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THE HEYTING DOCTRINES

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0. Introduction.

The notion of hyperdoctrine was introduced in [1] as a generalization of some concepts from logic and category theory. In this paper, we define a slightly different notion of a Heyting doctrine. This notion seems to suit better for describing intuitionistic first-order theories and their models (we also obtain a correspondence with interpretations in the sense of Tarski in 1.7 which would be difficult to formulate in the language of hyperdoctrines). The Heyting doctrine differs from the hyperdoctrine in the following points.

1) The category $\mathbf{T}$ of types and terms is not assumed to be cartesian closed, but only to have finite products.

2) For any type X the category $\mathbf{P}(X)$ of attributes of the type X and deductions is required to be a Heyting category (i.e. to be cartesian closed and to have finite coproducts).

3) The mappings Σ, Π assigning functors to terms are required to be functors.

The exact definitions are given in § 1.

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The notion of a Heyting model, defined later, corresponds to the semantical and syntactical models used in logic. The Heyting doctrines and their models form a category $\mathcal{D}$. The logical concept of a language induces a categorical notion of a type. The types with their morphisms form a category Δ . We construct a functor $\mathcal{U} : \mathcal{D} \longrightarrow \Delta$ which in a sense forgets a part of the structure of $\mathcal{D}$. We prove that $\mathcal{U}$ has a left adjoint $L : \Delta \longrightarrow \mathcal{D}$ which is "free" in a sense that assigns to any logical language (type) the theory (Heyting doctrine) with this language and no proper axioms.

I wish to express here my thanks to A. Pultr for his guidance and encouragements.

1. <u>Preliminaries.</u>

In this paragraph we recall some basic notions and facts which will be used throughout the paper and introduce some definitions.

1.1. A functor $R : A \longrightarrow B$ satisfies the solution set condition, if for any object $b \in |B|$ there is a set of objects of A $\mathcal{L}_b \subseteq |A|$ such that for any $a \in |A|$ and $v : b \longrightarrow aR$ there are an object $a' \in \mathcal{L}_b$ and morphisms $u : a' \longrightarrow a$, $w : b \longrightarrow a'R$ such that the diagram

$$\begin{tikzcd} & a'R \arrow[d, "uR"] \\ b \arrow[ur, "w"] \arrow[r, "v"] & aR \end{tikzcd}$$

commutes.

The well known Freyd's adjoint functor theorem states that a functor $R : A \longrightarrow B$ where A is a complete and locally small category has a left adjoint iff it preserves all the limits in A and satisfies the solution set condition.

1.2. An adjoint situation will be mostly presented as a system consisting of two categories A, B functors $A \xrightarrow{R} B \xrightarrow{L} A$ and natural transformations $\eta : B \longrightarrow LR$, $\mu : RL \longrightarrow A$ such that $\eta L \circ L\mu = L$, $R\eta \circ \mu R = R$.

An adjoint morphism consists of two adjoint situations $(A_1, B_1, R_1, L_1, \eta_1, \mu_1)$, $(A_2, B_2, R_2, L_2, \eta_2, \mu_2)$ and functors $F : A_1 \longrightarrow A_2$, $G : B_1 \longrightarrow B_2$ commuting with all the structure (i.e. $FR_2 = R_1G$, $GL_2 = L_1F$, $F\mu_2 = \mu_1F$, $G\eta_2 = \eta_1G$). The adjoint situations with small categories and the adjoint morphisms form a category that we denote $\mathcal{A}$. It is easy to see that $\mathcal{A}$ is complete.

1.3. A Heyting category $\underline{A}$ is a category A together with the following adjoint situations.

a) A right adjoint $1_A : 1 \longrightarrow A$ and a left adjoint $0_A : 1 \longrightarrow A$ to the unique $A \longrightarrow 1$.

b) A right adjoint $\wedge_A : A \times A \longrightarrow A$ and a left adjoint $\vee_A : A \times A \longrightarrow A$ to the diagonal functor $A \longrightarrow A \times A$.

c) A right adjoint $(\ )^a : A \longrightarrow A$ to the functor $a \times (\ ) : A \longrightarrow A$ for every object $a \in |A|$.

Any Heyting or Boolean algebra yields in a natural way a thin Heyting category. Moreover, thin Heyting categories are just Heyting algebras.

A Heyting functor consists of Heyting categories

$\underline{A}$, $\underline{B}$ and a functor $F: A \longrightarrow B$ preserving all the structure i.e. such that

a) $(1, F), (F, 1)$ are adjoint morphisms,

b) $(F \times F, F), (F, F \times F)$ are adjoint morphisms,

c) For any $a \in |A|$

$$\begin{tikzcd}
A \arrow[r, "(\ )^{a}"] \arrow[d, "F"'] & A \arrow[r, "a \times (\ )"] \arrow[d, "F"] & A \arrow[d, "F"] \\
A \arrow[r, "(\ )^{aF}"] & A \arrow[r, "aF \times (\ )"] & A
\end{tikzcd}$$

is an adjoint morphism.

The small Heyting categories and their functors form a category $\mathcal{U}$. The completeness of $\mathcal{U}$ follows immediately from that of $\mathbf{Cat}$ (the category of small categories) and $\mathcal{A}$.

1.4. A Heyting triple consists of Heyting categories $\underline{A}$, $\underline{B}$, Heyting functor $F^{\flat} : \underline{A} \longrightarrow \underline{B}$ and two adjoint situations $(B, A, F^{\forall}, F^{\flat}, \dots)$, $(A, B, F^{\flat}, F^{\exists}, \dots)$. (Thus $F^{\forall}$ is a right and $F^{\exists}$ a left adjoint to $F^{\flat}$.)

If $\underline{A} \xrightarrow{\underline{F}} \underline{B} \xrightarrow{\underline{G}} \underline{C}$ are Heyting triples, then the adjoint situations $(C, A, G^{\forall}F^{\forall}, F^{\flat}G^{\flat}, \dots)$, $(A, C, F^{\flat}G^{\flat}, G^{\exists}F^{\exists}, \dots)$ determine a Heyting triple $\underline{FG}$: $: \underline{A} \longrightarrow \underline{C}$. The small Heyting categories and Heyting triples form a category $\mathcal{U}_3$.

1.5. A Heyting doctrine is a couple (T, P) where T is a category with finite products and $P : T^* \longrightarrow \mathcal{U}_3$ is a functor.

1.6. The two basic examples mentioned in [1] (p.291 and 292) appear in the language of Heyting doctrines as follows.

1.6.1. For every intuitionistic first-order theory with equality $Te = (L, A)$ (L is its language and A is the set of its proper axioms) we define the corresponding Heyting doctrine $D_{Te} = (T_L, P_A)$.

a) The objects of the category T_L are natural numbers.

b) For every $n, m \in |T_L|$ $\langle n, m\rangle_{T_L}$ is the set of all m-tuples of terms $t = (t_0, \dots t_{m-1}) : n \longrightarrow m$ whose free variables are contained in the list $\{x_0, \dots x_{n-1}\}$. The composition in T_L is defined by the substitution.

c) For every $n \in |T_L|$ the objects of the Heyting category nP_A are all formulas, whose free variables are contained in the list $\{x_0, \dots x_{n-1}\}$. nP_A is made in a category by the preorder $\vdash$ (to be deducible) and in a Heyting category by well known logical operations.

d) If $t = (t_0, \dots t_{m-1}) : n \longrightarrow m$ is a morphism in T_L then $tP_A : mP_A \longrightarrow nP_A$ is defined as follows: If $\varphi \in |nP_A|$, $\psi \in |mP_A|$ then

$$\psi(tP_A)^b = \psi(t_0, \dots t_{m-1}) ,$$

$$\varphi(tP_A)^{\exists} = (\exists\xi_0)\dots(\exists\xi_{n-1})(x_0 = t_0(\xi_0 \dots \xi_{n-1}) \wedge \dots x_{m-1} = t_{m-1}(\xi_0 \dots \xi_{n-1}) \wedge \varphi(\xi_0 \dots \xi_{n-1})) ,$$

$$\varphi(tP_A)^{\forall} = (\forall\xi_0)\dots(\forall\xi_{n-1})(x_0 = t_0(\xi_0 \dots \xi_{n-1}) \wedge \dots x_{m-1} = t_{m-1}(\xi_0 \dots \xi_{n-1}) \Rightarrow \varphi(\xi_0 \dots \xi_{n-1})) .$$

We factorize each nP_A so that $((ts)P_A)^{\exists} = (tP_A)^{\exists} \cdot (sP_A)^{\exists}$, $((ts)P_A)^{\forall} = (tP_A)^{\forall}(sP_A)^{\forall}$ for every $t, s \in T_L$ $\quad n \xrightarrow{t} m \xrightarrow{s} p$.

1.6.2. Further we define the semantical Heyting doctrine $\mathcal{D}_s = (\mathbf{Set}, P_s)$. Here $P_s : \mathbf{Set}^* \longrightarrow \mathcal{U}_3$ is the functor assigning to every $X \in |\mathbf{Set}|$ the set of all subsets of X ordered by inclusion. This is a Boolean algebra and therefore a Heyting category. If $f : X \longrightarrow Y$ is a mapping of sets and $X_0 \subseteq X$, $Y_0 \subseteq Y$ then

$Y_0(fP_s)^s = \{x \in X ; xf \in Y_0\} = Y_0 f^{-1}$,

$X_0(fP_s)^{\exists} = \{y \in Y ;$ there is an $x \in X$, so that $y = xf$ and $x \in X_0\}$,

$X_0(fP_s)^{\forall} = \{y \in Y ;$ for every $x \in X$, $y = xf$ implies $x \in X_0\}$.

1.7. A Heyting model $(F, \tau) : (T_1, P_1) \longrightarrow (T_2, P_2)$ consists of Heyting doctrines (T_1, P_1) , (T_2, P_2) , a functor $F : T_1 \longrightarrow T_2$ preserving finite products and a system of morphisms in $\mathcal{U}$ $\tau = (a\tau)_{a \in T_1}$ $a\tau : aP_1 \longrightarrow aFP_2$, such that for any morphism $f : a \longrightarrow b$ in T_1 the following two diagrams are adjoint morphisms (see 1.2).

$$\begin{tikzcd}
aP_1 \arrow[r, "a\tau"] \arrow[d, shift right, "(fP_1)^{\exists}"'] & aFP_2 \arrow[d, shift right, "(fFP_2)^{\exists}"'] \\
bP_1 \arrow[r, "b\tau"] \arrow[u, shift right, "(fP_1)^{s}"'] & bFP_2 \arrow[u, shift right, "(fFP_2)^{s}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
aP_1 \arrow[r, "a\tau"] \arrow[d, shift left, "(fP_1)^{\forall}"] & aFP_1 \arrow[d, shift left, "(fFP_2)^{\forall}"] \\
bP_1 \arrow[r, "b\tau"] \arrow[u, shift left, "(fP_1)^{s}"] & bFP_2 \arrow[u, shift left, "(fFP_2)^{s}"]
\end{tikzcd}$$

It is easy to see that for theories Te_1 , Te_2 Heyting models between their corresponding Heyting doctrines (T_1, P_1),

(T_2, P_2) are in one-to-one correspondence with interpretations in the sense of Tarski [3] and Heyting models from (T_1, P_1) to $(\mathbf{Set}, P_b)$ are in one-to-one correspondence with semantical models of $\mathbf{Te}_1$.

$$\text{If } (T_1,P_1) \xrightarrow{(F_1,\tau_1)} (T_2,P_2) \xrightarrow{(F_2,\tau_2)} (T_3,P_3)$$

are Heyting models, we define their composition (F, τ) as $(F,\tau) = (F_1F_2 , \tau_1 \circ F\tau_2)$ (for every $a \in |T_1|$ $a\tau : aP_1 \xrightarrow{a\tau_1} aF_1P_2 \xrightarrow{aF_1\tau_2} aF_1F_2P_3$) .
Since this composition is associative, we obtain the category $\mathcal{D}$ of small Heyting doctrines (the category of types is small) and Heyting models.

1.8. We may extend the category $\mathcal{D}$ into 2-category as follows:

if $(T_1,P_1) \underset{(F_2,\tau_2)}{\overset{(F_1,\tau_1)}{\rightrightarrows}} (T_1,P_1)$ are Heyting models, then 2-morphisms $\varphi : (F_1,\tau_1) \longrightarrow (F_2,\tau_2)$ are natural transformations $\varphi : F_1 \longrightarrow F_2$, such that for every $a \in |T_1|$ the diagram

$$\begin{tikzcd}
aP_1 \arrow[rr, "a\tau_1"] \arrow[drr, "a\tau_2"'] & & aF_1P_2 \\
& & aF_2P_1 \arrow[u, "(a\varphi P_2)^{b}"']
\end{tikzcd}$$

commutes.

There are no nontrivial 2-morphisms between interpretations. On the other hand, 2-morphisms between semantical models are mappings between their underlying sets, preserving all the structure.

2. The free Heyting doctrines.

<u>Definition</u>. A type is a triple (T, R, S) where T is a small category with finite products and $R : |T| \longrightarrow Set$, $S : |T|^{\cdot} \times |T| \longrightarrow Set$ are functors ($|T|$ is the discrete category of objects of T).
A type morphism is a triple $(F, \varphi, \tau) : (T_1, R_1, S_1) \longrightarrow (T_2, R_2, S_2)$ where $F : T_1 \longrightarrow T_2$ is a functor preserving finite products and $\varphi : R_1 \longrightarrow |F| R_2$; $\tau : S_1 \longrightarrow |F|^2 S_2$ are natural transformations.

We define the composition of type morphisms

$$(T_1, R_1, S_1) \xrightarrow{(F_1, \varphi_1, \tau_1)} (T_2, R_2, S_2) \xrightarrow{(F_2, \varphi_2, \tau_2)} (T_3, R_3, S_3)$$

as $(F, \varphi, \tau) : (T_1, R_1, S_1) \longrightarrow (T_3, R_3, S_3)$ where

$F = F_1 F_2$, $\varphi = \varphi_1 \circ |F_1| \varphi_2$, $\tau = \tau_1 \circ |F_1|^2 \tau_2$.

Thus the types and their morphisms form a category that we denote Δ .

<u>Definition</u>. The type of a logical language L is $\delta_L = (Set_o^*, R, S)$ where Set_o is the category of natural numbers and their mappings (natural numbers are regarded as sets $n = \{0, 1, \dots n-1\}$) and for every $n, m \in |Set_o|$
$nR =$ the set of all n-ary predicates of L ,
$(n, 1)S =$ the set of all n-ary function symbols of L ,
$(n, m)S = 0$ for $m \neq 1$.

<u>Definition</u>. The forgetful functor $\mathcal{U} : \mathcal{D} \longrightarrow \Delta$ is defined as follows:
a) For any Heyting doctrine (T, P)

$(T,P)\mathcal{U} = (T, i_T^* P u, |\langle -,- \rangle_T|)$,

where $i_T : |T| \longrightarrow T$ is the inclusion,

$u : \mathcal{U}_3 \longrightarrow \mathcal{S}et$ is the forgetful functor

$(\underline{A}u = |A|,\ \underline{F}u = |F^{s}|)$

and $|\langle -,- \rangle_T| : |T| \times |T| \xrightarrow{i_T^* \times i_T} T^* \times T \xrightarrow{\langle -,- \rangle_T} \mathcal{S}et$.

b) For any Heyting model $(F, \tau) : (T,P) \longrightarrow (T_1, P_1)$,

$(F,\tau)\mathcal{U} = (F, i_T^* \tau u, |\langle -,- \rangle_F|)$

where the natural transformation $\langle -,- \rangle_F : \langle -,- \rangle_T \longrightarrow$ $\longrightarrow \langle -F, -F \rangle_{T_1}$ is defined by the formula $f\langle a, b \rangle_F = fF$ for every morphism $f : a \longrightarrow b$ in the category T .

We shall prove that $\mathcal{U}$ has a left adjoint. For this purpose let us factorize the functor $\mathcal{U}$ as follows:

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "\mathcal{U}"] \arrow[dr, "\mathcal{U}_r"'] & & \Delta \\ & \Delta_r \arrow[ur, "\mathcal{U}_s"'] & \end{tikzcd}$$

Δ_r (the category of relation types) is the category whose objects are couples (T, R) where T is a small category with finite products and $R : |T| \longrightarrow \mathcal{S}et$ is a functor. Its morphisms are couples $(F, \varphi) : (T_1, R_1) \longrightarrow (T_2, R_2)$ where $F : T_1 \longrightarrow T_1$ is a functor preserving finite products, and $\varphi : R_1 \longrightarrow |F| R_2$ is a natural transformation. The composition is defined analogously to that of Δ . The functors $\mathcal{U}_r$, $\mathcal{U}_s$ are defined by formulas:

$(T,P)\mathcal{U}_r = (T, i_T^* P u)$, $(F,\tau)\mathcal{U}_r = (F, i_T^* \tau u)$

for any Heyting model $(F, \tau) : (T. P) \longrightarrow (T_1, P_1)$,

$$(T,R)\mathcal{U}_{\mathfrak{s}} = (T,R,|\langle -,-\rangle_T|) \qquad (F,\varphi)\mathcal{U}_{\mathfrak{s}} = (F,\varphi,|\langle -,-\rangle_F|)$$

for any morphism $(F,\varphi):(T,R)\longrightarrow(T_1,R_1)$ in the category $\Delta_{\mathcal{K}}$.

Now we shall prove that the functors $\mathcal{U}_{\mathcal{K}}, \mathcal{U}_{\mathfrak{s}}$ have left adjoints $L_{\mathcal{K}}, L_{\mathfrak{s}}$.

Lemma. The category $\mathfrak{D}$ is complete.

Proof: Let us prove the existence of equalizers. Let $(T_1,P_1)\underset{(F_2,\tau_2)}{\overset{(F_1,\tau_1)}{\rightrightarrows}}(T_2,P_2)$ be a diagram in $\mathfrak{D}$.

Denote $F:T\longrightarrow T_1$ an equalizer of F_1 and F_2 in $\mathbf{Cat}$. It is easy to show that T has finite products which are preserved by F . For every $a\in|T|$ let the following diagram be an equalizer in $\mathcal{U}$

$$aP\xrightarrow{a\tau}aFP_1\underset{(aF)\tau_2}{\overset{(aF)\tau_1}{\rightrightarrows}}aFF_1P_2 \quad .$$

For every $f:a\longrightarrow b\in T$ denote fP a Heyting triple obtained from the adjoint situations in fFP_1 , fFF_1P_2 by equalizers in $\mathcal{A}$. Now, it is easy to see that the diagram

$$(T,P)\xrightarrow{(F,\tau)}(T_1,P_1)\underset{(F_2,\tau_2)}{\overset{(F_1,\tau_1)}{\rightrightarrows}}(T_2,P_2)$$

is an equalizer in $\mathfrak{D}$.

Similarly we prove the existence of products.

Lemma. $\mathcal{U}_{\mathcal{K}}$ preserves limits.

Proof: Let $(T,P)\xrightarrow{(F,\tau)}(T_1,P_1)\underset{(F_2,\tau_2)}{\overset{(F_1,\tau_1)}{\rightrightarrows}}(T_2,P_2)$

be an equalizer in $\mathfrak{D}$ and let

$$(T',R') \xrightarrow{(F',\varphi')} (T_1, i_1^* P_1 u) \underset{(F_2, i_2^* \tau_2 u)}{\overset{(F_1, i_1^* \tau_1 u)}{\rightrightarrows}} (T_2, i_2^* P_2 u)$$

commute in Δ_{κ} .

Since $F'F_1 = F'F_2$ there is exactly one functor $H : T' \longrightarrow T$ such that $HF = F'$. For any $a \in |T'|$

$$aHP \xrightarrow{aH\tau} aHFP_1 \underset{aHF\tau_2}{\overset{aHF\tau_1}{\rightrightarrows}} aHFF_1P_2$$

is an equalizer in $\mathcal{U}$ and its image under u is an equalizer in $\mathcal{S}et$. Since the diagram

$$aR' \xrightarrow{a\varphi'} aHFP_1u \underset{aHF\tau_2 u}{\overset{aHF\tau_1 u}{\rightrightarrows}} aHFF_1P_2u$$

commutes in $\mathcal{S}et$, there is exactly one morphism $a\varkappa : aR' \longrightarrow aHPu$ such that $(a\varkappa)(aH\tau u) = a\varphi'$. Thus we have exactly one type morphism $(H,\varkappa) : (T',R') \longrightarrow (T, i^*Pu)$ such that $(F',\varphi') = (H,\varkappa)(F, i^*\tau u)$.

Similarly we prove that $\mathcal{U}_{\kappa}$ preserves products.

Lemma. The category Δ_{κ} is locally small.

Proof: It is easy to show that if $(F,\varphi) : (T,R) \longrightarrow (T_1,R_1)$ is a monomorphism in Δ_{κ} then F is a monomorphism in $\mathcal{C}at$ and φ is a monomorphism in $\mathcal{S}et^{|T|}$. Since these categories are locally small, we obtain the statement.

Lemma. The category $\mathfrak{D}$ is locally small.

This is an easy consequence of the preceding two lemmas.

Lemma. $\mathcal{U}_{\kappa}$ satisfies the solution set condition.

Proof: The solution set for a relation type (T,R) is the set of all Heyting doctrines (T_2, P_2) such that

$card(T_2) \leq card(T)$ and for every $a_2 \in |T_2|$ $card(a_2 P_2) \leq max\{card(aR) ; a \in |T|\}$.

Theorem. The functor $\mathcal{U}_h$ has a left adjoint L_h : $: \Delta_h \longrightarrow \mathfrak{D}$. This is a consequence of Freyd's adjoint functor theorem and preceding lemmas.

We have defined, for every logical language L , a category T_L with the following properties.

a) There exists an inclusion functor $\mathcal{J} : Set_0^* \longrightarrow T_L$ defined as follows.

$$m\mathcal{J} = n \quad \text{for every } m \in |Set_0| ,$$

$$f\mathcal{J} = (x_{0f}, \dots x_{(m-1)f}) \quad \text{for every mapping}$$

$f : n \longrightarrow m$.

b) If $G : Set_0^* \longrightarrow A$ preserves finite products and for every function symbol f (from L) $a_f : nF \longrightarrow 1F$ is a morphism in A (different from projections) then there is a unique $\overline{G} : T_L \longrightarrow A$ such that $G = \mathcal{J}\overline{G}$ and $f\overline{G} = a_f$ for any function symbol f .

We shall now formalize and generalize this adjoint situation.

Definition. a) Cat_p is the category of small categories with finite products and finite products preserving functors.

b) Δ_f (the category of function types) is the category whose objects are couples (T, S) where T is a small category with finite products and $S : |T| \times |T| \longrightarrow Set$ is a functor. Its morphisms are couples (F, τ) : $: (T_1, S_1) \longrightarrow (T_2, S_2)$ where $F : T_1 \longrightarrow T_2$ is a functor preserving finite products and $\tau : S_1 \longrightarrow |F|^2 S_2$ is a natural transformation.

The composition is defined analogously to that of Λ .

c) The functor $\mathcal{U}_f : \mathit{Cat}_p \longrightarrow \Delta_f$ is defined by the formula

$$T\mathcal{U}_f = (T, |\langle -,- \rangle_T|) \qquad F\mathcal{U}_f = (F, |\langle -,- \rangle_F|)$$

for every morphism $F : T \longrightarrow T_1$ in Cat_p .

Lemma. The category Cat_p is complete and locally small.

The proof follows immediately from the properties of the category of small categories.

Lemma. $\mathcal{U}_f$ preserves limits.

Proof: Let $T \xrightarrow{F} T_1 \overset{F_1}{\underset{F_2}{\rightrightarrows}} T_2$ be an equalizer in Cat_p . We will prove that for any $a_1, a_2 \in |T|$ the diagram

$$\langle a_1, a_2 \rangle_T \xrightarrow{\langle a_1, a_2 \rangle_F} \langle a_1F, a_2F \rangle_{T_1} \overset{\langle a_1F, a_2F \rangle_{F_1}}{\underset{\langle a_1F, a_2F \rangle_{F_2}}{\rightrightarrows}} \langle a_1FF_1, a_2FF_1 \rangle_{T_2}$$

is an equalizer in Set .

Let $G : M \longrightarrow \langle a_1F, a_2F \rangle_{T_1}$ be a mapping and $G \langle a_1F, a_2F \rangle_{F_1} = G \langle a_1F, a_2F \rangle_{F_2}$.

Since for any $m \in M$, $(mG)F_1 = (mG)F_2$ there exists exactly one $\alpha : a_1 \longrightarrow a_2$ with $\alpha F = mG$.

Thus, we have obtained the required mapping $M \longrightarrow$ $\longrightarrow \langle a_1, a_2 \rangle_T$. Now, it is easy to see that

$$(T, |\langle -,- \rangle_T|) \xrightarrow{(F, |\langle -,- \rangle_F|)} (T_1, |\langle -,- \rangle_{T_1}|) \overset{(F_1, |\langle -,- \rangle_{F_1}|)}{\underset{(F_1, |\langle -,- \rangle_{F_2}|)}{\rightrightarrows}} (T_2, |\langle -,- \rangle_{T_2}|)$$

is an equalizer in Δ_f .

Lemma. $\mathcal{U}_f$ satisfies the solution set condition.

Proof: The solution set for a function type $(\mathcal{A}, \mathcal{S})$ is the set of all small categories with finite products, and the cardinality less or equal to $card(\mathcal{A})$.

Theorem. The functor $\mathcal{U}_f$ has a left adjoint $L_f : \Delta_f \longrightarrow Cat_{fl}$.

This is a consequence of Freyd's adjoint functor theorem and preceding lemmas.

Lemma. The functor $\mathcal{U}_h : \Delta_{h} \longrightarrow \Delta$ is a right adjoint.

Proof: Let L_f be the functor from the preceding theorem. We define a functor $L_h : \Delta \longrightarrow \Delta_h$ as follows:

$(T, R, S)L_h = ((T, S)L_f, R)$; $(F, \varphi, \tau)L_h = ((F, \tau)L_f, \varphi)$.

It is easy to show that L_h is a left adjoint to $\mathcal{U}_h$.

Theorem. The functor $\mathcal{U} : \mathcal{D} \longrightarrow \Delta$ is a right adjoint.

Proof: Namely to the composition $L = L_{h} \cdot L_{h}$.

Corrolary. It is easy to show that for any logical language L and for the Heyting doctrine (T_L, P_0) (corresponding to the theory $(L, 0)$)

$$\langle (T_L, P_0), - \rangle_{\mathcal{D}} \approx \langle \mathcal{S}_L, -\mathcal{U} \rangle_{\Delta} .$$

Thus, we have isomorphic Heyting doctrines

$$(T_L, P_0) \approx (\mathcal{S}_L)L .$$

R e f e r e n c e s :

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