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ON ORTHOGONAL CONJUGATE NETS IN E^4

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Abstract: We prove that orthogonal conjugate nets satisfying globally certain conditions are situated on a sphere.

Key words: Surface, conjugate net, sphere.

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Let $M \subset E^4$ be a surface, let ∂M be its boundary; everything be of class C^∞. On M, be given two unit tangent vector fields v_1, v_2 such that $v_1(m)$ is orthogonal to $v_2(m)$ for each $m \in M$. We say that v_1, v_2 generate an orthogonal conjugate net on M if $v_1 v_2 \in T(M)$, $T(M)$ being the tangent bundle of M. This being the case, introduce the normal vector fields

$$w_1 = (v_1 v_1)^N, \quad w_2 = (v_2 v_2)^N, \tag{1}$$

$(v_i v_i)^N$ being the normal part of $v_i v_i$. We are going to prove the following

Theorem. Let $M \subset E^4$ be a surface, and let $v_1 v_2 \in T(M)$ generate an orthogonal conjugate net. Suppose: (i) $K > 0$ on M, K being the Gauss curvature; (ii) on M,

(2) $\langle v_1(w_1 - w_2), v_1 w_2 \rangle + \langle v_2 w_1, v_2(w_2 - w_1) \rangle \leq 0$;

(iii) <u>on</u> ∂M , <u>we have</u> $w_1 = w_2$. <u>Then</u> M <u>is a part of a sphere</u>.

<u>Proof</u>. To each point $m \in M$ associate an orthonormal frame $(m; v_1, v_2, v_3, v_4)$. Then

$$(3) \qquad dm = \omega^1 v_1 + \omega^2 v_2 , \quad dv_i = \omega_i^j v_j ;$$

$$\omega_i^j + \omega_j^i = 0 , \quad d\omega^i = \omega^j \wedge \omega_j^i , \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j ;$$

$$(4) \qquad \omega^3 = \omega^4 = 0 .$$

From (4), $\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0$, $\omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0$, and we get the existence of functions $a_1, \ldots, b_3$ such that

$$(5) \qquad \omega_1^3 = a_1\omega^1 + a_2\omega^2 , \quad \omega_1^4 = b_1\omega^1 + b_2\omega^2 ,$$

$$\omega_2^3 = a_2\omega^1 + a_3\omega^2 , \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 .$$

Thus

$$(6) \qquad v_1 v_1 = (.)v_2 + a_1 v_3 + b_1 v_4 , \quad v_1 v_2 = (.)v_1 + a_2 v_3 + b_2 v_4 ,$$

$$v_2 v_1 = (.)v_2 + a_2 v_3 + b_2 v_4 , \quad v_2 v_2 = (.)v_1 + a_3 v_3 + b_3 v_4 .$$

The vector fields v_1, v_2 generating an orthogonal conjugate net, we have

$$(7) \qquad a_2 = b_2 = 0$$

and

$$(8) \qquad w_1 = a_1 v_3 + b_1 v_4 , \quad w_2 = a_3 v_3 + b_3 v_4 .$$

The moving frames may be chosen in such a way that $w_1 - w_2$

and $v_3 + v_4$ are linearly dependent, i.e.,

$$(9) \qquad a_1 - a_3 = b_1 - b_3 .$$

From (5),(7) and (9), we get

$$(10) \quad \begin{aligned} &(da_1 - b_1\omega_3^4) \wedge \omega^1 + (a_1 - a_3)\omega_1^2 \wedge \omega^2 = 0 , \\ &(a_1 - a_3)\omega_1^2 \wedge \omega^1 + (da_3 - b_3\omega_3^4) \wedge \omega^2 = 0 , \\ &(db_1 + a_1\omega_3^4) \wedge \omega^1 + (a_1 - a_3)\omega_1^2 \wedge \omega^2 = 0 , \\ &(a_1 - a_3)\omega_1^2 \wedge \omega^1 + (db_3 + a_3\omega_3^4) \wedge \omega^2 = 0 \end{aligned}$$

and the existence of functions $\alpha_1 , \ldots, \beta_4$ such that

$$(11) \quad \begin{aligned} da_1 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2 , \\ db_1 + a_1\omega_3^4 &= \beta_1\omega^1 + \alpha_2\omega^2 , \\ da_3 - b_3\omega_3^4 &= \alpha_3\omega^1 + \alpha_4\omega^2 , \\ db_3 + a_3\omega_3^4 &= \alpha_3\omega^1 + \beta_4\omega^2 , \\ (a_1 - a_3)\omega_1^2 &= \alpha_2\omega^1 + \alpha_3\omega^2 \end{aligned}$$

and

$$(12) \qquad 2(a_1 - a_3)\omega_3^4 = (\beta_1 - \alpha_1)\omega^1 + (\alpha_4 - \beta_4)\omega^2 .$$

Consider the invariant 1-form

$$(13) \qquad \tau = (a_1 - a_3)^2\omega_1^2 = (a_1 - a_3)(\alpha_2\omega^1 + \alpha_3\omega^2) ;$$

we get

$$(14) \quad d\tau = \{(\alpha_1 + \beta_1)\alpha_3 + (\alpha_4 + \beta_4)\alpha_2 - 2\alpha_2^2 - 2\alpha_3^2 - \\ - (a_1 - a_3)^2 K\}\omega^1 \wedge \omega^2$$

because of

$$(15)\quad d\omega_1^2 = -K\,\omega^1 \wedge \omega^2 \quad , \text{ i.e., } \quad K = a_1a_3 + b_1b_3 \; .$$

The second order invariants of the considered orthogonal conjugate net being

$$(16)\quad J_1 = \langle w_1 , w_1 \rangle = a_1^2 + b_1^2 \; , \; J_2 = \langle w_2 , w_2 \rangle = a_3^2 + b_3^2 \; ,$$

$$K = \langle w_1 , w_2 \rangle \; ,$$

we get

$$(17)\quad \begin{aligned} v_1w_1 &= - J_1\, v_1 + \alpha_1\, v_3 + \beta_1 v_4 \quad , \\ v_1w_2 &= - Kv_1 + \alpha_3\,(v_3 + v_4) \; , \\ v_2w_1 &= - Kv_2 + \alpha_2\,(v_3 + v_4) \; , \\ v_2w_3 &= - J_2v_2 + \alpha_4\, v_3 + \beta_4\, v_4 \end{aligned}$$

from (3) and (11). Thus

$$(18)\quad \langle v_1(w_1 - w_2),\, v_1\, w_2 \rangle + \langle v_2w_1,\, v_2(w_2 - w_1)\rangle =$$

$$= (\alpha_1 + \beta_1)\alpha_3 + (\alpha_4 + \beta_4)\alpha_2 - 2\alpha_2^2 - 2\alpha_3^2 +$$

$$+ (J_1 + J_2 - 2\,K)K \; .$$

From (13),(14),(18) and $J_1 + J_2 - 2\,K = 2(a_1 - a_3)^2$, we get the integral formula

$$(19)\quad \int_{\partial M} (a_1 - a_3)(\alpha_2\, \omega^1 + \alpha_3\, \omega^2\,) =$$

$$= \int_M \{\langle v_1(w_1 - w_2), v_1w_2 \rangle + \langle v_2w_1,\, v_2(w_2 - w_1)\rangle -$$

$$- 3(a_1 - a_3)^2\, K\,\}\,\omega^1 \wedge \omega^2 \quad .$$

Because of (iii), we have $a_1 = a_3$ on ∂M ; $K > 0$ and (2) imply

(20) $$a_1 = a_3$$

on M , and (11) reduce to

(21) $$da_1 - b_1 \omega_3^4 = 0 , \quad db_1 + a_1 \omega_3^4 = 0 .$$

We see that $J_1 = J_2 = K = a_1^2 + b_1^2 = \text{const.}$, and we are able to choose the moving frames in such a way that $w_1 = w_2 = a_1 v_3$, i.e., $b_1 = 0$. Then $\omega_3^4 = 0$ from (21), and

(22) $$dm = \omega^1 v_1 + \omega^2 v_2 , \quad dv_1 = \omega_1^2 v_2 + a_1 \omega^1 v_3 ,$$
$$dv_2 = - \omega_1^2 v_1 + a_1 \omega^2 v_3 ,$$
$$dv_3 = - a_1 \omega^1 v_1 - a_1 \omega^2 v_2 , \quad dv_4 = 0 .$$

Because of

(23) $$d(m + \frac{1}{a_1} v_3) = 0 ,$$

M is a part of the sphere $(m + \frac{1}{a_1} v_3 , \frac{1}{|a_1|})$ situated in the hyperplane spanned by v_1, v_2, v_3 .

For the details on the local differential geometry of surfaces in E^4 see my paper On the existence of parallel normal vector fields of surfaces in E^4 (to be published in Czech.Math.Journal).

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