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## COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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## ON A CLASS OF TANGENTIAL CAUCHY-RIEMANN MAPS

Alois ŠVEC, Olomouc

Abstract: We present conditions for a tangential Cauchy-Riemann map  $f: S^3 \rightarrow H^1$ ,  $S^3 \subset H^2$  a unit hyper-sphere and  $H^N$  a Hermitian space, to be constant.

Key words: Hermitian space, tangential Cauchy-Riemann map, integral formula, harmonic map.

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Be given a Hermitian plane  $H^2$ , i.e., the Euclidean space  $E^4$  together with an endomorphism  $J: V(E^4) \rightarrow V(E^4)$ ,  $J^2 = -\text{id.}$ ,  $V(E^4)$  being the vector space of  $E^4$ , such that  $\langle Ju, Jv \rangle = \langle u, v \rangle$  for each  $u, v \in V(E^4)$ . By a frame of  $H^2$  we mean each frame  $\varphi = \{m; v_1, v_2, v_3, v_4\}$  of  $E^4$  such that  $v_2 = Jv_1$ ,  $v_4 = Jv_3$ . Let  $\varphi$  be a field of frames of  $H^2$ ; then

$$(1) \quad dm = \omega^1 v_1 + \omega^2 v_2 + \omega^3 v_3 + \omega^4 v_4,$$

$$dv_1 = \omega^2 v_2 + \omega^3 v_3 + \omega^4 v_4,$$

$$dv_2 = -\omega^1 v_1 + \omega^3 v_3 + \omega^4 v_4,$$

$$dv_3 = -\omega^1 v_1 - \omega^2 v_2 + \omega^4 v_4,$$

$$dv_4 = -\omega^1 v_1 - \omega^2 v_2 - \omega^3 v_3$$

together with the integrability conditions ( $i, j, k = 1, \dots, 4$ )

$$(2) \quad d\omega^i = \omega^j \wedge \omega^i_j, \quad d\omega^j_i = \omega^k_i \wedge \omega^j_k.$$

From  $dv_2 = Jdv_1$ ,  $dv_4 = Jdv_3$ , we get

$$(3) \quad \omega_1^3 = \omega_2^4, \quad \omega_2^3 = -\omega_1^4.$$

Let  $S^3 \subset H^2$  be a unit hypersphere. Let us associate to each of its points  $m$  a frame  $\varphi$  in such a way that  $m + v_4$  is its center; evidently, we get

$$(4) \quad \omega^4 = 0, \quad \omega_1^4 = \omega^1, \quad \omega_2^4 = \omega^2, \quad \omega_3^4 = \omega^3.$$

Further, let  $H^1$  be a Hermitian straight line, i.e., the Euclidean plane  $E^2$  endowed by an endomorphism  $J: V(E^2) \rightarrow V(E^2)$ ,  $J^2 = -id.$ ,  $\langle Jx, Jy \rangle = \langle x, y \rangle$  for  $x, y \in V(E^2)$ .

For its frames  $\sigma = \{n; w_1, w_2\}$ ,  $w_2 = Jw_1$ , we may write

$$(5) \quad dn = \varphi^1 w_1 + \varphi^2 w_2, \quad dw_1 = \varphi_1^2 w_2, \quad dw_2 = \\ = -\varphi_1^2 w_1;$$

$$(6) \quad d\varphi^1 = -\varphi^2 \wedge \varphi_1^2, \quad d\varphi^2 = \varphi^1 \wedge \varphi_1^2, \quad d\varphi_1^2 = 0.$$

Be given a mapping  $f: S^3 \rightarrow H^1$ . Let us write  $\tau^1 := f^* \varphi^1$ ,  $\tau^2 := f^* \varphi^2$ ,  $\tau_1^2 := f^* \varphi_1^2$ . Then

$$(7) \quad \tau^1 = a_1^1 \omega^1 + a_2^1 \omega^2 + a_3^1 \omega^3, \quad \tau^2 = a_1^2 \omega^1 + a_2^2 \omega^2 + \\ + a_3^2 \omega^3$$

and the differential  $f_* : T_m(S^3) \rightarrow T_{f(m)}(H^1)$ ,  $m \in S^3$ , is given by

$$(8) \quad f_* v_1 = a_1^1 w_1 + a_1^2 w_2.$$

Let  $\tau_m \subset T_m(S^3)$  be the plane given by  $J\tau_m = \tau_m$ ;  $\tau_m$  is spanned by  $v_1$  and  $v_2$ . The mapping  $f$  is said to satisfy the tangential Cauchy-Riemann condition, see [2], if  $J \circ f_* = f_* \circ J$ ; let us call it TCR. It is easy to see that  $f$  is TCR

if and only if

$$(9) \quad a_1^1 - a_2^2 = 0, \quad a_1^2 + a_2^1 = 0,$$

i.e., (7) are of the form

$$(10) \quad \alpha^1 = a\omega^1 - b\omega^2 + c\omega^3, \quad \alpha^2 = b\omega^1 + a\omega^2 + e\omega^3.$$

By the exterior differentiation,

$$(11) \quad \begin{aligned} & \{da + b(\omega_1^2 - \alpha_1^2)\} \wedge \omega^1 - \{db - a(\omega_1^2 - \alpha_1^2)\} \wedge \omega^2 + \\ & + (dc - e\alpha_1^2) \wedge \omega^3 = -2c\omega^1 \wedge \omega^2 - b\omega^1 \wedge \omega^3 - \\ & - a\omega^2 \wedge \omega^3, \\ & \{db - a(\omega_1^2 - \alpha_1^2)\} \wedge \omega^1 + \{da + b(\omega_1^2 - \alpha_1^2)\} \wedge \omega^2 + \\ & + (de + c\alpha_1^2) \wedge \omega^3 = -2e\omega^1 \wedge \omega^2 + a\omega^1 \wedge \omega^3 - \\ & - b\omega^2 \wedge \omega^3, \end{aligned}$$

and we get the existence of functions  $A, \dots, F: S^3 \rightarrow \mathbb{R}$  such that

$$(12) \quad \begin{aligned} da + b(\omega_1^2 - \alpha_1^2) &= (A - e)\omega^1 + (B + c)\omega^2 + C\omega^3, \\ db - a(\omega_1^2 - \alpha_1^2) &= (c - B)\omega^1 + (A + e)\omega^2 + D\omega^3, \\ dc - e\alpha_1^2 &= (C - b)\omega^1 - (D + a)\omega^2 + E\omega^3, \\ de + c\alpha_1^2 &= (D + a)\omega^1 + (C - b)\omega^2 + F\omega^3. \end{aligned}$$

Further,

$$(13) \quad \begin{aligned} & \{dA + B(\alpha_1^2 - 2\omega_1^2)\} \wedge \omega^1 + \{dB - A(\alpha_1^2 - 2\omega_1^2)\} \wedge \\ & \wedge \omega^2 + \{dc + D(\omega_1^2 - \alpha_1^2)\} \wedge \omega^3 = \\ & = -4c\omega^1 \wedge \omega^2 + (B - F + c)\omega^1 \wedge \omega^3 + (E - A + e)\omega^2 \wedge \\ & \wedge \omega^3, \\ & - \{dB - A(\alpha_1^2 - 2\omega_1^2)\} \wedge \omega^1 + \{dA + B(\alpha_1^2 - 2\omega_1^2)\} \wedge \end{aligned}$$

$$\begin{aligned}
& \wedge \omega^2 + \{dD - C(\omega_1^2 - \omega_1^2)\} \wedge \omega^3 = \\
& = -4D\omega^1 \wedge \omega^2 + (A + E + e)\omega^1 \wedge \omega^3 + (B + F - c)\omega^2 \wedge \\
& \quad \wedge \omega^3, \\
& \{dC + D(\omega_1^2 - \omega_1^2)\} \wedge \omega^1 - \{dD - C(\omega_1^2 - \omega_1^2)\} \wedge \omega^2 + \\
& \quad + (dE - F\omega_1^2) \wedge \omega^3 = \\
& = -2(E + e)\omega^1 \wedge \omega^2 - (2D + a)\omega^1 \wedge \omega^3 - (2C - b)\omega^2 \wedge \\
& \quad \wedge \omega^3, \\
& \{dD - C(\omega_1^2 - \omega_1^2)\} \wedge \omega^1 + \{dC + D(\omega_1^2 - \omega_1^2)\} \wedge \omega^2 + \\
& \quad + (dF + E\omega_1^2) \wedge \omega^3 = \\
& = -2(F - c)\omega^1 \wedge \omega^2 + (2C - b)\omega^1 \wedge \omega^3 - (2D + a)\omega^2 \wedge \\
& \quad \wedge \omega^3,
\end{aligned}$$

this implying the existence of functions  $K, \dots, S: S^3 \rightarrow \mathbb{R}$   
satisfying

$$\begin{aligned}
(14) \quad & dA + B(\omega_1^2 - 2\omega_1^2) = (L - 2D)\omega^1 - (K - 2C)\omega^2 + M\omega^3, \\
& dB - A(\omega_1^2 - 2\omega_1^2) = -(K + 2C)\omega^1 - (L + 2D)\omega^2 + N\omega^3, \\
& dC + D(\omega_1^2 - \omega_1^2) = (M + B - F + c)\omega^1 + (N - A + E + \\
& \quad + e)\omega^2 + P\omega^3, \\
& dD - C(\omega_1^2 - \omega_1^2) = -(N - A - E - e)\omega^1 + (M + B + F - \\
& \quad - c)\omega^2 + Q\omega^3, \\
& dE - F\omega_1^2 = (P - 2D - a)\omega^1 - (Q + 2C - b)\omega^2 + \\
& \quad + R\omega^3, \\
& dF + E\omega_1^2 = (Q + 2C - b)\omega^1 + (P - 2D - a)\omega^2 + \\
& \quad + S\omega^3.
\end{aligned}$$

Let  $\tilde{\phi} = \{m; \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$ ,  $\tilde{\sigma} = \{n; \tilde{w}_1, \tilde{w}_2\}$  be another fields of frames, let

$$(15) \quad v_1 = \cos \alpha \cdot \tilde{v}_1 - \sin \alpha \cdot \tilde{v}_2, \quad v_2 = \sin \alpha \cdot \tilde{v}_1 + \cos \alpha \cdot \tilde{v}_2,$$

$$v_3 = \tilde{v}_3, \quad v_4 = \tilde{v}_4;$$

$$w_1 = \cos \beta \cdot \tilde{w}_1 - \sin \beta \cdot \tilde{w}_2, \quad w_2 = \sin \beta \cdot \tilde{w}_1 +$$

$$+ \cos \beta \cdot \tilde{w}_2.$$

Then the direct calculation yields

$$(16) \quad \tilde{\omega}^1 = \cos \alpha \cdot \omega^1 + \sin \alpha \cdot \omega^2, \quad \tilde{\omega}^2 = -\sin \alpha \cdot \omega^1 +$$

$$+ \cos \alpha \cdot \omega^2, \quad \tilde{\omega}^3 = \omega^3, \quad \tilde{\omega}_1^2 = \omega_1^2 + d\alpha;$$

$$\tilde{\varphi}^1 = \cos \beta \cdot \omega^1 + \sin \beta \cdot \omega^2, \quad \tilde{\varphi}^2 = -\sin \beta \cdot \varphi^1 +$$

$$+ \cos \beta \cdot \varphi^2;$$

$$(17) \quad \tilde{a} = \cos(\alpha - \beta) \cdot a - \sin(\alpha - \beta) \cdot b, \quad \tilde{b} = \sin(\alpha - \beta) \cdot a +$$

$$+ \cos(\alpha - \beta) \cdot b,$$

$$\tilde{c} = \cos \beta \cdot c + \sin \beta \cdot e, \quad \tilde{e} = -\sin \beta \cdot c + \cos \beta \cdot e;$$

$$(18) \quad \tilde{A} = \cos(2\alpha - \beta) \cdot A + \sin(2\alpha - \beta) \cdot B, \quad \tilde{B} = -\sin(2\alpha - \beta) \cdot$$

$$A + \cos(2\alpha - \beta) \cdot B,$$

$$\tilde{C} = \cos(\alpha - \beta) \cdot C - \sin(\alpha - \beta) \cdot D, \quad \tilde{D} = \sin(\alpha - \beta) \cdot C +$$

$$+ \cos(\alpha - \beta) \cdot D,$$

$$\tilde{E} = \cos \beta \cdot E + \sin \beta \cdot F, \quad \tilde{F} = -\sin \beta \cdot E + \cos \beta \cdot F.$$

Lemma. The 2-form

$$(19) \quad \Omega = 2(aC + bD)\omega^1 \wedge \omega^2 - \{a(2B - F) + b(2A - E) + cC -$$

$$- cD - ae - be\} \omega^1 \wedge \omega^3 +$$

$$+ \{a(2A - E) - b(2B + F) + cC + eD + ae - bc\} \omega^2 \wedge \omega^3$$

on  $S^3$  is invariant and we have

(20)  $d\Omega = 4(a^2 + b^2 + A^2 + B^2 + C^2 + D^2 + eE - cF) dv$ ,  
 $dv = \omega^1 \wedge \omega^2 \wedge \omega^3$  being the volume element of  $S^3$ . The function

$$(21) \quad T = eE - cF$$

is invariant as well.

Theorem 1. Be given a TGR mapping  $f: S^3 \rightarrow H^4$  satisfying  $T \geq 0$ . Then  $f$  is a constant mapping.

Proof. From the integral formula  $\int_{S^3} d\Omega = 0$ , we get  $a = b = A = B = C = D = 0$ . From (12<sub>1</sub>),  $0 = -e\omega^1 + c\omega^2$ , i.e.,  $c = e = 0$  and  $\omega^1 = \omega^2 = 0$ . QED.

The harmonicity of a TGR mapping  $f: S^3 \rightarrow H^4$  should imply its constancy. Let us now prove a generalization of this assertion. Be given an arbitrary mapping  $f: S^3 \rightarrow H^4$ ,

$$(22) \quad \omega^\alpha = a_1^\alpha \omega^1 \quad (\alpha = 1, 2).$$

From this,

$$(23) \quad (da_1^\alpha - a_j^\alpha \omega_i^j + a_i^\beta \omega_\beta^\alpha) \wedge \omega^1,$$

and we get the existence of functions  $a_{1j}^\alpha = a_{j1}^\alpha: S^3 \rightarrow \mathbb{R}$  such that

$$(24) \quad da_1^\alpha - a_j^\alpha \omega_i^j + a_i^\beta \omega_\beta^\alpha = a_{1j}^\alpha \omega^j.$$

To each point  $m \in S^3$ , let us associate the vector

$$(25) \quad t = \sigma^{ij} a_{ij}^\alpha v_\alpha \in T_{f(m)}(H^4);$$

the mapping  $t$  is called the tension field, see [1] or [3] resp. The mapping  $f$  is called harmonic if  $t = 0$  on  $S^3$ . Now, let  $f$  be our TGR map. From (24) and (12), it is easy to see that

$$(26) \quad a_{11}^1 = A - e, \quad a_{22}^1 = -A - e, \quad a_{33}^1 = E, \quad a_{12}^1 = B, \quad a_{13}^1 = C,$$

$$a_{23}^1 = -D,$$

$$a_{11}^2 = -B + C, \quad a_{22}^2 = B + C, \quad a_{33}^2 = F, \quad a_{12}^2 = A, \quad a_{13}^2 = D,$$

$$a_{23}^2 = C,$$

i.e.,

$$(27) \quad t = (E - 2C)w_1 + (F + 2C)w_2.$$

For each  $m \in S^3$ , consider the vectors

$$(28) \quad t_1 = f_* v_3 = cw_1 + ew_2, \quad t_2 = f'_*(f_* v_3) = -ew_1 + cw_2,$$
$$t_3 = v_3(f_* v_3) = Dw_1 + Fw_2.$$

Then

$$(29) \quad t = t_3 + 2t_2$$

and

$$(30) \quad \langle t_2, t_3 \rangle = -T,$$

this presenting the geometrical description of  $T$ .

Theorem 2. Let  $f: S^3 \rightarrow H^4$  be a TCR map and let there exist functions  $p, q: S^3 \rightarrow \mathbb{R}$ ,  $q \geq 0$  on  $S^3$ , such that

$$(31) \quad t_3 + pt_1 + qt_2 = 0.$$

Then  $f$  is a constant mapping.

Proof. All we have to do is to prove  $T \leq 0$ . From (31),  $E + pc - qc = F + pe + qc = 0$ ,  $T = q(e^2 + c^2)$ , and we are done.

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