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NONLINEAR PERTURBATIONS OF LINEAR OPERATORS HAVING NULL-SPACE WITH STRONG UNIQUE CONTINUATION PROPERTY

(Preliminary Communication)

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Abstract: We are concerned with the existence of one or multiple solutions of various problems for nonlinear differential equations which can be reduced to an abstract operator equation of the form $Lu + G(u) = f$ in a real Hilbert space H, with $L: H \supset G(L) \longrightarrow H$ being linear and noninvertible, $G: H \longrightarrow H$ nonlinear and $f \in H$ given.

Key words: Nonlinear operator equations, strong unique continuation property.

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Let Q denote a bounded domain in R^N ($N \geq 1$), and let $H = L^2(Q)$, with norm $\|\cdot\|$ and inner product $(.,.)$. Let $L: H \supset D(L) \longrightarrow H$ be a closed linear operator with dense domain $D(L)$ and closed range $R(L)$. We assume that 0 is an eigenvalue of L and of its adjoint operator L^*, and that for the corresponding eigenspaces,

$$N(L) = N(L^*)$$

and $\dim N(L) < +\infty$. Hence H admits the orthogonal decomposition

$$H = N(L) \oplus R(L).$$

We set $H_1 := N(L)$, $H_2 := R(L)$, and denote by P_i the orthogonal projection of H onto H_i $(i = 1,2)$. An element $f \in H$ may thus be decomposed into $f = f_1 + f_2$, where $f_i = P_i f$. The restriction $\tilde{L} := L|_{H_2}$, with $D(\tilde{L}) = D(L) \cap H_2$, is an algebraic isomorphism in H_2. Its inverse (the so-called right inverse of L) will be denoted by T. We assume that $T: H_2 \longrightarrow H_2$ is compact.

Our main assumption on the functions in N(L) is the following "strong unique continuation property":

(SUCP): $N(L) \subset L^{\infty}(Q)$, and there exists $\rho > 0$ such that for the function φ :

$$\varepsilon \longmapsto \varphi(\varepsilon) = \sup_{\substack{w \in N(L) \\ \|w\|_{\infty}=1}} \operatorname{meas} \{x \in Q : |w(x)| < \varepsilon\},$$

$\varphi(\varepsilon) = O(\varepsilon^{\rho})$ as $\varepsilon \longrightarrow 0+$.

Remark. The usually imposed "unique continuation property" demands that the only function $w \in N(L)$ vanishing on a set of positive measure in Q is $w = 0$. As $\dim N(L) < +\infty$ this implies that $\varphi(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0+$. Thus the (SUCP) prescribes the speed of convergence.

Let $g: R \longrightarrow R$ be a continuous function with finite limits

$$g_{\pm} := \lim_{s \to \pm\infty} g(s).$$

Without restriction we may assume $g_- \leqq 0 \leqq g_+$. Suppose there exists $\delta > 0$ such that

$$g(s) \geqq g_+ \qquad \forall\ s \geqq \delta$$

$$g(s) \leqq g_- \qquad \forall\ s \leqq -\delta.$$

For $a \geqq \delta$ we set (with the same ρ as in (SUCP))

$$\gamma(a)_+ := \liminf_{\xi \to +\infty} \xi^{1+\rho} \min_{s \in [a,\xi]} (g(s) - g_+),$$

$$\gamma(a)_- := \liminf_{\xi \to +\infty} \xi^{1+\rho} \min_{s \in [-\xi,-a]} (g_- - g(s)).$$

Let $G: H \longrightarrow H$ be the Nemytskii operator associated with g:

$$G(u)(x) := g(u(x)), \quad x \in Q,$$

for any function u defined on Q. The mapping G is continuous and has bounded range in H.

Let

$$S := \{ f_1 \in H_1 : (f_1, w) \leq \int_Q (g_+ w^+ - g_- w^-) dx, \quad \forall\, w \in H_1 \} .$$

Here $w^+ (w^-)$ is the positive (negative) part of the function w, i.e. $w = w^+ - w^-$. Note that $S \subset H_1$ is nonempty, bounded, closed and convex.

Theorem 1. Suppose

(A) $D(L) \subset L^\infty(Q)$, and $T: H_2 \longrightarrow L^\infty(Q)$ is continuous.

Suppose further that either

(α) the functions in N(L) have constant sign in Q and $\gamma(a)_+ = \gamma(a)_- = +\infty$ (for a suitable $a \geq \sigma$), or

(β) the functions in N(L) change sign in Q and at least one of $\gamma(a)_+$, $\gamma(a)_-$ is $= +\infty$ (for a suitable $a \geq \sigma$).

Then to each $f_2 \in H_2$ there exists an open set $S_{f_2} \subset H_1$, $S_{f_2} \supset S$, such that

(i) the equation

(1) $$Lu + G(u) = f$$

has at least one solution for $f = f_1 + f_2$ with $f_1 \in S_{f_2}$;

(ii) the equation (1) has at least two solutions for $f = f_1 + f_2$ with $f_1 \in S_{f_2} \setminus S$.

Theorem 2. Under the assumptions of Theorem 1, the range of $L + G$ is closed in H.

Hence the assertion (i) of Theorem 1 is in fact valid for $f_1 \in \overline{S_{f_2}}$. A variant of Theorem 1 is

Theorem 3. Instead of (A) let the following regularity assumption be satisfied:

(A′) There exists $m > 0$ such that for any solution $u \in H$ of $Lu = f$ with $f \in L^{\infty}(Q)$ we have $u \in L^{\infty}(Q)$ and

$$\| P_2 u \|_{\infty} \leq m \| f \|_{\infty} .$$

Suppose either (α) or (β). Then the assertions of Theorem 1 hold, provided $f_2 \in L^{\infty}(Q)$.

It is possible to apply the above abstract theorems to a large variety of examples, such as the boundary value problem for ordinary and elliptic differential equations, and the problem of existence of periodic solutions of the nonlinear heat equation and the nonlinear telegraph equation. The proofs and the investigation of these applications will appear elsewhere.

The main part of the results was obtained while the second author was visiting the Charles University. The detailed paper has been submitted to "Nonlinear Analysis. Theory and applications".

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