$ with

$$(1.6) \qquad (Af, \phi) = |A| (f, A^{-1}\phi)$$

and $|A|$ be the determinant of the matrix A.

A generalized function $f \in S'$ will be said to be periodic with the matrix of period A if and only if for every $\phi \in S$ and every $\underline{k} \equiv (k_1, \dots, k_n)$, k_j integers $j = 1, \dots, n$, we have

$$(f, \phi) = (f, \psi)$$

with

$$\psi(\underline{x}) = \varphi(\underline{x} - A\underline{k}) .$$

A closed set K will be said to be a support of $f \in S'$ if and only if $(f, \phi) = 0$ for all $\phi \in S$ and $\phi = 0$ on some neighborhood of K; it will be written $K = \mathrm{sup}\, f$ 1).

A continuous function $g(x)$ will be said to be a multiplier if $g\phi \in S$ for every $\phi \in S$ and $g\phi_s \to 0$ if $\phi_s \to 0$, $s = 1, 2, \dots$ with the convergence in the topology of S. A function $f_0 \in S'$ will be said to be a convolutor if

$$f_0 * \phi = (f_0(\xi), \phi(\underline{x} + \xi)) = \psi(\underline{x}) \in S$$

for every $\phi \in S$ and if $\phi_m \to 0$ in topology of S then $f_0 * \phi_m \to 0$ in the topology of S. If $g(x)$ is a multi-

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1) We emphasize that the support in our sense does not mean the minimal support. In the literature very often the notion support means the minimal support. But this is not our case.

plier then $F^{-1}(g) = f_1$ is a convolutor and

$$F(f*f_1) = F(f) \quad F(f_1) .$$

Let $f \in S'$ have a compact support then $F(f)$ is a multiplier.

Lemma 1.1. Let $\omega \in S'$ and $\Omega_{\underline{a}} = E[\,|x_j| \leq a_j\,;\ j = 1,\dots,m\,]$, $supp.\ \omega \subset \Omega_{\underline{a}}$. Then $F(\omega)(\underline{\sigma})$ is a function which could be continuated analytically in the complex space $(s_1,\dots,s_m)$, $s_j = \sigma_j + i\tau_j$ and for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ and $q(\varepsilon) \geq 0$ that

$$(1.7)\ |F(\omega)(\sigma + i\tau)| \leq (1 + \|\sigma\|^2)\ C e^{(a_1+\varepsilon)|\tau_1| + \dots + (a_m+\varepsilon)|\tau_m|}$$

See [17],vol.2,Ch.III.,§ 2.2.

Lemma 1.2. Let $f(\underline{s})$, $s = (s_1,\dots,s_m)$, $s_j = \sigma_j + i\tau_j$ is an entire function of m complex variables such that for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ and $q(\varepsilon) \geq 0$ that

$$(1.8)\ |f(\underline{s})| \leq C(\varepsilon)(1 + \|\sigma\|^2)\ e^{(a_1+\varepsilon)|\tau_1| + \dots + (a_m+\varepsilon)|\tau_m|} .$$

Then there exists $\omega \in S'$ with $supp.\ \omega \subset \Omega_{\underline{a}}$, $\underline{a} = (a_1,\dots,a_m)$ such that $f(s)$ is analytic continuation of $F(\omega)(\underline{\sigma})$ in the space of complex variables $(s_1,\dots,s_m) = \underline{s}$, $s_j = \sigma_j + i\tau_j$. See [17],vol.2,Ch.III,§ 4.

2. The net function

Definition 2.1. The set $L \subset R_m$, $L = E[\underline{k} = (k_1,\dots,k_m)$, k_j integer] will be said to be a nornal net. Let $\underline{A}$ be a linear mapping R_m on R_m by a matrix A. Then the set $L_A = \underline{A}L$ will be said to be a $\underline{A}$-net.

Theorem 2.1. Let a function $\varphi \in S'$ with compact support be given. Further let $c(k_1, \dots, k_n)$ be a function defined on the normal net L, and let there exist $0 \leq \gamma < \infty$ and $C > 0$ with $|c(\underline{k})| \leq C\|\underline{k}\|^{\gamma}$. Defining

$$(2.1) \qquad f = \sum_{\underline{k} \in L} c(\underline{k})\,\varphi(\underline{x} - A\underline{k}) \in S' \quad ;$$

the sum is convergent in the usual sense of the theory of generalized functions and the Fourier transform $F(f)$ is

$$(2.2) \qquad F(f) = F(\varphi) \sum_{\underline{k} \in L} c(\underline{k})\, e^{i\langle A\underline{k}, \underline{x}\rangle}$$

with $F(\varphi)$ as a multiplier. The sum in (2.2) is convergent in the sense of the theory of generalized functions.

Proof. 1. Because $\varphi \in S'$ has by assumption a compact support, the series (2.1) converges obviously in the sense of generalized functions.

2. Because φ has compact support $F(\varphi)$ is a multiplicator (see [17], vol.2,Ch.3, § 3,p.4 and p.7). The series in (2.2) obviously converges as a generalized function.

Let $\psi \in S$; then

$$\left(F(\varphi)(\underline{x}) \sum_{\underline{k}} c(\underline{k})\, e^{i\langle A\underline{k}, \underline{x}\rangle},\quad \psi(\underline{x})\right) =$$

$$= \left(\sum_{\underline{k}} c(\underline{k})\, e^{i\langle A\underline{k}, \underline{x}\rangle},\ \overline{\overline{F(\varphi)(\underline{x})}}\ \psi(\underline{x})\right) =$$

$$= \sum_{\underline{k}} c(\underline{k})\left(e^{i\langle A\underline{k}, \underline{x}\rangle},\ \overline{\overline{F(\varphi)(\underline{x})}}\ \psi(\underline{x})\right) =$$

$$= \sum_{\underline{k}} c(\underline{k})\left(F(\varphi)(\underline{x}),\ e^{-i\langle A\underline{k}, \underline{x}\rangle}\,\psi(\underline{x})\right) .$$

Now put $\phi = F^{-1}(\psi)$. We have

$$F^{-1}(e^{-i\langle A\underline{k},\underline{x}\rangle}\psi(\underline{x})) =$$

$$= (2\pi)^{-n}\int_{R_n} e^{-i\langle\underline{\sigma},\underline{x}\rangle}e^{-i\langle A\underline{k},\underline{x}\rangle}\psi(x)\,dx = \phi(\underline{\sigma}+A\underline{k}) .$$

So we have

$$(F(g)(\underline{x})\sum_{\underline{k}} c(\underline{k})\, e^{i\langle A\underline{k},\underline{x}\rangle},\quad F(\phi)) =$$

$$= (2\pi)^n \sum_{\underline{k}} c(\underline{k})(g(\underline{\sigma}),\ \psi(\underline{\sigma}+A\underline{k}))$$

and so

$$(2.3)\qquad F(f) = F(g)\sum_{\underline{k}} c(\underline{k})\, e^{i\langle A\underline{k},\underline{x}\rangle} .$$

3. The spaces

Definition 3.1. The space $W_2^{\alpha}(R_n)$, $\alpha \geq 0$ will be the space of all functions $f \in S'$ that

$$(3.1)\qquad |F(f)|^2(1+\|\underline{x}\|^{2\alpha}) \in L_1(R_n)$$

and

$$(3.2)\qquad (2\pi)^n \|f\|^2_{W_2^{\alpha}(R_n)} = \| |F(f)|^2(1+\|\underline{x}\|^{2\alpha})\|_{L_1(R_n)} .$$

The spaces $W_2^{\alpha}(R_n)$ are the fractional Sobolev spaces. Obviously $W_2^{\alpha}(R_n) \supset W_2^{\beta}(R_n)$ for $0 \leq \alpha \leq \beta$ and $W_2^{0}(R_n) = L_2(R_n)$. The norm introduced in (3.2) is equivalent with the more common norm used in $W_2^{\alpha}(R_n)$

$$(3.3)\qquad \|u\|^2_{W_2^{\alpha}(R_n)} = (2\pi)^{-n}\int |F(x)|^2(1+\|\underline{x}\|^2)^{\alpha}\, d\underline{x} .$$

4. Some approximation theorems

Definition 4.1. The function $\chi(x)$ will be said to be a trigonometrical polynomial with periodicity matrix $B = (A^T)^{-1} 2\pi$ [1] if it is possible to write the function $\chi(x)$ as a finite sum

$$\chi(\underline{x}) = \sum_{\underline{k}} a(\underline{k})\, e^{i\langle A\underline{k}, \underline{x}\rangle},$$

$\underline{k} \equiv (k_1, \dots, k_n)$.

Theorem 4.1. Let us have $\omega_s(\underline{x}) \in S'$, $s = 1, 2, \dots$ $\dots k$, with compact support. Further let a regular matrix A be given. Let there exist trigonometric polynomials χ_s, $s = 1, \dots, k$ with periodicity matrix $B = (A^T)^{-1} 2\pi$ such that

$$\Lambda(\underline{x}) = \sum_{j=1}^{k} \lambda_j(\underline{x})\, \chi_j(\underline{x}) \tag{4.1}$$

with

$$\lambda_j = F(\omega_j)$$

has the following properties

1)

$$\Lambda(0) \neq 0 \tag{4.2}$$

2)

$$|\Lambda(x - 2\pi (A^T)^{-1}\underline{k})| \leq Z(\underline{k})\, \|\underline{x}\|^t, \quad t \geq 0 \tag{4.3}$$

for all $\underline{x}$ such that

$$\|\underline{x}\| \leq \|(A^T)^{-1}\| \cdot \pi n^{1/2} \tag{4.4}$$

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1) A^T means A transposed.

and $\underline{k} \equiv (k_1, \ldots, k_n)$, k_i integers

3)

(4.5) $$\sum_{\underline{k}} Z^2(\underline{k}) \cdot \|\underline{k}\|^{2\bar{\alpha}} = D < \infty, \quad \bar{\alpha} \geq 0 .$$

Then there exists an operator $A(h)$ which maps $W_2^{\beta}(R_n)$ into $W_2^{\bar{\alpha}}(R_n)$, $0 \leq \alpha \leq \bar{\alpha} \leq \beta$, such that

1)

(4.6) $$A(h)(f) = \sum_{j=1}^{\kappa} \sum_{\underline{k}} c_j(h, f, \underline{k})\, \omega_j\left(\frac{\underline{x} - hA\underline{k}}{h}\right)$$

2)

(4.7) $$\|f - A(h)f\|_{W_2^{\alpha}(R_n)} \leq K h^{\mu} \|f\|_{W_2^{\beta}(R_n)}$$

where

(4.8) $$\mu = \min(t - \alpha, \beta - \alpha)$$ and

K does not depend on h.

3) There exists a constant L such that if Q is the compact support of $f \in W_2^{\beta}(R_n)$ then $A(h)f$ has compact support Q^* such that $Q^* \subset Q_{Lh}$ where Q_{Lh} is the Lh neighborhood of Q.

Proof. [1]) 1. Let

$$\begin{aligned} \varkappa_1(\underline{x}) &= R e^{-\frac{1}{\|\underline{x}\|^2 - 1}} && \text{for } \|\underline{x}\| \leq 1 \\ &= 0 && \text{for } \|\underline{x}\| \geq 1 \end{aligned}$$

and R is chosen in such a way that

(4.9) $$\int_{R_n} \varkappa_1(\underline{x})\, d\underline{x} = 1 .$$

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1) In all that follows C will be a general constant, with different values on different places.

Placing $\phi_1(\underline{x}) = F(\varkappa_1)$ we have $\phi_1(\underline{x}) \in S$ and because of (4.9) we have $\phi_1(0) = 1$. Now let $P(\underline{x})$ be a trigonometrical polynomial with the periodicity matrix $2\pi(A^T)^{-1}$ such that we have

$$(4.10) \qquad |\phi(\underline{x}) - 1| \leq C\|\underline{x}\|^t \qquad \text{for } \|\underline{x}\| \leq 1$$

with

$$(4.11) \qquad \phi(\underline{x}) = \phi_1(\underline{x}) P(\underline{x}) .$$

Obviously we have $\phi(\underline{x}) \in S$. Let us put

$$\varkappa(\underline{x}) = F^{-1}\phi .$$

Because $\varkappa_1(\underline{x})$ has compact support, $\varkappa(x)$ has a compact support, too.

Now let $f \in W_2^{\beta}(R_m)$ and let Q be the support of f. Let us denote

$$(4.12) \qquad f_h = F^{-1}(Ff \cdot \phi(\underline{x}h)) .$$

Then f_h also has compact support which is in Kh neighborhood of Q, where K is a proper constant independent of h.

Let us show now that

$$(4.13) \qquad \|f_h - f\|_{W_2^{\alpha}(R_m)} \leq C h^{\mu} \|f\|_{W_2^{\beta}(R_m)}$$

where μ is given by (4.8). In fact we have

$$(4.14) \quad \|f_h - f\|^2_{W_2^{\alpha}(R_m)} = (2\pi)^{-m} \int_{R_m} |F(f)|^2 |1-\phi(\underline{x}h)|^2 (1+\|\underline{x}\|^{2\alpha})\, d\underline{x} .$$

We may write

$$(4.15) \quad \int_{R_m} |F(f)|^2 |1-\phi(\underline{x}h)|^2 (1+\|\underline{x}\|^{2\alpha})\, d\underline{x} = \int_{\|\underline{x}h\| \leq 1} \ldots + \int_{\|\underline{x}h\| > 1} \ldots$$

Because of (4.10) we have also

$$|\Phi(\underline{x})-1| \le C\,\|\underline{x}\|^{s} \qquad \text{for every } 0 \le s \le t$$

and $\|\underline{x}\| \le 1$.

Therefore putting $s = \mu$, we have

$$(4.16) \qquad \int_{\|\underline{x}h\|\le 1} \dots \le C \int |F(f)|^2 \|\underline{x}\|^{2\mu} h^{2\mu} (1+\|\underline{x}\|^{2\alpha})\, d\underline{x} \le$$

$$\le C h^{2\mu} \int |F(f)|^2 (1+\|\underline{x}\|^{2\alpha+2\mu})\, d\underline{x} \le C h^{2\mu} \|f\|^2_{W_2^\beta(R_m)}$$

because $\qquad 2\alpha + 2\mu \le 2\beta$.

So we have

$$(4.17) \qquad \int_{\|\underline{x}h\|\le 1} \dots \le C h^{2\mu} \|f\|^2_{W_2^\beta(R_m)} \quad .$$

Because $\Phi(\underline{x}) - 1$ is bounded we have

$$(4.18) \qquad \int_{\|\underline{x}h\|\ge 1} \dots \le C \int_{\|\underline{x}h\|\ge 1} |Ff|^2 (1+\|\underline{x}\|^{2\alpha}) \frac{\|\underline{x}\|^{2\mu}}{\|\underline{x}\|^{2\mu}}\, d\underline{x} \le$$

$$\le C h^{2\mu} \|f\|^2_{W_2^\beta(R_m)} \quad .$$

2. Let us now select a trigonometric polynomial $P_1(\underline{x})$ (with matrix of periodicity $2\pi(A^T)^{-1}$) such that

$$(4.19) \qquad |\Lambda(\underline{x}) P_1(\underline{x}) - 1| \le C\,\|\underline{x}\|^{t}$$

for all $\underline{x}$, $\|\underline{x}\| \le \|(A^T)^{-1}\|\,\pi\, m^{1/2}$.

Let us now put

$$(4.20) \qquad \xi = \sum_{\underline{k}} \xi_h(\underline{x} - \tfrac{1}{h}(A^T)^{-1}\underline{k}\, 2\pi) \cdot P_1(\underline{x}h)$$

where $\qquad \xi_h = Ff \cdot \Phi(\underline{x}h) = Ff_h$.

Because $\phi \in S$ the series is obviously convergent in $L_2(\Omega_A^h)$ where

(4.21) $\Omega_A^h = E[\underline{x},\ A^T\underline{x} \equiv (x_1,\dots,x_n),\ |x_i| \leq \frac{\pi}{h}]$

and ξ_h is periodic with matrix of periodicity $\frac{1}{h}(A^T)^{-1}\underline{k}\,2\pi$.

Let us write

(4.22) $$\xi_h = P_1(\underline{x}h)[\xi_h + \sum_{\underline{k},\underline{k}\neq 0} \xi_h(\underline{x} - \frac{1}{h}(A^T)^{-1}\underline{k}\,2\pi)]$$

$$= P_1(\underline{x}h)[\xi_h + \xi_h^*] \ .$$

Let us now show that

(4.23) $$\int_{\Omega_A^h} |\xi_h^*(\underline{x})|^2(1 + \|\underline{x}\|^{2\alpha})\,d\underline{x} \leq C h^{2\alpha}\|f\|^2_{W_2^\beta(R_n)}$$

In fact we have

(4.24) $$[\int_{\Omega_A^h} |\xi_h^*(\underline{x})|^2(1+\|\underline{x}\|^{2\alpha})d\underline{x}]^{1/2} \leq C h^{-\alpha}[\int_{\Omega_A^h} |\xi_h^*(\underline{x})|^2 d\underline{x}]^{1/2}$$

$$\leq (\sum_{\underline{k},\underline{k}\neq 0} \int_{\Omega_A^h} |\xi_h(\underline{x} - \frac{1}{h}(A^T)^{-1}\underline{k}\,2\pi)|^2 d\underline{x})^{1/2} C h^{-\alpha} \leq$$

$$\leq C h^{-\alpha}\sum_{\underline{k},\underline{k}\neq 0}(\int_{\Omega_A^h} |F(f)(\underline{x} - \frac{1}{h}(A^T)^{-1}\underline{k}\,2\pi)|^2 |\phi(\underline{x}h - (A^T)^{-1}\underline{k}\,2\pi)|^2 d\underline{x})^{1/2}$$

$$\leq C h^{-\alpha}\sum_{\underline{k},\underline{k}\neq 0}\|\underline{k}\|^{-p}(\int_{\Omega_A^h}|(Ff)(\underline{x} - \frac{1}{h}(A^T)^{-1}\underline{k}\,2\pi)|^2)^{1/2} \leq$$

$$\leq C h^{-\alpha}\sum_{\underline{k},\underline{k}\neq 0}\|\underline{k}\|^{-p}(\int_{R_n - \Omega_A^h}|(Ff)(\underline{x})|^2 d\underline{x})^{1/2} \leq$$

$$\leq C h^{-\alpha}\sum_{\underline{k},\underline{k}\neq 0}\|\underline{k}\|^{-p} h^\beta \|f\|_{W_2^\beta(R_n)}$$

Because $\phi \in S$, for every $p > 0$ we have for $x \in \Omega_A^h$

$$|\phi(\underline{x}h - (A^T)^{-1}\underline{k}\, 2\pi)| \leq C_{(p)} \|\underline{k}\|^{-p}, \quad \underline{k} \neq 0$$

and so we have chosen p such that the series $\sum_{\underline{k}, \underline{k}\neq 0} \|\underline{k}\|^{-p}$ is convergent.

3. Obviously the functions $\xi_{h,j} = \chi_j(xh)\xi_h(\underline{x})$, $j = 1, 2, \ldots, r$ are periodic functions with matrix of periodicity $(2\pi)(A^T)^{-1}\frac{1}{h}$. So we may write

$$\xi_{h,j} = \sum_{\underline{k}} C_j^h(\underline{k})\, e^{ih\langle A\underline{k}, \underline{x}\rangle} \tag{4.25}$$

Let Q be support of f. Obviously there exists a constant K such that $Q_{Kh} = E[\underline{x},\ \varphi(\underline{x}, Q) \leq Kh]$ is a support of the function $F^{-1}(\xi_h(\underline{x})\chi_j(\underline{x}h)P_1(\underline{x}h))$. Therefore

$$\int_{R_n} e^{-i\langle \underline{x}, \underline{y}\rangle} \xi_h(\underline{x})\chi_j(\underline{x}h)P_1(\underline{x}h)\, d\underline{x} = 0 \tag{4.26}$$

for all $\underline{x}$ outside of Q_{Kh}. So in (4.25), $C_j^h(\underline{k}) = 0$ for all $\underline{k}$ such that $hA\underline{k}$ are outside of Q_{Kh}.

Let us define

$$g(\underline{x}) = h^{-n} \sum_{j=1}^{r} \sum C_j^h(\underline{k})\, \omega_j\left(\frac{\underline{x} - hA\underline{k}}{h}\right) \tag{4.27}$$

Using Theorem 2.1 we may easily show that

$$(Fg)(\underline{x}) = \xi_h(\underline{x})\Lambda(\underline{x}h) = P_1(\underline{x}h)\Lambda(\underline{x}h)\xi_h(\underline{x}) + P_1(\underline{x}h)\Lambda(\underline{x}h)\xi^*(\underline{x}). \tag{4.28}$$

Let us estimate $\|f_h - g\|_{W_2^\alpha(R_n)}$, we have

$$\|f_h - g\|^2_{W_2^\alpha(R_n)} \leq C\Big[\int_{\Delta_A^h} |1 - P_1(\underline{x}h)\Lambda(\underline{x}h)|^2 |\xi_h(\underline{x})|^2 (1 + \|\underline{x}\|^{2\alpha})\, d\underline{x} + \tag{4.29}$$

$$+ \int_{\Omega_A^h} |P_1(\underline{x}h)\Lambda(\underline{x}h)|^2 |\xi_h^*|^2 (1+\|\underline{x}\|^{2\alpha}) d\underline{x} +$$

$$+ \sum_{\underline{k}\neq 0} \int_{\Omega_A^h} |\xi_h(\underline{x})|^2 |\Lambda(\underline{x}h - 2\pi(A^T)^{-1}\underline{k}|^2 (1+\|\underline{x}+\frac{2\pi}{h}(A^T)^{-1}\underline{k}\|^{2\alpha}) d\underline{x}$$

$$+ \int_{R_n-\Omega_A^h} |\xi_h(\underline{x})|^2 (1+\|\underline{x}\|^{2\alpha}) d\underline{x} \,] = C\,[\,I_1 + I_2 + I_3 + I_4\,] \quad .$$

Because of (4.9) we have

$$(4.30) \qquad |\Lambda(\underline{x})P_1(\underline{x}) - 1| \leq C\|\underline{x}\|^{\mu}$$

and therefore

$$(4.31)\quad I_1 \leq C h^{2\mu} \int_{\Omega_A^h} \|\underline{x}\|^{2\mu}(1+\|\underline{x}\|^{2\alpha})|Ff|^2 d\underline{x} \leq C h^{2\mu} \|f\|_{W_2^{\beta}(R_n)} \ .$$

Because of (4.3) and the boundedness of $P_1(\underline{x}h)\Lambda(\underline{x}h)$ (independently on h) we have

$$(4.32) \qquad I_2 \leq C h^{2\mu} \|f\|_{W_2^{\beta}(R_n)} \quad .$$

Further

$$I_3 = I_{3,1} + I_{3,2} \qquad \text{where}$$

$$I_{3,1} = \sum_{\underline{k}\neq 0} \int_{\Omega_A^h} |P_1(\underline{x}h)|^2 |\xi_h(x)|^2 |\Lambda(\underline{x}h - 2\pi(A^T)^{-1}\underline{k}|^2 \ . \\ \cdot (1+\|\underline{x}+\frac{2\pi}{h}(A^T)^{-1}\underline{k}\|^{2\alpha}) d\underline{x}$$

$$I_{3,2} = \sum_{\underline{k}\neq 0} \int_{\Omega_A^h} |P_1(\underline{x}h)|^2 |\xi_h^*(\underline{x})|^2 |\Lambda(\underline{x}h - 2\pi(A^T)^{-1}\underline{k}|^2 \ . \\ \cdot (1+\|\underline{x}+\frac{2\pi}{h}(A^T)^{-1}\underline{k}\|^{2\alpha}) d\underline{x} \quad .$$

Because of (4.3) we have

$$I_{3,1} \leq C h^{2\mu} \sum_{\underline{k}\neq 0} \int_{\Omega_A^h} |\xi_h(\underline{x})|^2 Z^2(\underline{k}) \|\underline{x}\|^{2\mu+2\alpha} h^{2\alpha} (1+\|\underline{x}+\frac{2\pi}{h}(A^T)^{-1}\underline{k}\|^{2\alpha}) d\underline{x}$$

$$\leq C h^{2\mu} \sum Z^2(\underline{k}) \|\underline{k}\|^{2\alpha} \int_{\Omega_A^h} (Ff)^2 \|\underline{x}\|^{2\mu+2\alpha} d\underline{x} \quad .$$

Because of $2\mu + 2\alpha \leq 2\beta$ and (4.5) we have

$$I_{3,1} \leq C h^{2\mu} \| f \|^2_{W_2^\beta(R_n)} \quad .$$

We have analogously

$$I_{3,2} \leq C h^{2\mu} \int_{\Omega_A^h} | \xi^*(\underline{x})|^2 \|\underline{x}\|^{2\beta} d\underline{x} \quad .$$

By the same way as in (4.23)we may show that

$$\int_{\Omega_A^h} | \xi^*_h(\underline{x})|^2 \|\underline{x}\|^{2\beta} d\underline{x} \leq C \| f \|^2_{W_2^\beta(R_n)} \quad .$$

So we have

$$(4.33) \qquad I_3 \leq C h^{2\mu} \| f \|^2_{W_2^\beta(R_n)} \quad .$$

Further we have

$$I_4 \leq C \int_{R-\Omega_A^h} |Ff|^2 (1+\|\underline{x}\|^{2\alpha}) d\underline{x} \leq C \int_{R-\Omega_A^h} |Ff|^2 (1+\|\underline{x}\|^{2\beta}) \frac{1+\|\underline{x}\|^{2\alpha}}{1+\|\underline{x}\|^{2\beta}} d\underline{x} \leq$$

$$\leq C h^{2\mu} \| f \|^2_{W_2^\beta(R_n)}$$

and therefore

$$(4.34) \qquad I_4 \leq C h^{2\mu} \| f \|^2_{W_2^\beta(R_n)} \quad .$$

So we have

$$\| f_h - g \|_{W_2^\alpha(R_n)} \leq C h^{\mu} \| f \|_{W_2^\beta(R_n)} \quad .$$

Because of (4.13) we have

$$\| f - g \|_{W_2^\alpha(R_n)} \leq C h^{\mu} \| f \|_{W_2^\beta(R_n)}$$

q.e.d.

Let there be given the functions $\omega_j \in S'$, $j = 1, \ldots, r$ with compact support. Let us introduce

$$(4.36) \qquad P(h, \alpha, \beta, \omega_1, \ldots, \omega_r)$$

$$= \sup_{\|f\|_{W_2^\beta(R_n)} \leq 1} \inf_{\substack{c_j^h(\underline{k}),\ c_j^h(\underline{k}) \leq D(h)\|\underline{k}\|^s \\ s < \infty,\ \text{arbitr.}}} \|f - \sum_{j=1}^{r} \sum_{\underline{k}} c_j^h(\underline{k}) \omega_j\left(\frac{x - hA\underline{k}}{h}\right)\|_{W_2^\alpha(R_n)}.$$

The value P describes the approximation property of ω_j with respect to the spaces $W_2^\alpha(R_n)$ and $W_2^\beta(R_n)$.

Theorem 4.1 says that assuming (4.2) - (4.5) we have

$$P(h, \alpha, \beta, \omega_1, \ldots, \omega_r) \leq K h^{\mu} .$$

Let us now study further questions.

Theorem 4.2. Let there be given the functions $\omega_s \in S'$, $s = 1, \ldots, r$ with compact support, and A be a nonsingular matrix. Then there exists a constant $C > 0$ such that

$$(4.37) \qquad P(h, \alpha, \beta, \omega_1, \ldots, \omega_r) \geq C h^{\beta - \alpha}, \quad \beta > \alpha .$$

Proof. Define

$$(4.38) \quad \Gamma_h(f) = \inf_{q_j(\underline{x})} \int_{R_n} |F(f)(\underline{x}) - \sum_{j=1}^{r} q_j(\underline{x}) F(\omega_j)(h\underline{x})|^2 \|\underline{x}\|^{2\alpha} d\underline{x}$$

with $q_j(\underline{x}) \in S'$, periodic with matrix of periodicity $(A^T)^{-1} \frac{2\pi}{h}$ and $f \in W_2^\alpha(R_n)$, $\beta > \alpha$.

Obviously

$$0 \leq \Gamma_h(f) \leq C \|f\|^2_{W_2^\alpha(R_n)} .$$

Now put $h = 1$ and select $g \in W_2^\beta(R_n)$ such that

$$\Gamma_1(g) = \Gamma > 0, \quad \|g\|_{W_2^\beta(R_n)} = 1 .$$

Such function clearly exists.

Now put $f_h(\underline{x}) = g(\frac{x}{h})$; then we have

$\|f_h\|^2_{W_2^\beta(R_n)} \leq C h^{n-2\beta}$.

On the other hand

$$\Gamma_h(f_h) = \Gamma_1(g)\, h^{n-2\alpha} \quad .$$

So because

$$P^2(h, \alpha, \beta, \omega_1, \dots, \omega_k) \geq C \frac{\Gamma_h(f_h)}{\|f_h\|^2_{W_2^\beta(R_n)}}$$

we have

$$P(h, \alpha, \beta, \omega_1, \dots, \omega_k) \geq C h^{\alpha-\beta}$$

and the theorem is proved.

Let us now study the case $k = 1$.

Theorem 4.3. Let there be given a function $\omega \in S'$ with compact support and let be given a nonsingular matrix A . Further let there exist a $C > 0$ such that

$$P(h, \alpha, \beta, \omega) \leq C h^\gamma \tag{4.39}$$

for all $1 \geq h > 0$.

Let further $\Lambda(\underline{x}) = F(\omega)$, $\Lambda(0) \neq 0$. Then for every $\underline{k} \in L$

$$|\Lambda(\underline{x} - 2\pi(A^T)^{-1}\underline{k})| \leq D(\underline{k})\|\underline{x}\|^{\alpha+\gamma} \tag{4.40}$$

provided that $\|\underline{x}\| \leq d(\underline{k})$, $d(\underline{k}) > 0$.

Proof. Let $\rho_0 > 0$, $K_{\rho_0} = E[\underline{x}, \|\underline{x}\| \leq \rho_0]$ and $K_{\rho_0} \subset \Omega_A^{h_0}$, $h_0 = 1$. Let ϕ be the characteristic function of $K_{\rho_0/2}$ and put $f = F^{-1}(\phi)$.

By (4.36) there exist coefficients $C^{h}(\underline{k})$

$$|C^{h}(\underline{k})| \leq C(h)\,\|\underline{k}\|^{\ell(h)}, \quad 0 \leq \ell(h) < \infty ,$$

such the for

$$(4.41) \qquad \phi^{h}(\underline{x}) = f - \sum_{\underline{k}} C^{h}(\underline{k})\,\omega\left(\frac{\underline{x} - hA\underline{k}}{h}\right)$$

we have

$$(4.42) \qquad \|\phi^{h}(\underline{x})\|_{W_2^{\alpha}(R_n)} \leq C h^{\gamma}$$

and C does not depend on h .

By Theorem 2.1 we have

$$(4.43) \qquad F(\phi^{h})(\underline{x}) = \phi(\underline{x}) - G^{h}(\underline{x})\Lambda(\underline{x}h)$$

and $G^{h}(\underline{x}) \in S'$ is a generalized periodic function with matrix of periodicity $(A^{T})^{-1}\,\frac{2\pi}{h}$. Because $supp\ \phi \subset \Omega_A^1$ we have for all $h < 1$

$$(4.44) \qquad \|\phi^{h}(\underline{x})\|^2_{W_2^{\alpha}(R_n)} = \int_{\Omega_A^h} |\phi(\underline{x}) - G^{h}(\underline{x})\Lambda(\underline{x}h)|^2(1 + \|\underline{x}\|^{2\alpha})d\underline{x}$$

$$+ \sum_{\substack{\underline{k} \\ \underline{k} \neq 0}} \int_{\Omega_A^h} |G^{h}(\underline{x})|^2\,|\Lambda(\underline{x}h - 2\pi(A^{T})^{-1}\underline{k})|^2(1 + \|\underline{x} + \frac{2\pi}{h}(A^{T})^{-1}\underline{k}\|^{2\alpha})\,d\underline{x} .$$

Because $\omega(\underline{x})$ has compact support $\Lambda(\underline{x})$ is continuous (see Lemma 1.1) at $\underline{x} = 0$. Because $\Lambda(0) \neq 0$ by the assumption there exists $H > 0$ such that

$$(4.45) \qquad \eta_2 > |\Lambda(\underline{x}h)| > \eta_1, \quad 0 < \eta_1 < \eta_2 < \infty$$

for $\underline{x} \in \Omega_A$ and $h < H$

By (4.42) we have

$$(4.46) \qquad \int_{K_{\varrho_0/2}} \left| \frac{1}{\Lambda(\underline{x}h)} - G^{h}(\underline{x}) \right|^2 d\underline{x} \leq C h^{2r}$$

because ϕ is the characteristic function of $K_{\varrho_0/2}$ and also

$$(4.47) \qquad \int_{K_{\varrho_0/2}} |G^{h}(\underline{x})|^2 |\Lambda(\underline{x}h - 2\pi(A^T)^{-1}\underline{k})|^2 d\underline{x} \leq C h^{2r+2\alpha}$$

for all $\underline{k} \neq 0$.

Define now

$$(4.48) \qquad \Lambda(\underline{x} - 2\pi(A^T)^{-1}\underline{k}) = \Lambda_{\underline{k}}(\underline{x}) .$$

By Lemma 1.1 the function $\Lambda_{\underline{k}}(\underline{x}) = \Lambda_{\underline{k}}(x_1, \dots, x_m)$ is analytic entire function of m variables $x_1, \dots, x_m$. So we may write

$$(4.49) \qquad \Lambda_{\underline{k}}(\underline{x}) = \sum_{j=0} \sum_{\ell_1+\dots+\ell_m=j} a^{\underline{k},j}_{\ell_1\dots\ell_m} x_1^{\ell_1}, \dots, x_m^{\ell_m}$$

and the series converges absolutely in a $K_{\varrho_0/2}$. So we may write

$$(4.50) \qquad \Lambda_{\underline{k}}(\underline{x}h) = \sum_{p=0}^{\infty} h^p \sum_{\ell_1+\dots+\ell_m=p} a^{\underline{k},p}_{\ell_1\dots\ell_m} x_1^{\ell_1}, \dots, x_m^{\ell_m}$$

$$= \sum_{p=0}^{\infty} h^p \psi_p(x_1, \dots, x_m, \underline{k}) .$$

Now put

$$(4.51) \qquad b(p, \underline{k}) = \int_{K_{\varrho_0/2}} \psi_p^2(\underline{x}, \underline{k}) \, d\underline{x} .$$

Let $0 \leq q(\underline{k})$ be an integer such that

$$b(p, \underline{k}) = 0 \text{ for all } 0 \leq p \leq q(\underline{k})$$

and

$$b(q(k), k) \neq 0 .$$

From (4.47) we have

$$\int_{K_{\varphi_0/2}} |G^h(\underline{x})|^2 |\Lambda_k(\underline{x}, h)|^2 d\underline{x} \leq C(k)\, h^{2\gamma+2\alpha}$$

and hence

(4.52) $$h^{2q(k)} \int_{K_{\varphi_0/2}} |G^h(\underline{x})|^2 |\psi_{q(k)}|^2 d\underline{x} \leq$$

$$\leq C(k)\, h^{2\alpha+2\gamma} + o(h^{2q(k)}) \int_{K_{\varphi_0/2}} |G^h(\underline{x})|^2 d\underline{x} .$$

But by (4.46)

(4.53) $$G^h(\underline{x}) = \frac{1}{\Lambda(\underline{x}h)} + \chi(\underline{x}, h)$$

with

$$\int_{K_{\varphi_0/2}} |\chi(\underline{x}, h)|^2 d\underline{x} \leq C(k)\, h^{2\gamma} .$$

So $\quad h^{2q(k)} b(q(k), k) \leq C(k)\, h^{2\gamma+2\alpha} + o(h^{2q(k)})$

and

(4.54) $$q(k) \geq \gamma + \alpha .$$

The theorem 4.3 follows from (4.54) and (4.50).

5. A closer analysis of the one dimensional case

Now we shall study in more detail the case $m = 1$ and $\kappa = 1$. Let us prove the following theorem.

Theorem 5.1. Let $\omega(x) \in S'$ and $\omega(x)$ have compact support. Further let $F(\omega) = \Lambda(x)$ fulfill

(5.1) $$\Lambda(0) \neq 0$$

(5.2) $$\Lambda(2\pi k + x) \leq \|x\|^t D(k)$$

for $\|x\| \leq d(k)$, $d(k) > 0$ (see also the 4.3). Then $\langle -\frac{t'}{2} + \varepsilon , \frac{t'}{2} - \varepsilon \rangle$, $\varepsilon > 0$ cannot be a support of $\omega(x)$, where $t' = \min [\ell ; \ \ell$ integral , $\ell \geq t]$.

Proof. By Lemma 1.1, it is possible to continue $\Lambda(x) = F(\omega)$, in the complex plane $z = x + iy$ and

(5.3) $$|\Lambda(z)| \leq (1 + |x|^{\ell}) C e^{a|y|} .$$

The function $\Lambda(z)$ has zero of order t' ($t' = \min [\ell ;$ ℓ integer $\ell \geq t]$) at the points $2\pi k$, $k = \ldots, -2, -1, 1, 2, \ldots$, because of (5.2).

Let us introduce the function

(5.4) $$\phi_{t'}(z) = \sin^{t'}(\tfrac{1}{2} z) .$$

The function

(5.5) $$\psi(z) = \frac{z^{t'}\Lambda(z)}{\phi_{t'}(z)}$$

is an entire function and because of (5.1) we have $\psi(0) \neq 0$.

We have

(5.6) $$|\phi_{t'}(x + iy)| \geq |sh^{t'} \frac{1}{2} y| .$$

So for $|y| > 1$ we have

$|\Psi(z)| \leq (1 + |x|^{\ell'}) C \cdot e^{(a - t'/2 + \varepsilon)|y|}$, $\varepsilon > 0$ arbitrary

and also for $|y| \leq 1$

(5.8) $$|\Psi(x)| \le (1+|x|^{q'})\,C\ .$$

So if $a + \varepsilon \le \frac{t'}{2}$ then $\Psi(x)$ is a polynomial and if $a < \frac{t'}{2}$ then $\Psi(x) = 0$ which contradicts with (5.5) and (5.1). Finally, by the use of Lemma 1.1, the theorem is proved.

From Theorem 5.1 it is obvious that the function

$$\varphi_t(x) = F^{-1}\left(\frac{1}{\varkappa^{t'}}\,\Phi_{t'}(x)\right) \in S'$$

fulfills (5.1) and (5.3) and has minimal support.

The functions $\varphi_t(x)$ have been studied by Schoenberg and called B-splines. For numerical construction see [14].

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