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REMARK ON SURFACES IN E^4 SATISFYING CERTAIN RELATIONS BETWEEN COVARIANT DERIVATIVES OF THE MEAN AND GAUSS CURVATURES

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Abstract: We show under which conditions surfaces with constant mean or Gauss curvature are, globally, a part of a 2-dimensional sphere in E .

Key words: Surface, mean and Gauss curvatures, sphere.

AMS: 53C45

This contribution gives several results concerning the global characterization of the 2-dimensional sphere among surfaces in E^4, under the supposition that at least one of the curvatures H and K is constant.

Let M be a surface in E^4 and ∂M its boundary. Let $\{U_\alpha\}$ be an open covering of M such that in each domain U_α, there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ with $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$ where T(M), N(M) denote the tangent and normal bundle of M, respectively. Then

$$(1)\qquad \begin{aligned} dM &= \omega^1 v_1 + \omega^2 v_2, \\ dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4, \\ dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4, \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4, \\ dv_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 \quad ; \end{aligned}$$

(2) $d\omega^i = \omega^k \wedge \omega_k^i,\ d\omega_i^j = \omega_i^k \wedge \omega_k^j,\ \omega_i^j + \omega_j^i = 0$

$$(i,j,k = 1,2,3,4),$$

$$\omega^3 = \omega^4 = 0.$$

Differentiating the last equation of (2) and applying the Cartan's lemma, we get the existence of real-valued functions $a_i, b_i, c_i (i = 1,2)$ on each U_α such that

(3) $$\omega_1^3 = a_1\omega^1 + b_1\omega^2,\quad \omega_2^3 = b_1\omega^1 + c_1\omega^2,$$

$$\omega_1^4 = a_2\omega^1 + b_2\omega^2,\quad \omega_2^4 = b_2\omega^1 + c_2\omega^2.$$

As always, denote

(4) $$H = (a_1 + c_1)^2 + (a_2 + c_2)^2,$$

$$K = a_1c_1 - b_1^2 + a_2c_2 - b_2^2$$

the mean and Gauss curvature of M, respectively.

Let F be a real function on M. According to [1], p. 16, we define its covariant derivatives $F_i, F_{ij} = F_{ji}$ $(i,j = 1,2)$ with respect to the given field of orthonormal frames over U_α by

(5) $$dF = F_1\omega^1 + F_2\omega^2,$$

$$dF_1 - F_2\omega_1^2 = F_{11}\omega^1 + F_{12}\omega^2,\ dF_2 + F_1\omega_1^2 = F_{21}\omega^1 + F_{22}\omega^2.$$

The proof of the following theorem is based on the maximum principle of this form:

Let M be a surface in E^4 and ∂M its boundary. Let F be a real-valued function on M and F_i, F_{ij} (o,j = 1,2) its covariant derivatives. Let $F \geq 0$ on M, $F = 0$ on ∂M, F satisfy in U_α the equation

$$a_{11}F_{11} + 2a_{12}F_{12} + a_{22}F_{22} + a_1F_1 + a_2F_2 + a_oF = a$$

where $a_o \leq 0$, $a \geq 0$ and the quadratic form $a_{ij}x^ix^j$ is positive definite. Then $F = 0$ on M.

Now, we are going to prove this

Theorem 1. Let M be a surface in E^4 and ∂M its boundary. Let H_{ij}, K_{ij} $(i, j, = 1,2)$ be covariant derivatives of H, K, respectively. Let

(i) ∂M consist of umbilical points;

(ii) $H_{11} + H_{22} - 4(K_{11} + K_{22}) \geq 0$ on M.

Then M is a part of a 2-dimensional sphere in E^4.

Proof. Consider the function

$$(6) \quad f = H - 4K = (a_1 - c_1)^2 + (a_2 - c_2)^2 + 4b_1^2 + 4b_2^2$$

which is non-negative on M and equals to zero at the umbilical points $(a_1 = c_1,\ a_2 = c_2,\ b_1 = 0,\ b_2 = 0)$ of M. From (5) and (6) we have immediately

$$(7) \qquad f_{ij} = H_{ij} - 4K_{ij} \ (i, j = 1,2)$$

and hence

$$f_{11} + f_{22} = H_{11} + H_{22} - 4(K_{11} + K_{22}).$$

Using (ii) and applying the maximum principle we obtain $f = 0$ on M.

The following results are direct consequences of the Theorem 1:

Corollary. Let M be a surface in E^4, ∂M its boundary. Let

(i) ∂M consist of umbilical points;

and let be satisfied one of these conditions:

(ii) $H = \text{const}$, *and* $K_{11} + K_{22} \leq 0$ *on* M;

(iii) $K = \text{const}$, *and* $H_{11} + H_{22} \geq 0$ *on* M;

(iv) $H = \text{const}$, $K = \text{const}$ *on* M.

Then M is a part of a 2-dimensional sphere in E^4.

Next, we are going to prove a generalization of the preceding Theorem 1.

Theorem 2. Let M be a surface in E^4 *and* ∂M *its boundary. Let S be a positive definite symmetric quadratic tensor field on M with components* S_{ij} $(i,j = 1,2)$. *Let*

(i) ∂M *consist of umbilical points;*

(ii) $S_{11}H_{11} + 2S_{12}H_{12} + S_{22}H_{22} - 4(S_{11}K_{11} + 2S_{12}K_{12} + S_{22}K_{22}) \geq 0$ *on* M.

Then M is a part of a 2-dimensional sphere in E^4.

Proof. Because of (7), we have

$$S_{11}f_{11} + 2S_{12}f_{12} + S_{22}f_{22} = S_{11}H_{11} + 2S_{12}H_{12} + S_{22}H_{22} - 4(S_{11}K_{11} + 2S_{12}K_{12} + S_{22}K_{22})$$

and the assertion follows immediately by means of the maximum principle.

Remark. In what follows, we shall show that the result of the Theorem 2 contains the most general condition expressed by means of the covariant derivatives of the functions H, K, which enables to prove, using the maximum principle, that the given surface is a part of a sphere in E^4.

According to [3], we have

$$(8) \quad f_{11} = 2(a_1 - c_1)(A_1 - C_1) + 2(a_2 - c_2)(A_2 - C_2) + 8(b_1B_1 + b_2B_2) + 2(\alpha_1 - \gamma_1)^2 + 2(\alpha_2 - \gamma_2)^2 +$$

$$+ 8(\beta_1^2 + \beta_2^2) - [k + 4(a_1 b_2 - b_1 a_2)] k -$$

$$- 2 [(a_1 - c_1)c_1 + (a_2 - c_2)c_2 - 4(b_1^2 + b_2^2)] K,$$

$$f_{12} = 2(a_1 - c_1)(B_1 - D_1) + 2(a_2 - c_2)(B_2 - D_2) +$$

$$+ 8(b_1 C_1 + b_2 C_2) + 2(\alpha_1 - \gamma_1)(\beta_1 - \delta_1) + 2(\alpha_2 - \gamma_2)\cdot$$

$$\cdot(\beta_2 - \delta_2) + 8(\beta_1 \gamma_1 + \beta_2 \gamma_2) + 4 [(a_1 + c_1)b_1 +$$

$$+ (a_2 + c_2)b_2] K,$$

$$f_{22} = 2(a_1 - c_1)(C_1 - E_1) + 2(a_2 - c_2)(C_2 - E_2) +$$

$$+ 8(b_1 D_1 + b_2 D_2) + 2(\beta_1 - \delta_1)^2 + 2(\beta_2 - \delta_2)^2 +$$

$$+ 8(\gamma_1^2 + \gamma_2^2) - [k + 4(b_1 c_2 - c_1 b_2)] k +$$

$$+ 2 [(a_1 - c_1)a_1 + (a_2 - c_2)a_2 + 4(b_1^2 + b_2^2)] K;$$

$$H_{11} = 2(a_1 + c_1)(A_1 + C_1) + 2(a_2 + c_2)(A_2 + C_2) +$$

$$+ 2(\alpha_1 + \gamma_1)^2 + 2(\alpha_2 + \gamma_2)^2 + [(a_1 + c_1)b_2 -$$

$$- (a_2 + c_2)b_1] k + 2 [(a_1 + c_1)c_1 + (a_2 + c_2) c_2] K$$

$$H_{12} = 2(a_1 + c_1)(B_1 + D_1) + 2(a_2 + c_2)(B_2 + D_2) +$$

$$+ 2(\alpha_1 + \gamma_1)(\beta_1 + \delta_1) + 2(\alpha_2 + \gamma_2)(\beta_2 + \delta_2),$$

$$H_{22} = 2(a_1 + c_1)(C_1 + E_1) + 2(a_2 + c_2)(C_2 + E_2) +$$

$$+ 2(\beta_1 + \delta_1)^2 + 2(\beta_2 + \delta_2)^2 - [(a_1 + c_1)b_2 -$$

$$- (a_2 + c_2)b_1] k + 2 [(a_1 + c_1)a_1 + (a_2 + c_2)a_2] K;$$

$$K_{11} = (c_1 A_1 - 2b_1 B_1 + a_1 C_1) + (c_2 A_2 - 2b_2 B_2 + a_2 C_2) +$$

$$+ 2(\alpha_1 \gamma_1 - \beta_1^2) + 2(\alpha_2 \gamma_2 - \beta_2^2) + \frac{3}{2}(a_1 b_2 - b_1 a_2)k +$$

$$+ [(a_1 c_1 - 2b_1^2) + (a_2 c_2 - 2b_2^2)] K,$$

$$K_{12} = (c_1 B_1 - 2b_1 C_1 + a_1 D_1) + (c_2 B_2 - 2b_2 C_2 + a_2 D_2) +$$

$$+ (\alpha_1 \delta_1 - \beta_1 \gamma_1) + (\alpha_2 \delta_2 - \beta_2 \gamma_2) -$$

$$- [(a_1 + c_1)b_1 + (a_2 + c_2)b_2] k,$$

$$K_{22} = (c_1C_1 - 2b_1D_1 + a_1E_1) + (c_2C_2 - 2b_2D_2 + a_2E_2) +$$
$$+ 2(\beta_1 \sigma_1 - \gamma_1^2) + 2(\beta_2 \sigma_2 - \gamma_2^2) + \tfrac{3}{2}(b_1c_2 - c_1b_2)k +$$
$$+ [(a_1c_1 - 2b_1^2) + (a_2c_2 - 2b_2^2)] k;$$

where

$$k = (a_1 - c_1)b_2 - (a_2 - c_2)b_1,$$

the functions $\alpha_i, \ldots, \sigma_i$, $A_i, \ldots, E_i$ $(i = 1,2)$ being determined by the prolongation procedure of the system (3). For being possible to use the maximum principle,we must be able to determine the functions S_{ij}, x_{ij}, y_{ij} in such a way that the equation

$$(9) \quad S_{11}f_{11} + 2S_{12}f_{12} + S_{22}f_{22} = x_{11}H_{11} + 2x_{12}H_{12} + x_{22}H_{22}$$
$$+ 2(y_{11}K_{11} + 2y_{12}K_{12} + y_{22}K_{22}) + \Phi$$

would not contain $A_i, \ldots, E_i$. Inserting (8) into (9), we obtain the system of equations

$$(a_1 + c_1)x_{11} + c_1y_{11} = (a_1 - c_1)S_{11},$$
$$(a_1 + c_1)x_{12} - b_1y_{11} + c_1y_{12} = 2b_1S_{11} + (a_1 - c_1) S_{12},$$
$$(a_1 + c_1)x_{11} + (a_1 + c_1)x_{22} + a_1y_{11} - 4b_1y_{12} + c_1y_{22} =$$
$$= - (a_1 - c_1)S_{11} + 8b_1S_{12} + (a_1 - c_1)S_{22},$$
$$(a_1 + c_1)x_{12} + a_1y_{12} - b_1y_{22} = - (a_1 - c_1)S_{12} + 2b_1S_{22},$$
$$(a_1 + c_1)x_{22} + a_1y_{22} = - (a_1 - c_1)S_{22};$$

$$(a_2 + c_2)x_{11} + c_2y_{11} = (a_2 - c_2)S_{11},$$
$$(a_2 + c_2)x_{12} - b_2y_{11} + c_2y_{12} = 2b_2S_{11} + (a_2 - c_2)S_{12},$$
$$(a_2 + c_2)x_{11} + (a_2 + c_2)x_{22} + a_2y_{11} - 4b_2y_{12} + c_2y_{22}$$
$$= - (a_2 - c_2)S_{11} + 8b_2S_{12} + (a_2 - c_2) S_{22},$$

$$(a_2 + c_2)x_{12} + a_2y_{12} - b_2y_{22} = -(a_2 - c_2)S_{12} + 2b_2S_{22},$$

$$(a_2 + c_2)x_{22} + a_2y_{22} = -(a_2 - c_2)S_{22}.$$

Hence

$$(y_{11} + 2S_{11}):(y_{12} + 2S_{12}):(y_{22} + 2S_{22}) = 1:0:1$$

so that

$$y_{11} = -2S_{11} + \lambda, \; y_{12} = -2S_{12}, \; y_{22} = -2S_{22} + \lambda$$

and

$$(a_1 + c_1)x_{11} = (a_1 + c_1)S_{11} - c_1\lambda, \quad (a_2 + c_2)x_{11} = (a_2 + c_2)S_{11} - c_2\lambda,$$

$$(a_1 + c_1)x_{12} = (a_1 + c_1)S_{12} + b_1\lambda, \quad (a_2 + c_2)x_{12} = (a_2 + c_2)S_{12} + b_2\lambda,$$

$$(a_1 + c_1)x_{22} = (a_1 + c_1)S_{22} - a_1\lambda, \quad (a_2 + c_2)x_{22} = (a_2 + c_2)S_{22} - a_2\lambda,$$

the function λ satisfying the conditions

$$(a_1c_2 - c_1a_2)\lambda = 0,$$

$$[(a_1b_2 - b_1a_2) - (b_1c_2 - c_1b_2)]\lambda = 0.$$

From the last two equations it follows that these two cases are possible:

1. $\lambda \neq 0$. Then $a_1c_2 - c_1a_2 = 0$, $(a_1b_2 - b_1a_2) - (b_1c_2 - c_1b_2) = 0$

which means that $M \subset E^3$, see [2]. This case is not considered in this contribution.

2. $\lambda = 0$. In this case

$$x_{11} = S_{11}, \quad x_{12} = S_{12}, \quad x_{22} = S_{22},$$

$$y_{11} = -2S_{11}, \quad y_{12} = -2S_{12}, \quad y_{22} = -2S_{22}$$

and, according to (8), $\Phi = 0$. This yields our assertion.

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