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NONLINEAR EIGENVALUE PROBLEMS

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Abstract: Let g be a continuously differentiable functional on a real Banach space V and f' - in one sense - the limit of continuously differentiable functionals on V with domain $D(f') := \{u \in V : f'(u) \in V^*\}$.
The existence of a solution of the nonlinear eigenvalue problem

$$f'(u) = \lambda g'(u)$$

with $\lambda \in R^1$ and $u \in D(f') \cap M_c(g)$ is proved, where the level surface is defined by $M_c(g) := \{u \in V ; g(u) = c\}$.
Application to a nonlinear elliptic eigenvalue problem is given.

Key words: Variational problem, nonlinear eigenvalue problem, regularization method, elliptic differential equation, boundary condition.

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Let V be a real Banach space, f and g two functionals defined on V which are once continuously differentiable with the derivatives f' and g' respectively. Let c be a real number and define the level surface $M_c(g) : \{u \in V : g(u) = c\}$. Then the critical points of f with respect to $M_c(g)$ are (under suitable restrictions) solutions of the eigenvalue problem

(1) $$f'(u) = \lambda g'(u)$$

with some $\lambda \in \mathbb{R}^1$. This reduction of the eigenvalue problem (1) to the problem of extremizing a functional f on the level surface $M_c(g)$ is used to prove the existence of a solution for (1) (see e.g. [5 - 8]).

It is the purpose of the present note to prove the existence of a solution u_0 for (1) with $u_0 \in M_c(g)$ (Theorem 1) under the assumption that f' is the limit of the derivatives of a sequence of functionals on V . In particular, $f'(u)$ must not be defined for all u of $M_c(g)$. The proof of the theorem is based on regularization methods, recently used by the author in studying nonlinear integral equations [9] and nonlinear elliptic boundary value problems [10]. Theorem 1 generalizes results of Browder [5, 6] and Hess [8]. As an application of Theorem 1 we obtain a result (Theorem 2) on nonlinear elliptic eigenvalue problems which strengthens the corresponding statements in [1,2,4 - 6] (see also [3]).

1. Let V be a real separable reflexive Banach space with dual V^*. The pairing between V and V^* shall be denoted by $(\cdot , \cdot)$. By $\longrightarrow$ and $\rightharpoonup$ we will denote strong and weak convergence respectively.

A mapping $T : V \longrightarrow V^*$ is said to satisfy Condition (S): if $\{u_m\} \subset V$ is weakly convergent in V to u_0 and if $(Tu_m - Tu_0, u_m - u_0) \longrightarrow 0$, then u_m converges strongly to u_0 .

A mapping $T : V \longrightarrow V^*$ is said to satisfy Condition (S+): if $\{u_m\} \subset V$ is weakly convergent to u_0 in V and if $\limsup_n (Tu_n - Tu_0, u_n - u_0) \leq 0$, then u_n converges strongly to u_0.

To prove an existence theorem for the eigenvalue problem (1) we use regularization methods. Therefore we introduce

Assumption 1. Let $\varepsilon_0 > 0$. For each $\varepsilon \in]0, \varepsilon_0]$ suppose that $f_1, f_2(\varepsilon, \cdot), g$ are functionals on V satisfying the following conditions: (a) $f_1, f_2(\varepsilon, \cdot)$ and g are C^1-functions on V with the derivatives f_1', $f_2'(\varepsilon, \cdot)$ and g' respectively. (b) g is weakly continuous and g' is a compact mapping from V to V^*. (c) Set $f(\varepsilon, u) := f_1(u) + f_2(\varepsilon, u)$ with the derivative $f'(\varepsilon, u) = f_1'(u) + f_2'(\varepsilon, u)$. Suppose that f_1' and $f_2'(\varepsilon, \cdot)$ map bounded sets into bounded sets, that f_1' satisfies Condition (S+) and $f'(\varepsilon, \cdot)$ Condition (S) and that $f(\varepsilon, u) \to \infty$ as $\|u\| \to \infty$ uniformly with respect to $\varepsilon \in]0, \varepsilon_0]$. (d) Let there exist a constant $c > 0$ such that for all u in $M_c(g) := \{u \in V : g(u) = c\}$, $(g'(u), u) > 0$ and for each $R > 0$, there exists $c(R) > 0$ such that $(g'(u), u) \geq c(R)$ for u in $M_c(g)$ with $\|u\| \leq R$. By a theorem of Browder ([6]; Theorem 15) it follows

Proposition 1. Suppose that Assumption 1 holds. Then for each $\varepsilon \in]0, \varepsilon_0]$, $f(\varepsilon, \cdot)$ assumes its minimum on the set $M_c(g)$ at a point u_ε which is a solution of the eigenvalue equation

$$(2) \qquad f'(\varepsilon, u) = f_1'(u) + f_2'(\varepsilon, u) = \lambda_\varepsilon g'(u)$$

for some real number λ_ε.

We define for all $\varepsilon \in \,]0, \varepsilon_0]$, $u, v \in V$,
$B(\varepsilon, u, v) := (f_2'(\varepsilon, u), v)$.

The problem to be studied is obtained by the limiting process $\varepsilon \to 0$. Hence we formulate

<u>Assumption 2</u>. (a) Let there exist a constant $\mathcal{C}_1 > 0$ and an element $v_0 \in M_c(g)$ such that for each $\varepsilon \in \,]0, \varepsilon_0]$

$$f(\varepsilon, v_0) \leq \mathcal{C}_1 .$$

(b) Suppose that there exists a constant $c_0 > 0$ such that for all $u \in M_c(g)$ and each $\varepsilon \in \,]0, \varepsilon_0]$

$$0 \leq (f_2'(\varepsilon, u), u) \leq c_0 f_2(\varepsilon, u) .$$

(c) Suppose that any sequences $\{\varepsilon_m\}$ and $\{u_{\varepsilon_m}\} \subset V$ satisfying $\varepsilon_m \to 0$, $u_{\varepsilon_m} \rightharpoonup u_0$ in V and $0 \leq B(\varepsilon_m, u_{\varepsilon_m}, u_{\varepsilon_m}) \leq \mathcal{C}_2$ with some constant $\mathcal{C}_2 > 0$ imply the existence of $B(0, u_0, \varphi)$ for all $\varphi \in W$, where W is a dense subset of V ; furthermore there exists a subsequence $\{m'\}$ such that

$$B(\varepsilon_{m'}, u_{\varepsilon_{m'}}, \varphi) \to B(0, u_0, \varphi)^{(+)}$$

for all $\varphi \in W$. If in addition $B(0, u_0, u_0)$ exists, then it exists a subsequence (also denoted by m') such that

$$B(0, u_0, u_0) \leq \underline{\lim}_{m'} B(\varepsilon_{m'}, u_{\varepsilon_{m'}}, u_{\varepsilon_{m'}}) .$$

We set
$D(f') := \{u \in V : B(0, u, \cdot) : V \to \mathbb{R}^1$ is linear and continuous$\}$.

(+) The referee has remarked, that under this assumption, it then follows that the whole sequence converges.

Then for $u \in D(f')$ there exists $f_2'(0,u) \in V^*$ such that for all $v \in V$

$$B(0,u,v) = (f_2'(0,u),v) \ .$$

The eigenvalue equation to be studied may then be written in the form

$$(3) \qquad f_1'(u) + f_2'(0,u) = \lambda g'(u)$$

with some real number λ and $u \in M_c(g) \cap D(f')$.

We now state our main theorem:

Theorem 1. Suppose that the assumptions 1,2 are true. Then there exist at least one real number λ_0 and one $u_0 \in M_c(g) \cap D(f')$ satisfying (3).

Proof. By Proposition 1 and Assumption 2(a) it follows

$$f(\varepsilon, u_\varepsilon) \leq f(\varepsilon, v_0) \leq \mathcal{C}_1$$

from which by Assumption 1(c) there exists a constant $R >$ > 0 such that for each ε in $]0, \varepsilon_0]$, $\|u_\varepsilon\|_V \leq R$. Therefore there exists a sequence ε_n , such that

$$(4) \qquad \varepsilon_n \to 0 , \ u_{\varepsilon_n} \rightharpoonup u_0 \text{ in } V \ .$$

Further we obtain by Assumptions 2(b), 1(c)

$$(5) \quad 0 \leq (f_2'(\varepsilon, u_\varepsilon), u_\varepsilon) \leq c_0 f_2(\varepsilon, u_\varepsilon) \leq c_0 (f(\varepsilon, u_\varepsilon) + $$
$$+ |f_1(u_\varepsilon)|) \leq c_0 (\mathcal{C}_1 + \mathcal{C}_3) =: \mathcal{C}_2$$

with some constant $\mathcal{C}_3 > 0$. Hence it follows by Assumption 1(d)

$$\mathcal{C}_4 \geq \|f_1'(u_\varepsilon)\| \|u_\varepsilon\| + (f_2'(\varepsilon, u_\varepsilon), u_\varepsilon) \geq |(f'(\varepsilon, u_\varepsilon), u_\varepsilon)|$$
$$= |\lambda_\varepsilon| (g'(u_\varepsilon), u_\varepsilon) \geq |\lambda_\varepsilon| c(R)$$

with some constant $\mathcal{C}_4 > 0$, i.e. $|\lambda_\varepsilon| \le \mathcal{C}_4 / c(R)$.
Therefore by virtue of Assumptions 1(b,c) and 2(c), (4) and (5) there exists a subsequence n' such that
$\lambda_{\varepsilon_{n'}} \to \lambda_0$, $f_1'(u_{\varepsilon_{n'}}) \rightharpoonup w_1$ in V^*, $g'(u_{\varepsilon_{n'}}) \rightharpoonup w_2$ in V^*
and

$$B(\varepsilon_{n'}, u_{\varepsilon_{n'}}, \varphi) \to B(0, u_0, \varphi)$$

for all $\varphi \in W$. From

$$(f'(\varepsilon_{n'}, u_{\varepsilon_{n'}}), \varphi) = \lambda_{\varepsilon_{n'}} (g'(u_{\varepsilon_{n'}}), \varphi)$$

it therefore follows for each φ in W

$$(w_1, \varphi) + B(0, u_0, \varphi) = \lambda_0 (w_2, \varphi) .$$

Hence

$$B(0, u_0, \varphi) = (\lambda_0 w_2 - w_1, \varphi)$$

for each φ in W. Since W is dense in V and $\lambda_0 w_2 - w_1 \in V^*$ it follows therefore that $B(0, u_0, \varphi)$ can be defined for all $\varphi \in V$ and $B(0, u_0, \varphi) = (f_2'(0, u_0), \varphi)$, i.e.

$$f_2'(0, u_0) = \lambda_0 w_2 - w_1$$

and $u_0 \in D(f')$. Further we obtain by Assumption 2(c)

$$\limsup_{n'} (f_1'(u_{\varepsilon_{n'}}), u_{\varepsilon_{n'}}) =$$

$$= \limsup_{n'} \{\lambda_{\varepsilon_{n'}} (g'(u_{\varepsilon_{n'}}), u_{\varepsilon_{n'}}) - (f_2'(\varepsilon_{n'}, u_{\varepsilon_{n'}}), u_{\varepsilon_{n'}})\}$$

$$= \lambda_0 (w_2, u_0) - \lim_{n'} B(\varepsilon_{n'}, u_{\varepsilon_{n'}}, u_{\varepsilon_{n'}})$$

$$\le \lambda_0 (w_2, u_0) - B(0, u_0, u_0) = (\lambda_0 w_2 - f_2'(0, u_0), u_0) = (w_1, u_0) .$$

Therefore by Assumption 1(c), $u_{\varepsilon_{n'}}$ converges strongly to u_0. Hence we obtain by the continuity of f_1' and g' that $u_0 \in M_c(g)$ and

$$f_1'(u_0) + f_2'(0, u_0) = \lambda_0 \, g'(u_0)$$

proving Theorem 1.

Remark. (1) Assumption 2(c) is also used in [10], where nonlinear boundary value problems are studied.
(2) Theorem 1 is a generalization of a theorem of Browder ([6], Theorem 15). It shall be remarked that the domain of Definition $D(f')$ of the operator $f_2'(0, \cdot)$ is a subset of V.

2. To apply Theorem 1 to nonlinear elliptic eigenvalue problems we make the following assumptions.

Let Ω be an open bounded subset of R^n with sufficiently smooth boundary $\partial\Omega$ such that the imbedding theorems of Sobolev hold (see e.g. [6]). We consider the Sobolev space $V := \overset{o}{W}_{m,p}$ with $1 < p < \infty$ and denote by $[u, v] := \int_\Omega u(x) v(x)\, dx$. In the following we shall use the notations of Browder [6].

Assumptions 3 (see [6]): (a) Suppose that $F : \Omega \times R^{s_m} \rightarrow R^1$, $G : \Omega \times R^{s_{m-1}} \rightarrow R^1$. For each fixed ξ in R^{s_m}, $F(\cdot, \xi)$ is measurable on Ω and for almost all x in Ω, $F(x, \cdot)$ is once continuously differentiable on R^{s_m}. For each fixed η in $R^{s_{m-1}}$, $H(\cdot, \eta)$ is measurable on Ω and for almost all x in

Ω, $H(x,\cdot)$ is once continuously differentiable on R^{sm-1}. The functions F and G satisfy the following inequalities:

$$|F(x,\xi)| \le c(\xi_b)(x) + c_1(\xi_b) \sum_{m-n/p \le |\alpha| \le m} |\xi_\alpha|^{s_\alpha},$$

$$|G(x,\eta)| \le c(\eta_b)(x) + c_1(\eta_b) \sum_{m-n/p \le |\beta| \le m-1} |\eta_\beta|^{t_\beta},$$

where $s_\alpha^{-1} = p^{-1} - n^{-1}(m-|\alpha|)$, $s_\alpha < \infty$, $t_\beta < s_\beta$, b is the greatest integer less than $m-n/p$, $\xi_b := \{\xi_\alpha : |\alpha| \le b\}$, c_1 is a continuous function from R^{s_b} to R^1, and c is a continuous function from R^{s_b} to L^p.

(b) Set $F_\alpha := \partial F / \partial \xi_\alpha$ for $|\alpha| \le m$ and $G_\beta := \partial G / \partial \eta_\beta$ for $|\beta| \le m-1$. Suppose that

$$|F_\alpha(x,\xi)| \le c_\alpha(\xi_b)(x) + c_1(\xi_b) \sum_{m-n/p \le |\beta| \le m} |\xi_\beta|^{p_{\alpha\beta}},$$

$$|G_\alpha(x,\eta)| \le c_\alpha(\eta_b)(x) + c_1(\eta_b) \sum_{m-n/p \le |\beta| \le m-1} |\eta_\beta|^{p_{\alpha\beta}},$$

where c_α are continuous functions from R^{s_b} to L^{p_α} and the exponents p_α and $p_{\alpha\beta}$ satisfy the inequalities

$$p_\alpha = p' \; (p^{-1} + p'^{-1} = 1) \quad \text{for } |\alpha| = m,$$

$$p_\alpha > s'_\alpha \; (s_\alpha^{-1} + s_\alpha'^{-1} = 1) \quad \text{for } m - n/p \le |\alpha| < m,$$

$$p_\alpha = 1 \quad \text{for } |\alpha| < m - n/p$$

and

$$p_{\alpha\beta} \le p-1 \text{ for } |\alpha| = |\beta| = m,$$

$$p_{\alpha\beta} < s_\beta s_\alpha'^{-1} \text{ for } m - n/p \le |\alpha|, |\beta| \le m,$$

$|\alpha| + |\beta| < 2m$,

$r_{\alpha\beta} \le s_\beta$ for $|\alpha| < m - n/p$, $m - n/p \le |\beta| \le m$.

(c) If $\xi = (\eta, \zeta_m)$ is the division of ξ into its m-th order components ζ_m and the corresponding $(m-1)$-st order jet η, then for each x in Ω and each η in $R^{s_{m-1}}$

$$\sum_{|\alpha|=m} [F_\alpha(x,\eta,\zeta_m) - F_\alpha(x,\eta,\zeta'_m)]\,[\zeta_\alpha - \zeta'_\alpha] > 0$$

for $\zeta_m \neq \zeta'_m$.

(d) There exist two continuous functions c_2 and c_3 from R^{s_ℓ} to R^1 with $c_2(\eta_\ell) \ge \tilde{c}_2 > 0$ for each η_ℓ, and two constants $c_4 > 0$ and c_5 such that for all x, ζ_m and η

$$\sum_{|\alpha|\le m} F_\alpha(x,\xi)\,\xi_\alpha \ge c_4 |\xi|^p - c_5 ,$$

$$\sum_{|\alpha|=m} F_\alpha(x,\xi)\,\xi_\alpha \ge c_2(\eta_\ell)\,|\zeta_m|^p - $$

$$- c_3(\eta_\ell) \sum_{m-n/p \le |\beta| \le m-1} |\xi_\beta|^{t_\beta} .$$

(e) Let there exist a constant $\alpha > 0$ such that for each $x \in \Omega$ and each $\eta \in R^{s_{m-1}}$ the following inequality holds:

$$\sum_{|\beta|\le m-1} G_\beta(x,\eta)\,\eta_\beta \ge \alpha\, G(x,\eta) .$$

<u>Assumption 4</u>. (a) Let $H : \bar{\Omega} \times R^{s_{m-1}} \to R^1$ be a nonnegative continuous function such that for all x in Ω, $H(x,\cdot)$ is once continuously differentiable on $R^{s_{m-1}}$.

(b) Set $H_\alpha := \partial H / \partial \eta_\alpha$ and

$$L_\alpha(\varepsilon, x, \eta) := \frac{H_\alpha(x,\eta)}{(1+\varepsilon H(x,\eta))^2} \quad \text{with } \varepsilon \in]0,1]$$

and $|\alpha| \le m-1$. Suppose that for each ε in $]0,1]$, each x in Ω and each η in $R^{s_{m-1}}$ the following inequalities hold:

$$|L_\alpha(\varepsilon, x, \eta)| \le c_\alpha(\eta_\&)(x) + c_1(\eta_\&) \sum_{m-n/p \le |\beta| \le m-1} |\eta_\beta|^{p_{\alpha\beta}} ,$$

$$c_0 H(x,\eta) \ge \sum_{|\alpha| \le m-1} H_\alpha(x,\eta)\eta_\alpha \ge 0$$

with some constant $c_0 > 0$.

(c) Suppose that there exist a constant $c_6 \ge 0$ and a function $R : \Omega \times R^{s_{m-1}} \times R^{s_{m-1}} \longrightarrow R^1$ such that for each η, η' in $R^{s_{m-1}}$ and each x in Ω

$$\Big| \sum_{|\beta| \le m-1} H_\beta(x,\eta)\eta'_\beta \Big| \le c_6 \sum_{|\beta| \le m-1} H_\beta(x,\eta)\eta_\beta + R(x,\eta,\eta') .$$

Further suppose that for each w in $W_{m^*,p}$ with $m^* > m + n/p$ the mapping $R(\eta(\cdot), \eta(w))$ defined by $R(\eta(u),\eta(w))(x) := R(x, \eta(u)(x), \eta(w)(x))$, is bounded and continuous from $W_{m-1,p}$ to L^1.

We define

$D := \{u \in \mathring{W}_{m,p}$: such that the form $\sum_{|\beta| \le m-1} [H_\beta(\cdot,\eta(u)), D^\beta v]$ is continuous with respect to v in $\mathring{W}_{m,p}\}$,

$$M_c(g) := \{u \in \mathring{W}_{m,p} : \int_\Omega G(x,\eta(u)(x))dx = c\} .$$

Let $c > \sigma$, then we ask for elements $\lambda_0 \in R^1$, $u_0 \in M_c(g) \cap D$ which satisfy the condition

$$(6)\begin{cases} \sum_{|\alpha|\le m}[F_\alpha(\cdot,\xi(u_0)),D^\alpha v]+\sum_{|\beta|\le m-1}[H_\beta(\cdot,\eta(u_0)),D^\beta v] \\ \quad = \lambda_0 \sum_{|\beta|\le m-1}[G_\beta(\cdot,\eta(u_0)),D^\beta v] \end{cases}$$

for each v in $\mathring{W}_{m,p}$.

Theorem 2. Suppose that $c > 0$ and that Assumptions 3,4 hold. Let exist an element $v_0 \in M_c(g)$ such that $\int_\Omega H(x,\eta(v_0)(x))dx < \infty$. Then Problem (6) has at least one solution $\lambda_0 \in R^1$, $u_0 \in M_c(g) \cap D$.

Proof. (a) For $\varepsilon \in]0,1]$ and $u, v \in \mathring{W}_{m,p}$ we define

$$f_1(u) := \int_\Omega F(x,\xi(u)(x))dx, \quad g(u) := \int_\Omega G(x,\eta(u)(x))dx ,$$

$$f_2(\varepsilon,u) := \int_\Omega H(x,\eta(u)(x))/(1+\varepsilon H(x,\eta(u))(x)))dx ,$$

$$a(u,v) := \sum_{|\alpha|\le m}[F_\alpha(\cdot,\xi(u)),D^\alpha v] ,$$

$$b(u,v) := \sum_{|\beta|\le m-1}[G_\beta(\cdot,\eta(u)),D^\beta v] ,$$

$$c(\varepsilon,u,v) := \sum_{|\beta|\le m-1}[H_\beta(\cdot,\eta(u))/(1+\varepsilon H(\cdot,\eta(u)))^2,D^\beta v] .$$

By a theorem of Browder (see [6], Lemma 7 and 3´) it follows: (α) f_1 , and g are once differentiable functionals on $\mathring{W}_{m,p}$ and their derivatives f_1' and g' satisfy the equations

$$(f_1'(u),v) = a(u,v), \quad (g'(u),v) = b(u,v)$$

for all $u, v \in \mathring{W}_{m,p}$.

(β) g is weakly continuous and g' is a continuous compact mapping from $\overset{\circ}{W}_{m,p}$ to $\overset{\circ}{W}{}^{*}_{m,p}$.

(γ) f_1' is a continuous mapping of $\overset{\circ}{W}_{m,p}$ into $\overset{\circ}{W}{}^{*}_{m,p}$, which maps bounded sets into bounded sets and satisfies Condition (S+).

(δ) $f_1(u) \to \infty$ as $\|u\| \to \infty$.

Along the lines of the theorem of Browder it also follows for each $\varepsilon \in]0,1]$:

(α) $f_2(\varepsilon,\cdot)$ is a once differentiable functional on $\overset{\circ}{W}_{m,p}$ and its derivative $f_2'(\varepsilon,\cdot)$ satisfies for all $u, v \in \overset{\circ}{W}_{m,p}$

$$(f_2'(\varepsilon,u),v) = c(\varepsilon,u,v) .$$

(β) $f_2'(\varepsilon,\cdot)$ is a continuous mapping of $\overset{\circ}{W}_{m,p}$ into $\overset{\circ}{W}{}^{*}_{m,p}$ which maps bounded sets into bounded sets.

(γ) $f'(\varepsilon,\cdot) := f_1'(\cdot) + f_2'(\varepsilon,\cdot)$ satisfies Condition (S+) and hence Condition (S), too.

Further by the assumption on H and by (δ) we have for each $\varepsilon \in]0,1]$

$$f(\varepsilon,u) = f_1(u) + f_2(\varepsilon,u) \geq f_1(u) \to \infty$$

as $\|u\| \to \infty$. Therefore it easily follows - by virtue of the above remarks - that Assumptions 1(a,b,c) hold.

(b) Let $u \in M_c(g)$, then we obtain by Assumption 3(e)

$$(g'(u),u) = \sum_{|\beta|\leq m-1} [G_\beta(\cdot,\eta(u)), D^\beta u] \geq$$

$$\geq \alpha \int_\Omega G(x,\eta(u)(x))\,dx = \alpha g(u) = \alpha c > 0$$

which implies Assumption 1(d).

(c) By the assumption of Theorem 2 it follows for all

$\varepsilon \in]0,1]$

$$f(\varepsilon, v_0) = f_1(v_0) + f_2(\varepsilon, v_0) \leq f_1(v_0) + \\ + \int_\Omega H(x, \eta(v_0)(x))\,dx \leq \mathcal{C}_5$$

with some constant $\mathcal{C}_5$, proving Assumption 2(a).

(d) For each $\varepsilon \in]0,1]$ and each $u \in \overset{\circ}{W}_{m,p}$ we further have by Assumption 4(b)

$$0 \leq (f_2'(\varepsilon, u), u) = \sum_{|\beta| \leq m-1} [H_\beta(\cdot, \eta(u))/(1+\varepsilon H(\cdot, \eta(u)))^2, D^\beta u]$$

$$\leq c_0 \int_\Omega H(x, \eta(u)(x))/(1+\varepsilon H(x, \eta(u)(x)))dx = c_0 f_2(\varepsilon, u)$$

which proves Assumption 2(b).

(e) Assumption 2(c) follows along the lines of the proof of Theorem 2(c) in [10] and shall therefore be omitted. Hence Theorem 2 follows from Theorem 1.

<u>Remark</u>. (a) Assumption 4(c) can be replaced by a condition which is more useful in applications (see [10], Proposition 3).

(b) $f_2(0, u)$ is by Assumption not necessarily defined for all u in $M_c(g)$.

(c) Theorem 2 generalizes in one sense a theorem of Browder [6] (Theorem 17), who assumes $H(x, \eta) = 0$.

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