Commentationes Mathematicae Universitatis Carolinae

Charles R. Diminnie; Albert G. White

2-innerproduct spaces and Gâteaux partial derivatives

Commentationes Mathematicae Universitatis Carolinae, Vol. 16 (1975), No. 1, 115--119

Persistent URL: http://dml.cz/dmlcz/105609

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

16,1 (1975)

2-INNER PRODUCT SPACES AND GÂTEAUX PARTIAL DERIVATIVES

Charles R. DIMINNIE and Albert G. WHITE Jr., St.Bonaventure

Abstract: The purpose of this paper is to characterize 2-inner product spaces by means of partial derivatives of bifunctionals. If $(L,(\cdot,\cdot|\cdot))$ is a 2-inner product space with 2-norm defined by $\|x,y\| = (x,x\,y)^{\frac{1}{2}}$, then $(a,b\,|\,c) = \lim_{t\to 0^+} \frac{\|a+tb,c\|^2 - \|a,c\|^2}{2t}$

Key words and phrases: 2-inner product space, 2-norm space, Gâteaux partial derivative.

AMS: 46A99 Ref. Ž.: 7.972.2

In [4], R.A. Tapia discusses a characterization of inner-product spaces which involves the Gâteaux derivative of a certain functional. Several of the results of that paper are useful in studying 2-inner-product spaces as well. For definitions and basic results in 2-inner-product spaces and 2-normed spaces, see [2] and [3].

Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space of dimension > 1. If $F(x,y)$ is a real bifunctional on L, then the right partial derivative of F with respect to x at (x,y) in the direction of h, $F_{1+}(x,y)(h)$, is defined by

$$F_{1+}(x,y)(h) = \lim_{t\to 0^+} \frac{1}{t}\, F(x+th,y) - F(x,y)$$

Similar definitions are used for F_{1-} , F_{2+} , F_{2-} .

The partial derivative of F _with respect to_ x _in the direction of_ h , $F_1(x,y)(h)$, is defined by:

$$F_{1+}(x,y)(h) = F_1(x,y)(h) = F_{1-}(x,y)(h) ,$$

whenever the one-sided partials agree.

$F_2(x,y)(h)$ is defined similarly.

The following two results are easily proved from the above definitions.

Theorem 1. Let x,y , $h \in L$ and F be a real bifunctional on L .

1. If F is linear in its first variable, then $F_1(x,y)(h) = F(h,y)$.
2. If F is linear in its second variable, then $F_2(x,y)(h) = F(x,h)$.
3. If F is bilinear, then $F_1(x,y)(h) = F(h,y)$ and $F_2(x,y)(h) = F(x,h)$.

Theorem 2. If F is a symmetric bifunctional and $F_1(x,y)(h)$ exists, then $F_2(y,x)(h)$ exists also and $F_2(y,x)(h) = F_1(x,y)(h)$.

For the topics to follow, it is useful to consider a certain class of normed spaces associated with $(L, \|\cdot,\cdot\|)$. If $c \neq 0$, let L_c be the quotient space $L/V(c)$, where $V(c)$ is the subspace of L generated by c . For $a \in L$, let a_c denote the element of L_c determined by a. L_c is a vector space under the operations $a_c + b_c = (a+b)_c$ and $\alpha a_c = (\alpha a)_c$. Define $\| \cdot \|_c$ on L_c by $\| a_c \|_c = \| a,c \|$. By using the properties of $\| \cdot , \cdot \|$, particularly $\big| \| a,c \| - \| b,c \| \big| \leq \| a-b, c \|$, it is easily shown that $\| \cdot \|_c$ is a norm on L_c (see [1]).

The remainder of the discussion will be devoted to the bifunctional

$$F(x,y) = \frac{1}{2} \| x,y \|^2 . \tag{1}$$

If $c \neq 0$, F generates a functional F_c on L_c defined by

$$F_c(a_c) = F(a,c) = \frac{1}{2} \| a,c \|^2 = \frac{1}{2} \| a_c \|_c^2 . \tag{2}$$

If F^1_{c+}, F^1_{c-}, and F^1_c denote the Gâteaux derivatives of F_c, then it is easily seen that $F_{1+}(x,c)(h) = F^1_{c+}(x_c)(h_c)$, $F_{1-}(x,c)(h) = F^1_{c-}(x_c)(h_c)$, and $F_1(x,c)(h) = F^1_c(x_c)(h_c)$, whenever these derivatives exist.

For $a, b, c \in L$, define

$$[a, b \mid c] = F_{1+}(a,c)(b) . \tag{3}$$

Theorem 3. $[\cdot , \cdot \mid \cdot]$ has the following properties:

1. $[a,b \mid c]$ is defined for every $a, b, c \in L$.
2. $\| a,b \| = [a,a \mid b]^{1/2}$.

3. $|[a,b \mid c]| \leq \| a,c \| \; \| b,c \|$.

4. If L is a 2-inner-product space, with 2-inner-product $(\cdot,\cdot \mid \cdot)$, then $[a,b \mid c] = (a,b \mid c)$.

<u>Proof</u>. Properties 2 and 4 follow by direct computation.

1. $[a,b \mid 0] = \lim_{t \to 0^+} \frac{1}{2} [\frac{1}{2} \| a + tb , 0 \|^2 - \frac{1}{2} \| a,0 \|^2 = 0$.

If $c \neq 0$, then $F^1_{c+}(a_c)(b_c)$ exists for every $a, b \in L$ by Proposition 1 of [4]. Therefore, $[a,b \mid c] = F_{1+}(a,c)(b)$ exists, too. Hence, $[a,b \mid c]$ exists for every $a, b, c \in L$.

3. If $c = 0$, the result is obvious since $[a,b \mid 0] = 0$.

If $c \neq 0$, then by Proposition 1 of [4] ,

$$\begin{aligned} |[a,b \mid c]| &= |F_{1+}(a,c)(b)| \\ &= |F^1_{c+}(a_c)(b_c)| \\ &\leq \| a_c \|_c \; \| b_c \|_c \\ &= \| a,c \| \; \| b,c \| . \end{aligned}$$

The last theorem is a direct result of Theorem 1 of [4] and Theorem 6 of [2].

<u>Theorem 4</u>. The following are equivalent.

1. $(L, \| \cdot,\cdot \|)$ is a 2-inner-product space.
2. $[a,b \mid c]$ is linear in a .
3. $[a, b \mid c]$ is symmetric in a and b .

Remark. By Theorem 2, $[\cdot,\cdot|\cdot]$ could also have been defined by $[a,b|c] = F_{2+}(c,a)(b)$.

References

[1] C. DIMINNIE, S. GÄHLER, and A. WHITE: Strictly convex 2-normed spaces, Math.Nachr.59(1974),319-324.

[2] __________ , 2-inner product spaces, Demontratio Mathematica 6(1973),525-536.

[3] S. GÄHLER: Lineare 2-normierte Räume, Math.Nachr.28 (1965),1-45.

[4] R.A. TAPIA: A characterization of inner-product spaces, Bull.Amer.Math.Soc.79(1973),530-531.

St. Bonaventure University
St. Bonaventure
New York 14778
U.S.A.

(Oblatum 16.10.1974)