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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 1, 95--105

Persistent URL: http://dml.cz/dmlcz/105473

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14,1 (1973)

ON GENERATING OF RELATIONS

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Abstract: Given a family of relations R_i indexed by a set I and a relation τ on I one can form a new relation $\mathcal{R}(\tau, R_i)$ induced by R_i and τ. (If τ is an ordering then $\mathcal{R}(\tau, R_i)$ is the lexico-graphic product of R_i.) The question is studied how many R_i are necessary to generate a given relation R. This is related to preference-relations in psychology.

Key words: relation

AMS, Primary: 05C99 Ref. Ž. 8.83

Introduction. Let X be a set, let $(R_i)_{i\in I}$ be a family of binary relations on X. In addition, let τ be a binary relation on I. This system of data generates a new binary relation on X defined in the following way: $\langle x, y\rangle \in R$ if and only if there is $i \in I$ such that 1) $\langle x, y\rangle \in R_i$, 2) $\langle y, x\rangle \notin R_{i'}$ for every $\langle i', i\rangle \in \tau$, $i \neq i'$.
Thus we can, from relations R_i and τ of special character, obtain relation far more general. For example any antisymmetric relation is generated in this sense by means of quasielementary preferences R_i (by a quasielementary preference we understand $R \subset X^2$, what may be written as $R = A\times B$, where $A \subset X$, $B \subset X$ and $A \cap B = \emptyset$; in case of $A \cup B = X$ we say that R is an elementary preference) and a binar order τ. At this place we might recall the

motivation of the above construction: having an antisymmetric relation ϱ, we interpret $\langle x, y \rangle \in \varrho$ as "x is preferred to y" and try to represent ϱ by means of a system of simple decisions. For instance, the above theorem states that any antisymmetric relation on a set X may be obtained from a finite sequence of subsets $V_1, V_2, \dots, V_m$ of X, if we interpret them as properties for elements of X and where for $i < j$ we consider the i-th property more important than the j-th one. Hence we prefer x to y, whenever from the point of the most important property V_j we prefer x to y (V_j is the most important property if j is the first j such that x is compatible with y in V_j).

Given a certain class of relations $\mathcal{L}$ (elementary preferences, quasielementary preferences, linear orders, partial orders, etc.) and a relation R generated by means of relations from $\mathcal{L}$, and appropriate relation τ (τ linear order, respectively) we define the dimension (the linear dimension, respectively) of R with respect to $\mathcal{L}$ as the least number of $R_i \in \mathcal{L}$ which generate R via some relation (linear order, respectively). Very often we realize that the dimension of relations with respect to $\mathcal{L}$ grows beyond any limit merely ascertaining the number of elements in $\mathcal{L}$. (This is the case in both foregoing instances.) In 1970 a problem was put by professor Katětov: what is the behavior of dimension in cases which one cannot decide by merely comparing the numbers, namely what happens when we generate general relations from partial orders. In the sequel, we shall concern ourselves in the dimension with respect to partial orders only.

We shall show that there exist tournaments (trichotonomic relations) of an arbitrarily great dimension. First we proved that the linear dimension of tournaments grows beyond any limit. The general result was achieved by formulating a correspondence between the general and the special dimension of tournaments.

We would like to thank A. Pultr who not only acquainted us with the problem mentioned above, but helped us in the development of its solution.

A tournament T is a couple $\langle X, R \rangle$, where X is a finite set and R is a subset of X^2 such that the following holds:

$$x, y \in X \Longrightarrow (\langle x, y \rangle \in R \Longleftrightarrow \langle y, x \rangle \notin R).$$

<u>Definition 1</u>. Let X, I be nonempty sets. Let $\{R_i ; i \in I\}$ be a collection of partial ordering on X. Let τ be a relation on I. A relation R is said to be generated by $\{R_i \mid i \in I\}$ and by τ if for every couple $\langle x, y \rangle \in R$ there exists $i \in I$ such that the following holds:

1) $\langle x, y \rangle \in R_i$,

2) if $i' \in I$, $i \neq i'$ and $\langle i, i' \rangle \in I$, then $\langle y, x \rangle \notin R_{i'}$.

The relation R generated by τ and by $\{R_i, i \in I\}$ will be denoted by $\mathcal{R}(\tau, \{R_i\}_{i \in I})$.

<u>Proposition 1</u>. Let $G = \langle X, R \rangle$ be a graph. There exists a collection of partial orderings on X $\{R_i \mid i \in I\}$ and an index relation τ ($\tau \subset I^2$) so that $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$.

<u>Proof</u>. Put $I = R$, $\tau = \emptyset$, $R_{\langle x, y \rangle} = \Delta_X \cup \{\langle x, y \rangle\}$.

(We put $\Delta_X = \{\langle x, x\rangle / x \in X\}$.)

Proposition 2. A graph $G = \langle X, R\rangle$ has no 2-cycles iff $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$ where τ is a linear order of I.

Proof. Let $G = \langle X, R\rangle$ be a graph without 2-cycles. Put $I = R$, $R_{\langle x, y\rangle} = \Delta_X \cup \{\langle x, y\rangle\}$. Let τ be any linear order of I. Obviously $R = \mathcal{R}(\tau, \{R_i\})$.
If $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$ where τ is any linear order of I it obviously holds: $\{\langle x, y\rangle, \langle y, x\rangle\} \not\subset R$.

Definition 2. Let Φ be a class of relations. The Φ - dimension of a graph $G = \langle X, R\rangle$ ($dim_{\Phi} G$) is the least cardinality of an index-set I such that $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$, where $\tau \subset I^2$, $\tau \in \Phi$. We write $dim\, G = dim_{\Phi} G$ (the universal dimension of G) if Φ is the class of all relations. We write $dim_{\mathcal{L}} G = dim_{\Phi} G$ (the linear dimension of G) if Φ is a class of linear orders.
We write $dim_{N} G = dim_{\Phi} G$ (the acyclic dimension) if Φ is a class of graphs without cycles.

Remark. Every graph has the universal dimension according to the proposition 1. Every graph without 2-cycles has the linear dimension according to the proposition 2.

Notation. 1) The collection of all tournaments with n vertices is denoted by $\mathcal{T}_n$.

2) Put $D^T_{\mathcal{L}}(n) = \max_{T \in \mathcal{T}_n} \{dim_{\mathcal{L}} T\}$.

In the following the two main theorems will be proved:

Theorem 1. Let T be a tournament. If $dim\, T = k$, then $dim_{\mathcal{L}} T \leq 3^k$.

Theorem 2. Let us suppose $m > 1$. Then $D^T_{\mathcal{L}}(m) \geq$

$$\geq \frac{lg_2(m)}{lg_2(2\, lg_2(m)+1)}$$ (hence $D^T_{\mathcal{L}}(m)$ tends to infinity if m tends to infinity).

It follows from Theorem 1 and Theorem 2 that the set $\{dim\, T \mid T$ is a tournament$\}$ is unbounded as a subset of N.

A: Theorem 1:

Definition 3. Let $T = \langle X, R\rangle$ be a tournament, $R = \mathcal{R}(\tau, \{R_i\}_{i\in I})$. Let $\{M_1, M_2, M_3\}$ be a decomposition of I. Denote by $B\langle M_1, M_2, M_3\rangle$ the set of all $\langle x, y\rangle \in R$ which satisfy:

1) $\langle x, y\rangle \in R_i$ for every $i \in M_1$,

2) $\langle y, x\rangle \in R_i$ for every $i \in M_2$,

3) neither $\langle x, y\rangle \in R_i$ nor $\langle y, x\rangle \in R_i$ for every $i \in M_3$.

The set $\{B\langle M_1, M_2, M_3\rangle \mid \{M_1, M_2, M_3\}$ is a decomposition of $I\}$ is denoted by $\mathfrak{B}$.

Convention. The symbol $B\langle M_1, M_2, M_3\rangle$ will denote in the following the set defined in Definition 3.

We shall write B instead of $B\langle M_1, M_2, M_3\rangle$ when there is no danger of confusion.

Remark. $B\langle M_1, M_2, M_3\rangle$ is a relation on X without cycles.

Proposition 3. Let $T = \langle X, R\rangle$ be a tournament, $R = \mathcal{R}(\tau, \{R_i\}_{i\in I})$. Let $\{M_1, M_2, M_3\}$ and $\{M'_1, M'_2, M'_3\}$ be decompositions of I. Put $B = B\langle M_1, M_2, M_3\rangle, B' = B\langle M'_1, M'_2, M'_3\rangle$. Then $B \neq \emptyset, B' \neq \emptyset$ implies either $M_1 \not\subset M'_2$ or $M'_1 \not\subset M_2$.

Proof. Let us suppose $M_1 \subset M'_2$ and $M'_1 \subset M_2, B \neq \emptyset, B' \neq \emptyset$.

There exists a couple $\langle x, y\rangle \in B$ and consequently there exists $i_0 \in I$ such that $\langle x, y\rangle \in R_{i_0}$ (hence $i_0 \in M_1$). Further $\langle i, i_0\rangle \in \tau \Rightarrow \langle y, x\rangle \notin R_i$ for any $i \in I$. As $B' \neq \emptyset$ there exists $\langle x', y'\rangle \in B'$. Since $i_0 \in M_1$ and $M_1 \subset M_2'$, it holds $\langle y', x'\rangle \in R_{i_0}$. As $\langle x', y'\rangle \in R$ and $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$ and T is a tournament, there exists $i_1 \in M_1'$ such that $\langle i_1, i_0\rangle \in \tau$.

However, $M_1' \subset M_2$, therefore $\langle y, x\rangle \in R_{i_1}$ which is a contradiction with the properties of i_0 .

<u>Definition 4</u>. Let $\varphi \subset X^2$ be a relation without cycles. We define $\overline{\varphi} = \bigcap \{\sigma \mid \varphi \subset \sigma \subset X^2$, σ is a partial order$\}$. Obviously $\overline{\varphi}$ is a partial ordering.

<u>Definition 5</u>. Let $T = \langle X, R\rangle$ be a tournament, $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$. Let $\mathcal{B}$ be a set defined in Definition 3. We define the relation $\rho \subset \mathcal{B}^2 : \langle B, B'\rangle \in \rho \Longleftrightarrow B^{-1} \cap \overline{B}' \neq \emptyset$.

<u>Proposition 4</u>. Presumptions are the same as in Definition 5. The relation ρ defined in Definition 5 is a relation without cycles.

<u>Proof</u>. In the way of contradiction, let $\{B_1, \dots, B_k\}$ be a subset of $\mathcal{B}$ such that: $\langle B_i, B_{i+1}\rangle \in \rho$ for $i = 1, \dots, k-1$, $\langle B_k, B_1\rangle \in \rho$. It is $\overline{B_{i+1}} \cap B_i^{-1} \neq \emptyset$ and $\overline{B}_1 \cap B_k^{-1} \neq \emptyset$, hence $B_i \neq \emptyset$ for $i = 1, \dots, k$.
The following holds for $i = 1, \dots, k-1$ and $i \in I$ according to the Definition 3:

$$(B_{i+1} \subset R_i) \Rightarrow (\overline{B_{i+1}} \subset R_i) \Rightarrow (B_i^{-1} \cap R_i \neq \emptyset) \Rightarrow (B_i^{-1} \subset R_i) .$$

Shortly:

(I) $\quad B_{i+1} \subset R_i \Rightarrow B_i^{-1} \subset R_i$, $\quad i = 1, \dots, k-1$.

We can obtain in the same way:

(II) $B_1 \subset R_i \Rightarrow B_k^{-1} \subset R_i$,

(III) $B_{i+1}^{-1} \subset R_i \Rightarrow B_i \subset R_i$, $i = 1, \dots, k-1$,

(IV) $B_1^{-1} \subset R_i \Rightarrow B_k \subset R_i$.

Statements (III) and (IV) can be obtained in a similar way as (I). First, we consider the case $k = 2p+1$. It holds according to I,II,III,IV:

$B_1 \subset R_i \Rightarrow B_{2p+1}^{-1} \subset R_i \Rightarrow B_{2p} \subset R_i \Rightarrow \dots \Rightarrow B_2 \subset R_i \Rightarrow B_1^{-1} \subset R_i$,

hence $B_1 \subset R_i \Rightarrow B_1^{-1} \subset R_i$ which is a contradiction (B_1 is a nonempty set and R_i is an antisymmetric relation). Secondly, let $k = 2p$. The following holds according to I, II,III,IV:

$B_1 \subset R_i \Rightarrow B_{2p}^{-1} \subset R_i \Rightarrow B_{2p-1} \subset R_i$ and inductively $B_2^{-1} \subset R_i$,

hence $B_1 \subset R_i \Rightarrow B_2^{-1} \subset R_i$.

It follows from (I) also: $B_2 \subset R_i \Rightarrow B_1^{-1} \subset R_i$. We put $B_1 = B\langle M_1^1, M_2^1, M_3^1 \rangle$, $B_2 = B\langle M_1^2, M_2^2, M_3^2 \rangle$. Consequently $M_1^1 \subset M_2^2$, $M_1^2 \subset M_2^1$, which is a contradiction (it follows from Proposition 3).

Proposition 5. Let $T = \langle X, R \rangle$ be a tournament, $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$. Let $\flat$ be the relation defined by Definition 5. Then $R = \mathcal{R}(\flat, \{\bar{B}_B\}_{B \in \mathcal{B}})$.

Proof. 1) If $\langle x, y \rangle \in R$, there exists $B \in \mathcal{B}$ such that $\langle x, y \rangle \in B$. If $\langle y, x \rangle \in \bar{B}'$ then $\langle B, B' \rangle \in \flat$ (Definition 5). As $\flat$ is an antisymmetric relation, it is

$\langle B', B\rangle \notin \mathfrak{s}$. This implies that there exists no $B' \in \mathcal{B}$ such that both $\langle y, x\rangle \in \bar{B}'$ and $\langle B', B\rangle \in \mathfrak{s}$, consequently $\langle x, y\rangle \in \mathcal{R}(\mathfrak{s}, \{\bar{B}_B\}_{B\in\mathcal{B}})$.

2) As T is a tournament, it is sufficient for the proof of the statement to show the following:

$$\langle x, y\rangle \in R^{-1} \Longrightarrow \langle x, y\rangle \notin \mathcal{R}(\mathfrak{s}, \{\bar{B}_B\}_{B\in\mathcal{B}}) .$$

Let $\langle x, y\rangle \in R^{-1}$. There exists $B_0 \in \mathcal{B}$ such that $\langle y, x\rangle \in B_0$. If $\langle x, y\rangle \in \bar{B}$ for a $B \in \mathcal{B}$ then $\langle B_0, B\rangle \in \mathfrak{s}$ according to Definition 4, hence $\langle x, y\rangle \notin \mathcal{R}(\mathfrak{s}, \{\bar{B}\}_{B\in\mathcal{B}})$. Proposition 5 is proved.

<u>Proposition 6</u>. Let T be a tournament. If $dim\, T = k$, then $dim\, T \leq 3^k$.

<u>Proof</u>. The statement follows easily if we consider the remark under Definition 4, Proposition 4, Proposition 5 and an inequality $|\{\langle M_1, M_2, M_3\rangle \mid \{M_1, M_2, M_3\}$ is a decomposition of I, $|I| = k\}| \leq 3^k$.

<u>Proposition 7</u>. Let $G = \langle X, R\rangle$ be a graph. Let R be an antisymmetric relation. Then $dim_{\mathcal{L}}\, G = dim_N\, G$.

<u>Proof</u>. 1) Obviously $dim_N\, G \leq dim_{\mathcal{L}}\, G$.

2) Let $R = \mathcal{R}(\tau, \{R_i\}_{i\in I})$ where τ is an acyclic relation. There exists obviously a partial ordering u on I such that $u \supset \tau$ and $R = \mathcal{R}(u, \{R_i\}_{i\in I})$, hence $dim_{\mathcal{L}}\, G \leq dim_N\, G$.

Now a proof of Theorem 1 follows immediately from Propositions 5,6 and 7.

B: Theorem 2:

Notation. Let $\langle X, R\rangle$ be a graph. The maximal cardinality of a set Y such that $Y \subset X$ and $\langle Y, R \cap Y \times Y\rangle$ is a linear order is denoted by $\ell(R)$.
The maximal cardinality of a set Y such that $Y \subset X$ and $(Y \times Y) \cap R = \emptyset$ is denoted by $i\langle X, R\rangle$.

Let n be a positive integer. We define:
$L(n) = min\{\ell(R) \mid \langle X, R\rangle \in \mathcal{T}_n\}$. The following two propositions are well known and we state them without proofs.

Proposition 8. $L(n) \leq 2 . lg_2 n + 1$.

Proposition 9. Let $\langle X, R\rangle$ be a partially ordered set. Then $\ell(R) . i\langle X, R\rangle \geq card\, X$.

Notation. Let X be a set. Let Y be a subset of X. Let R be a subset of $X \times X$. We denote $R \cap (Y \times Y)$ by R/y.

Proposition 10. Let $\langle X, R\rangle$ be a graph. Let $R = \mathcal{R}(\tau, \{R_i\}_{i \in I})$ where τ is a linear order of I. Let Y be a subset of X. If there exists $i_0 \in I$ such that $R_{i_0}/y = \emptyset$, then $R/y = \mathcal{R}(\tau', \{R_i/y\}_{i \in I'})$ where $I' = I - \{i_0\}$ and $\tau' = \tau/_{I'}$.

Proof is trivial.

Notation. The symbol τ_p will denote a natural order of the set $\{1, \ldots, p\}$.

Proof of Theorem 2. Let $T_1 = \langle X_1, R_1\rangle$ be a tournament such that $card\, X_1 = n_1$, $\ell(R_1) = L(n_1)$ and $dim_{\mathcal{L}} T_1 = k$. Let us suppose that $R_1 = \mathcal{R}(\tau_k; \{C_i^1\}_{1 \leq i \leq k})$ and

$k < \dfrac{lg_2 n}{lg_2(2\, lg_2(n)+1}$ $(n = n_1)$. We shall construct tournaments $T_j = \langle X_j, R_j \rangle$ $(j = 2,\dots,k)$ where $X_j \subset X_{j-1}$, $R_j = \mathcal{R}(\tau_{k-j+1}, \{C^j_{j-1+i}\}_{1 \le i \le k-j+1})$ and $\operatorname{card} X_j = n_j$, by induction.

Suppose that the tournaments $T_1, \dots, T_j$ have been constructed. $R_j = \mathcal{R}(\tau_{k-j+1}, \{C^j_{j-1+i}\}_{1 \le i \le k-j+1})$, hence $\ell(R_j) \ge \ell(C^j_j)$. According to Proposition 9 there exists a set $X_{j+1} \subset X_j$ $(\operatorname{card} X_{j+1} = n_{j+1})$ such that $n_{j+1} \ge \dfrac{n_j}{\ell(R_j)}$ and $C^j_j / X_{j+1} = \emptyset$. Put $R_{j+1} = R_j / X_{j+1}$, $C^{j+1}_i = C^j_i / X_{j+1}$, $(i = j+1, \dots, k)$.

It holds $R_{j+1} = \mathcal{R}(\tau_{k-j}, \{C^{j+1}_{j+i}\}_{1 \le i \le k-j})$ according to Proposition 10. As $\ell(R_j) \le \ell(R_1) = L(n)$ for $j = 1, \dots, k$ and $n_{j+1} \ge \dfrac{n_j}{\ell(T_j)}$ it holds: $n_j \ge \dfrac{n}{L(n)^{j-1}}$.

Further $k < \dfrac{lg_2(n)}{lg_2(2lg_2(n)+1)}$, hence $n > (2\,g_2(n)+1)^k \ge L(n)^k$ (according to Proposition 8). Consequently $n_k > L(n)$. However, T_k is a tournament and a partially ordered set, hence T_k is a linearly ordered set, hence $n_k = \ell(R_k) \le L(n)$ which is a contradiction.

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(Oblatum 9.2.1973)

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