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NOTES ON RADICAL FILTERS OF IDEALS

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Abstract: Let R be a ring and $\mathfrak{M}$ be a non-empty set of left ideals of R. Denote by $\mathcal{F}(\mathfrak{M})$ the radical filter generated by $\mathfrak{M}$. In this paper we give a certain characterization of $\mathcal{F}(\mathfrak{M})$.

Key words: Radical filter, hereditary torsion class, hereditary radical.

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In the following, R will be an associative ring with unit and the word "module" means a unitary left R-module. Further, we shall denote by R-mod the category of all the R-modules and $\mathcal{S}(M)$ will be the set of all submodules in M for any $M \in R$-mod. Let $\mathfrak{M} \subseteq \mathcal{S}(R)$ be a non-empty subset. Consider the following six conditions for $\mathfrak{M}$.

(F_1) If $I \in \mathfrak{M}$, $K \in \mathcal{S}(R)$ and $I \subseteq K$, then $K \in \mathfrak{M}$.

(F_2) If $I \in \mathfrak{M}$ and $\lambda \in R$, then $(I:\lambda) = \{\varphi \mid \varphi \in R, \varphi\lambda \in I\} \in \mathfrak{M}$.

(F_3) If $I, K \in \mathfrak{M}$, then $I \cap K \in \mathfrak{M}$.

(F_4) If $I, K \in \mathfrak{M}$, then $I \cdot K \in \mathfrak{M}$. and

(F_5) If $I \in \mathfrak{M}, K \in \mathcal{S}(R), K \subseteq I$ and $(K:\lambda) \in \mathfrak{M}\ \forall \lambda \in I$, then $K \in \mathfrak{M}$.

(F_6) If $I \in \mathfrak{M}, K \in \mathcal{S}(R)$ and $(K:\lambda) \in \mathfrak{M}\ \forall \lambda \in I$, then $K \in \mathfrak{M}$.

The set $\mathfrak{M}$ is called a filter (a radical filter) if it satisfies the conditions (F_1), (F_2), (F_3) ((F_1), (F_2), (F_5)). As it is easy to show, any radical filter satisfies all the six conditions (F_1) ... (F_6). Recall that there is a one-to-one correspondence between radical filters and so called hereditary radicals. A hereditary radical is an arbitrary subfunctor of the identity κ having the following properties:

(i) $\kappa\left(M/\kappa(M)\right) = 0 \;\forall M \in R\text{-}mod$,

(II) $\kappa(N) = N \cap \kappa(M) \;\forall M \in R\text{-}mod \;\forall N \in \mathcal{S}(M)$.

If $\mathfrak{M}$ is a radical filter, then the subfunctor κ , given by $\kappa(M) = \{m \mid (0 : m) \in \mathfrak{M}\}$, is a hereditary radical. Conversely, if κ is a hereditary radical then $\{I \mid I \in \mathcal{S}(R), \kappa(R/I) = R/I\}$ is a radical filter. (For the proof see e.g. [4].) A non-empty class of modules $\mathfrak{M}$ is said to be a hereditary torsion class, if it is closed under submodules, homomorphic images, extensions and direct sums. In this case, the subfunctor κ, $\kappa(M) = \sum_{N \in \mathcal{S}(M) \cap \mathfrak{M}} N$ is a hereditary radical. Conversely, if κ is a hereditary radical then $\{M \mid \kappa(M) = M\}$ is a hereditary torsion class. Since the intersection of any set of radical filters is a radical filter, we can consider the complete lattice $\mathcal{L}(R)$ of all radical filters of the ring R . Finally, denote by $\mathcal{K}(R)$ the set of all the subsets $\mathfrak{M} \subseteq \mathcal{S}(R)$ which satisfy the conditions (F_1) and (F_2). It is obvious that $\mathcal{K}(R)$ is a sublattice in the lattice

$2^{\mathcal{S}(R)}$ of all subsets of $\mathcal{S}(R)$.

2. If $M \in R\text{-}mod$ and $K \in \mathcal{S}(M)$ then we denote by $\mathcal{E}^1(K,M)$ the set $\{N \mid N \in \mathcal{S}(M), K \subseteq N\}$ and by $\mathcal{E}^2(K,M)$ the set $\{N \mid N \in \mathcal{S}(M), K \subseteq N$ and N/K is essential in $M/K\}$. Further, $\mathcal{E}^3(K,M)$ will be $\mathcal{E}^2(K,M) \cup \{K\}$.

2.1. Lemma. Let $M \in R\text{-}mod$ and $K, L, N \in \mathcal{S}(M)$ be such that $K \subseteq L \subseteq N$. Then:

(i) $N \in \mathcal{E}^2(K,M)$ iff $N \cap X = K$ implies $X = K$ for arbitrary $X \in \mathcal{S}(M)$.

(ii) $N \in \mathcal{E}^2(L,M)$ implies $N \in \mathcal{E}^2(K,M)$.

(iii) $L \in \mathcal{E}^2(K,M)$ implies $N \in \mathcal{E}^2(K,M)$.

Proof. Obvious.

Before we proceed further, let us introduce the following notation. If $M \in R\text{-}mod$ and $\emptyset \neq \mathfrak{M} \subseteq \mathcal{S}(M)$, then by $r_{\mathfrak{M}}$ we shall mean the hereditary radical corresponding to the hereditary torsion class, which is generated by all the factor-modules M/N , $N \in \mathfrak{M}$. Further put $\mathfrak{A}(\mathfrak{M}) = \{S \mid S \in \mathcal{S}(M), \exists m \in M \setminus S$ $\forall n \in M\ \forall \lambda \in R \setminus (S:m)\ \forall N \in \mathfrak{M}\ \exists \sigma \in (N:n)$ such that $\sigma\lambda m \notin S\}$ and $\mathfrak{B}(\mathfrak{M}) = \mathcal{S}(\mathfrak{M}) \setminus \mathfrak{A}(\mathfrak{M})$. Thus $\mathfrak{B}(\mathfrak{M}) = \{S \mid S \in \mathcal{S}(M), \forall m \in M \setminus S\ \exists n \in M\ \exists \lambda \in R \setminus (S:m) \exists N \in \mathfrak{M}$ such that $(N:n) \subseteq (S:\lambda m)\}$.

2.2. Lemma. Let $M \in R\text{-}mod, A \in \mathcal{S}(M)$ and

$\emptyset \neq \mathfrak{M} \subseteq \mathcal{S}(M)$. Then $A \in \mathcal{A}(\mathfrak{M})$ iff there is $m \in M \setminus A$ such that

$$\mathrm{Hom}_R\left({}^{B}/_{N}, {}^{Rm+A}/_{A}\right) = 0$$

for all $N \in \mathfrak{M}$ and $B \in \mathcal{E}^1(N, M)$.

Proof. (i) Let $A \in \mathcal{A}(\mathfrak{M})$. Then there is $m \in M \setminus A$ such that $(N : n) \nsubseteq (A : \lambda m)$ for any $n \in M$, $N \in \mathfrak{M}$ and $\lambda \in R \setminus (A : m)$. If $\varphi : {}^{B}/_{N} \longrightarrow {}^{Rm+A}/_{A}$ is non-zero, then $\varphi(b + N) = \rho m + A \neq 0$ for some $b \in B$ and $\rho \in R$. Hence $\rho \in R \setminus (A : m)$ and $(N : b) \subseteq (A : \rho m)$, a contradiction.

(ii) Let A satisfy the condition of the lemma. If $(N : n) \subseteq (A : \lambda m)$ for some $N \in \mathfrak{M}$ and $\lambda \in R \setminus (A : m)$, then the mapping $\varphi : {}^{Rn+N}/_{N} \longrightarrow {}^{Rm+A}/_{A}$ defined by $\varphi(\rho n + N) = \rho\lambda m + A \ \forall \rho \in R$, is a non-zero homomorphism, a contradiction.

2.3. Lemma. Let $M \in R\text{-mod}$, $K \in \mathcal{S}(M)$ and $\emptyset \neq \mathfrak{M} \subseteq \mathcal{S}(M)$ be such that $K \in \mathcal{B}(\mathfrak{M})$. Then:

(i) $S \in \mathcal{E}^2(K, M)$, where ${}^{S}/_{K} = \varkappa_{\mathfrak{M}}\left({}^{M}/_{K}\right)$.

(ii) $\varkappa_{\mathfrak{M}}\left({}^{M}/_{K}\right) \neq 0$, provided $M \neq K$.

Proof. (i) Let $m \in M \setminus K$ be arbitrary. In view of Lemma 2.2, there is $N \in \mathfrak{M}$ and $B \in \mathcal{S}(M)$ such that $N \subseteq B$ and $\mathrm{Hom}_R\left({}^{B}/_{N}, {}^{Rm+K}/_{K}\right) \neq 0$. Since $\varkappa_{\mathfrak{M}}\left({}^{B}/_{N}\right) = {}^{B}/_{N}$, $\varkappa_{\mathfrak{M}}\left({}^{Rm+K}/_{K}\right) \neq 0$. However,

$\varkappa_{\mathfrak{m}}({}^{Rm+K}/_K) = {}^{Rm+K}/_K \cap {}^{S}/_K$, and consequently ${}^{S}/_K$ is essential in ${}^{M}/_K$.

(i) There is $m \in M \setminus K$, and hence (by Lemma 2.2) $Hom_R({}^{B}/_N, {}^{Rm+K}/_K) \neq 0$ for some $N \in \mathfrak{m}$ and $B \in \mathcal{E}^1(N,M)$. Thus $0 \neq \varkappa_{\mathfrak{m}}({}^{Rm+K}/_K) \subseteq \varkappa_{\mathfrak{m}}({}^{M}/_K)$.

2.4. Lemma. Let $M \in R\text{-}mod$, $K \in \mathcal{S}(M)$ and $\emptyset \neq \mathfrak{m} \subseteq \mathcal{S}(M)$. Then the following are equivalent:

(i) $\mathcal{E}^3(K,M) \cap \mathfrak{a}(\mathfrak{m}) \neq \emptyset$.

(ii) $\mathcal{E}^1(K,M) \cap \mathfrak{a}(\mathfrak{m}) \neq \emptyset$.

(iii) There are $A \in \mathcal{E}^3(K,M)$ and $S \in \mathcal{S}(M)$ such that $A \subsetneqq S$ and $\varkappa_{\mathfrak{m}}({}^{S}/_A) = 0$.

(iv) There are $A \in \mathcal{E}^1(K,M)$ and $S \in \mathcal{S}(M)$ such that $A \subsetneqq S$ and $\varkappa_{\mathfrak{m}}({}^{S}/_A) = 0$.

(v) $\varkappa_{\mathfrak{m}}({}^{M}/_K) \neq {}^{M}/_K$.

Proof. (i) implies (ii) and (iii) implies (iv) trivially. (i) implies (iii). Let $A \in \mathcal{E}^3(K,M) \cap \mathfrak{a}(\mathfrak{m})$. By Lemma 2.2, there is $m \in M \setminus A$ such that $Hom_R({}^{B}/_N, {}^{Rm+A}/_A) = 0$ for all $N \in \mathfrak{m}$ and $B \in \mathcal{E}^1(N,M)$. From this, one can easily derive $\varkappa_{\mathfrak{m}}({}^{Rm+A}/_A) = 0$. Now it is sufficient to put $S = {}^{Rm+A}/_A$.

Similarly we can prove (ii) implies (iv).

(iv) implies (v). If $\varkappa_{\mathcal{M}}(M/K) = M/K$, then $\varkappa_{\mathcal{M}}(S/A) = S/A$ for all $A, S \in \mathcal{E}^1(K,M)$ such that $A \subseteq S$.

(v) implies (i). Assume, on the contrary, that $K \in \mathcal{B}(\mathcal{M})$, and therefore, in view of Lemma 2,3, $S \in \mathcal{E}^2(K,M)$, where $S/K = \varkappa_{\mathcal{M}}(M/K)$. Using Lemma 2.3 again, we get $\varkappa_{\mathcal{M}}(M/S) \neq 0$, a contradiction.

<u>2.5. Theorem</u>. Let $\mathcal{M} \subseteq \mathcal{S}(R)$ be a non-empty subset. Then $\mathcal{F}(\mathcal{M}) = \{I \mid I \in \mathcal{S}(R), \mathcal{E}^1(I,R) \subseteq \mathcal{B}(\mathcal{M})\} = \{I \mid I \in \mathcal{S}(R), \mathcal{E}^3(I,R) \subseteq \mathcal{B}(\mathcal{M})\}$.

<u>Proof</u>. The theorem follows from Lemma 2.4, since $\mathcal{F}(\mathcal{M}) = \{I \mid I \in \mathcal{S}(R), \varkappa_{\mathcal{M}}(R/I) = R/I\}$.

<u>2.6. Corollary</u>. A non-empty subset $\mathcal{R} \subseteq \mathcal{S}(R)$ is a radical filter iff it satisfies the following condition:

$(F_{\mathcal{Y}})$ If $I \in \mathcal{S}(R)$ and $\forall K \in \mathcal{E}^1(I,R)\ \forall \varkappa \in R \setminus K\ \exists \sigma \in R\ \exists \lambda \in R \setminus (K:\varkappa)\ \exists L \in \mathcal{R}$ such that $(L:\sigma) \subseteq (K:\lambda\varkappa)$, then $I \in \mathcal{R}$.

<u>Proof</u>. This corollary is only a transcription of Theorem 2.5.

For a non-empty subset $\mathcal{M} \subseteq \mathcal{S}(R)$ put $\mathcal{C}(\mathcal{M}) = \{I \mid \exists \lambda \in R\ \exists K \in \mathcal{M}$ such that $(K:\lambda) \subseteq I\}$ and $\mathcal{D}(\mathcal{M}) = \{I \mid \forall \lambda \in R \setminus I\ \exists \wp \in R \setminus (I:\lambda)$ such that $(I:\wp\lambda) \in \mathcal{M}\}$.

2.7. Corollary. Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid I \in \mathcal{I}(R),\ \mathcal{E}^1(I,R) \subseteq \mathcal{D}(C(\mathfrak{M}))\} =$
$= \{I \mid I \in \mathcal{I}(R),\ \mathcal{E}^3(I,R) \subseteq \mathcal{D}(C(\mathfrak{M}))\}$.
In particular, if $\mathfrak{M}$ satisfies (F_1) and (F_2), then
$\mathcal{F}(\mathfrak{M}) = \{I \mid \mathcal{E}^1(I,R) \subseteq \mathcal{D}(\mathfrak{M})\} = \{I \mid \mathcal{E}^3(I,R) \subseteq \mathcal{D}(\mathfrak{M})\}$.

Proof. The corollary follows from Theorem 2.5, since $\mathcal{B}(\mathfrak{M}) = \mathcal{D}(C(\mathfrak{M}))$, as one may check easily.

As a very easy consequence of 2.7 and 2.1 we get the following well-known result (see [3]).

2.8. Corollary. Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset satisfying (F_1), (F_2) and let $\mathcal{E}^2(0,R) \subseteq \mathfrak{M}$. Then $\mathcal{F}(\mathfrak{M}) = \mathcal{D}(\mathfrak{M})$.

2.9. Corollary. Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset and let $\mathcal{H}(\mathfrak{M}) = \{I \mid I \in \mathcal{I}(R), \exists \lambda \in R \setminus I\ \exists N \in \mathfrak{M}\ \exists n \in R$ such that $(N : n) \subseteq (I : \lambda)\}$. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid \mathcal{E}^1(I,R) \setminus \{R\} \subseteq$
$\subseteq \mathcal{H}(\mathfrak{M})\}$.

Proof. (i) Let $I \in \mathcal{F}(\mathfrak{M})$, $I \neq R$. Then, by 2.6 (for $r = 1$), there are $n \in R$, $\lambda \in R \setminus (I : 1) = R \setminus I$ and $N \in$
$\in \mathfrak{M}$ with $(N : n) \subseteq (I : \lambda)$.

(ii) Let $I \in \mathcal{I}(R$ and $\{\mathcal{E}^1(I,R) \setminus \{R\}\} \subseteq \mathcal{H}(\mathfrak{M})$. Set $S/I = \kappa_{\mathfrak{M}}(R/I)$. If $S = R$, then obviously $I \in$
$\in \mathcal{F}(\mathfrak{M})$. Suppose $S \neq R$. By the hypothesis, there are $\lambda \in R \setminus S$, $N \in \mathfrak{M}$ and $n \in R$ such that $(N : n) \subseteq (S : \lambda)$. Thus $(S : \lambda) \in \mathcal{F}(\mathfrak{M})$ and $\lambda + S \in \kappa_{\mathfrak{M}}(R/S)$, a contradiction since $\kappa_{\mathfrak{M}}(R/S) = 0$.

2.10. Corollary. Let $I \in \mathcal{S}(R)$ be a two-sided ideal; $\varphi: R \longrightarrow R/I$ be the canonical epimorphism and $\mathcal{R} \subseteq \mathcal{S}(R/I)$ be a radical filter. Put $\mathfrak{X} = \{K \mid K \in \mathcal{S}(R),\ I \subseteq K$ and $\varphi(K) \in \mathcal{R}\}$. Then $\varphi(L) \in \mathcal{R}$ for all $L \in \mathcal{F}(\mathfrak{X})$.

Proof. Let $L \in \mathcal{F}(\mathfrak{X})$ be arbitrary and $K \in \mathcal{S}(R) \setminus \{R\}$ be such that $I \subseteq K$ and $\varphi(L) \subseteq \varphi(K)$. By 2.9, there are $N \in \mathfrak{X}$, $\varkappa \in R$ and $\sigma \in R \setminus K$ with $(N : \varkappa) \subseteq (K : \sigma)$. Since I is a two-sided ideal, $I \subseteq (N : \varkappa)$ and $I \subseteq (K : \sigma)$. Hence $\varphi((N:\varkappa)) = (\varphi(N) : \varphi(\varkappa)) \subseteq \varphi((K:\sigma)) = (\varphi(K) : \varphi(\sigma))$.

However, $\varphi(N) \in \mathcal{R}$ and $\varphi(\sigma) \notin \varphi(K)$. Thus we have proved $\{\mathcal{E}^{1}(\varphi(L)) \setminus \{R/I\}\} \subseteq \mathcal{H}(\mathcal{R})$, and therefore $\varphi(L) \in \mathcal{R}$ (by 2.9).

2.11. Corollary. The lattice $\mathcal{L}(R)$ is distributive, and it is complementary iff R is a semiartinian ring.

Proof. For $\mathcal{U}, \mathcal{V} \in \mathcal{K}(R)$ put $\mathcal{U} \rho \mathcal{V}$ iff $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{V})$. From 2.9 it is easy to see that ρ is a congruence relation on the lattice $\mathcal{K}(R)$ and that $\mathcal{K}(R)/\rho \cong \mathcal{L}(R)$. If, further, $\mathcal{L}(R)$ is complementary, then the radical filter $\mathcal{R}$ which is generated by all maximal left ideals possesses a complement $\mathcal{T}$, and consequently $\mathcal{R} = \mathcal{S}(R)$ (since $\mathcal{T} \cap \mathcal{R} = \{R\}$ implies $\mathcal{T} = \{R\}$). For the converse implication suppose that R is semiartinian and $\mathcal{U} \in \mathcal{L}(R)$ is an element. Denote $\mathcal{V} = \{I \mid I \in \mathcal{S}(R)$ is maximal and $I \in \mathcal{U}\}$ and

$\mathcal{Z} = \{I \mid I \in \mathcal{S}(R)$ is maximal and $I \notin \mathcal{U}$ or $I = R\}$. Obviously, $\mathcal{Z}, \mathcal{V} \in \mathcal{K}(R)$. Further, since R is semiartinian, $\mathcal{F}(\mathcal{V}) = \mathcal{U}$ and $\mathcal{F}(\mathcal{F}(\mathcal{U}) \cup \mathcal{F}(\mathcal{Z})) = \mathcal{S}(R)$. Finally, let $\mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{Z}) \neq \{R\}$. Then there is $I \in \mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{Z})$, $I \neq R$ is a maximal left ideal. By 2.9, $(I : \lambda) \in \mathcal{Z}$ for some $\lambda \in R \setminus I$. However, $1 = \rho\lambda + \alpha$, where $\rho \in R$ and $\alpha \in I$ are suitable, and so $I = (I : \rho\lambda) = ((I : \lambda) : \rho) \in \mathcal{Z}$. Thus $I \in \mathcal{Z} \cap \mathcal{V}$, a contradiction.

Let us note here that the preceding corollary was already proved before in [1] for the case of commuative noetherian rings.

3. In this paragraph we generalize some results from [2] to get a characterization of $\mathcal{F}(\mathcal{M})$, where $\mathcal{M}$ is a countable set of two-sided ideals. Let $\mathcal{M} = \{I_1, I_2, \dots\}$ be a countable subsystem of $\mathcal{S}(R)$. A sequence $\{\lambda_1, \lambda_2, \dots\}$ of elements from R will be called $\mathcal{M}$-regular if the set $\{i \mid \lambda_i \in I_j\}$ is infinite for any $j = 1, 2, \dots$. Denote by $\mathcal{U}(\mathcal{M})$ the set of all the $\mathcal{M}$-regular sequences and put $\mathcal{G}(\mathcal{M}) = \{I \mid \forall \{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathcal{M}) \forall \rho \in R\ \exists n \geq 1$ such that $\lambda_n \dots \lambda_1 \rho \in I\}$.

<u>3.1. Theorem</u>. Let $\mathcal{M} = \{I_1, I_2, \dots\}$ be a countable subsystem of $\mathcal{S}(R)$. Then:

(i) $\mathcal{G}(\mathcal{M})$ is a radical filter.

(ii) $\mathcal{G}(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$.

(iii) $\mathcal{G}(\mathcal{M}) = \mathcal{F}(\mathcal{M})$ provided every ideal from $\mathcal{M}$ is two-sided.

Proof. (i) The condition (F_1) is obvious. Now (F_2). Let $I \in \mathcal{G}(\mathfrak{M})$ and $\sigma \in R$ be arbitrary. If $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathfrak{M})$ and $\varrho \in R$, then (by the hypothesis) there is $n \geq 1$ such that $\lambda_n \dots \lambda_1 \varrho \sigma \in I$, i.e. $\lambda_n \dots \lambda_1 \varrho \in (I : \sigma)$. Finally (F_6). Let $I \in \mathcal{S}(R)$, $K \in \mathcal{G}(\mathfrak{M})$ and $(I : \varkappa) \in \mathcal{G}(\mathfrak{M})$ for each $\varkappa \in K$. Given $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathfrak{M})$ and $\varrho \in R$, there is $n \geq 1$ with $\lambda_n \dots \lambda_1 \varrho \in K$. However, the sequence $\{\lambda_{n+1}, \lambda_{n+2}, \dots\}$ is also $\mathfrak{M}$-regular and $(I : \lambda_n \dots \lambda_1 \varrho) \in \mathcal{G}(\mathfrak{M})$. Hence there is $m \geq 1$ such that $\lambda_{n+m} \dots \lambda_{n+1} \cdot 1 \in (I : \lambda_n \dots \lambda_1 \varrho)$ and so $\lambda_{n+m} \dots \lambda_n \dots \lambda_1 \varrho \in I$.

(ii) Suppose, on the contrary, that there exists $I \in \mathcal{G}(\mathfrak{M})$, $I \notin \mathcal{F}(\mathfrak{M})$. Hence (by (F_6)) there is $\lambda_1 \in I_1$ such that $(I : \lambda_1) \notin \mathcal{F}(\mathfrak{M})$. Further, $I_1 \cap I_2 \in \mathcal{F}(\mathfrak{M})$ and therefore there is $\lambda_2 \in I_1 \cap I_2$ such that $(I : \lambda_2 \lambda_1) = ((I : \lambda_1) : \lambda_2) \notin \mathcal{F}(\mathfrak{M})$. Repeating this argument, we get a sequence $\{\lambda_1, \lambda_2, \dots\}$ having the following properties:

(α) $\lambda_j \in I_1 \cap I_2 \cap \dots \cap I_j$ for every $j = 1, 2, \dots$,

(β) $(I : \lambda_j \dots \lambda_1) \notin \mathcal{F}(\mathfrak{M})$ for every $j = 1, 2, \dots$.

From (α) we see that $\{\lambda_1, \lambda_2, \dots\}$ is an $\mathfrak{M}$-regular sequence. Hence, by the hypothesis, $\lambda_n \dots \lambda_1 . 1 \in I$ for some $n \geq 1$, and consequently $(I : \lambda_n \dots \lambda_1) = R$, which is a contradiction with (β).

(iii) Obvious, since $I_j \in \mathcal{G}(\mathfrak{M})$ whenever I_j is a two-sided ideal.

3.2. Corollary. Let $\mathfrak{M} = \{I_1, \dots, I_n\}$ be a finite set of two-sided ideals. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid \forall \lambda_1, \lambda_2, \dots \in I_1 \cap \dots \cap I_n \ \exists m \geq 1$ such that $\lambda_m \dots \lambda_1 \in I\}$.

Proof. Denote by $\mathcal{J}$ the set defined above. From 3.1 it is obvious that $\mathcal{F}(\mathfrak{M}) \subseteq \mathcal{J}$. In order to prove the converse inclusion we need only to observe the following fact. If $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathfrak{M})$, then there exist $1 \leq \leq \ell_1 < \ell_2 < \ell_3 < \dots$ such that $\lambda_{\ell_j} \cdot \lambda_{\ell_j - 1} \dots \lambda_{\ell_{j-1}} \in I_1 \cap \dots \cap I_m$ for all $j = 1, 2, \dots$.

3.3. Corollary. Let $\mathfrak{M}$ be a finite set of two-sided ideals. Then $0 \in \mathcal{F}(\mathfrak{M})$ iff $\bigcap_{I \in \mathfrak{M}} I$ is right T-nilpotent.

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