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IDEAL EQUIVALENCES FOR ALMOST REALCOMPACT SPACES

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Abstract: If X is realcompact, Kaplansky's equality holds in C(X), that is, the intersection of the free maximal ideals coincides with the intersection of the free prime ideals. A systematic study is made of the validity of this and of other equivalences of ideals in C(X) when X is almost realcompact, a-real compact, and c-realcompact. Counterexamples are given where appropriate.

Key words and phrases: Realcompactness, almost realcompactness, ideals of continuous functions.

Classification: 54C40, 54D60

Let C(X) be the ring of real continuous functions on a completely regular Hausdorff space X, νX the Hewitt realcompactification of X and βX the Stone-Čech compactification of X. When X is realcompact, the ring C(X) has the following interesting property, discovered for compact spaces by Kaplansky and proved necessary for realcompact spaces by Gillman-Jerison [4]: the intersection of the free maximal ideals of C(X) is equal to the intersection of the free prime ideals of C(X). For $A \subset \beta X$ writing $M^A = \{f \in C(X): A \subset \subset cl_{\beta X} Z_X(f)\}$ and $O^A = \{f \in C(X): A \subset int_{\beta X} cl_{\beta X} Z_X(f)\}$, we see that M^A and O^A are ideals and Kaplansky's equality can be expressed as $M^{\beta X - X} = O^{\beta X - X}$. This property is not sufficient for realcompactness. Calling it μ-compactness, Mandelker [9] studied the property and introduced a strictly stronger property, ψ-compactness ($M^{\beta X - \nu X} = O^{\beta X - X}$), which is also necessary but not sufficient for realcompactness. In similar studies, [6],[12], η-compactness ($M^{\beta X - X} = M^{\beta X - \nu X}$) and λ-compactness ($M^{\nu X - X} = O^{\nu X - X}$) were defined. On the other hand, three generalizations of realcompactness, known as almost realcompactness, a-realcompactness and c-realcompactness, arose in different contexts [2],[3]. The purpose of this paper is to make a systematic study of the interrelationships between ideal equivalences given by μ-, ψ-, η- and λ-compactness properties and the generalizations of realcompactness named above. We show that, except for a single case, there is no logical connection between the first set of properties and the second set of generalizations. In particular, we answer, in the negative, a question of Riordan [12] on the equivalence of λ- and μ-compactness, and give examples of (i) an a-realcompact space which is not c-realcompact and (ii) a c-realcompact space which satisfies each greek-letter-compactness property but is not a-realcompact.

A summary of the interrelationships is exhibited in Table 1. Notation and terminology as in [4].

1.0. Throughout the paper X will denote a completely regular Hausdorff space. X is said to be realcompact provided it satisfies any one of the equivalent statements below:

1.0.1. Every free maximal ideal of C(X) is hyperreal.

Table 1

	real-compact	almost real-compact	a-real compact	c-real compact	ψ	μ	η	λ
realcompactness	+	+	+	+	+	+	+	+
almost realcompactness	M	+	+	+	+	+	+	M
a-realcompactness	M	Ψ	+	Ψ	Ψ	Ψ	Ψ	Ψ,M
c-realcompactness	M,H,S	H,S	H,S	+	S	S	?	M
ψ-compactness	M,H,P	H,P	H,P	P	+	+	+	M
μ-compactness	M,H,P,W	H,P,W	H,P,W	P,W	W	+	W	M
η-compactness	M,H,P,S	H,P	H,P	P	S	S	+	M
λ-compactness	H,P,W	H,P,W	H,P,W	P,W	W	+	W	+

Each space listed at the intersection of row A and column B is an example satisfying property A but not B. The symbol "+" appears if A $\Longrightarrow$ B holds.

LEGEND OF SPACES

Symbol	Space	Reference
M	Mrowka Space	3.1
Ψ	Isbell Space	3.2
S	Tychonoff Corkscrew	3.3
P	Non-realcompact P-space	3.4
W	Countable Ordinals	3.5
H	Mack-Johnson Space	4.0

1.0.2. Every z-ultrafilter on X with empty intersection contains a countable subfamily with empty intersection.

1.0.3. Every maximal cozero cover of X has a countable subcover. (A covering of X is said to be maximal if it does not have a finite subcover, and is maximal with respect to this property.)

1.0.4. $X = \upsilon X$.

The proofs of these equivalences may be found in [2] and [4].

1.1. The following definitions are generalizations of the specific formulations of realcompactness noted above.

1.1.1. Almost realcompact, if every ultrafilter of regularly closed subsets of X with empty intersection contains a countable subfamily with empty intersection;

1.1.2. a-realcompact, if every ultrafilter of closed subsets of X with empty intersection contains a countable subfamily with empty intersection;

1.1.3. c-realcompact, if, for every $p \in \beta X - X$ there exists a decreasing sequence $\{A_n | n \in N\}$ of regularly closed subsets of βX, with $p \in \cap\{A_n | n \in N\}$, while $\cap\{A_n \cap X | n \in N\} = \emptyset$.

As noted in the introduction, the space X is

1.1.4. μ-compact, if $O^{\beta X - X} = M^{\beta X - X}$

1.1.5. ψ-compact, if $O^{\beta X - X} = M^{\beta X - \upsilon X}$

1.1.6. η-compact, if $M^{\beta X - X} = M^{\beta X - X}$

1.1.7. λ-compact, if $O^{\upsilon X - X} = M^{\upsilon X - X}$.

1.2. Almost realcompactness is a closed, hereditary and productive property, and, in completely regular spaces, is preserved by perfect maps. a-realcompactness is closed, hereditary and is preserved by perfect maps [2]. The following implications are known:

1.2.1. X is realcompact $\Rightarrow$ X is almost realcompact [2]

1.2.2. X is almost realcompact $\Rightarrow$ X is a-realcompact [2]

1.2.3. X is almost realcompact $\Rightarrow$ X is c-realcompact [5]

The following characterizations of the ideals noted below are useful in the study of the properties 1.1.4. - 1.1.7.:

1.2.4. $O^{\beta X-X} = \{f \in C(X) \mid S_X(f)$ is compact$\}$ [4,7E]

1.2.5. $O^{\beta X-\upsilon X} = M^{\beta X-\upsilon X} = \{f \in C(X) \mid S_X(f)$ is pseudocompact$\}$ [6]

1.2.6. $M^{\upsilon X-X} = \{f \in C(X) \mid coz(f)$ is realcompact$\}$.

To prove 1.2.6., let $f \in M^{\upsilon X-X}$, and assume for a contradiction that $coz_X(f)$ is not realcompact. If f^{υ} denotes the continuous extension of f to υX, then $coz_{\upsilon X} f^{\upsilon} = \upsilon(coz_X f)$, [1]. Since $coz_X f$ is not realcompact, it follows that there is $p \in (coz_{\upsilon X} f^{\upsilon}) - coz_X f$. Thus $p \in \upsilon X - X \subseteq cl_{\beta X} Z_X(f) =$
$= cl_{\beta X} Z_{\upsilon X}(f^{\upsilon})$. (The first inclusion follows from the assumption, the second by [4,8.8].) This is impossible since $coz_{\upsilon X} f^{\upsilon} \cap Z_{\upsilon X}(f^{\upsilon}) = \emptyset$. Conversely, if $coz_X f$ is realcompact, then $coz_X f \cup cl_{\beta X} Z_X(f)$, as the union of a compact and a realcompact set, is realcompact, and contains X, hence must also contain υX. It follows that $\upsilon X - X \subseteq cl_{\beta X} Z_X(f)$. This completes the proof of 1.2.6.

1.2.7. $O^{\upsilon X-X} = \{f \in C(X) \mid S_X(f) = cl_{\upsilon X} S_X(f)\}$.

To prove 1.2.7., let $f \in O^{\upsilon X-X}$, and assume for a contradiction that $p \in cl_{\upsilon X}S_X(f)-S_X(f)$. Then $p \in \upsilon X-X$, and hence $p \in int_{\beta X}cl_{\beta X}Z(f)$. It follows that $int_{\beta X}cl_{\beta X}Z(f) \cap coz\ f \neq \emptyset$. This is impossible by [4, 6.5(IV)], since any $x \in coz\ f$ must satisfy $f(x) > r$ for some real number r, and hence is contained in the zero set $\{y \mid f(y) \geq \frac{r}{2}\}$, which is disjoint from $Z(f)$. Hence $O^{\upsilon X-X} \subseteq \{f \in C(X) \mid S_X(f) = cl_{\upsilon X}S_X(f)\}$. Conversely, if some $f \in C(X)$ satisfies $S_X(f) = cl_{\upsilon X}S_X(f)$, then $cl_{\beta X}S_X(f)-S_X(f) \subseteq \beta X-\upsilon X$, and $cl_{\beta X}S_X(f) \subseteq X \cup (\beta X-\upsilon X)$. Thus $\beta X-cl_{\beta X}S_X(f)$ is open in βX, and contained in $cl_{\beta X}Z(f)$. Then $\upsilon X-X \subseteq \beta X-cl_{\beta X}S_X(f) \subseteq int_{\beta X}cl_{\beta X}Z(f)$ and so $f \in O^{\upsilon X-X}$.

1.2.8. X is realcompact $\Rightarrow$ X is ψ-compact $\Rightarrow$ X is η-compact

1.2.9. X is realcompact $\Rightarrow$ X is λ-compact $\Rightarrow$ X is μ-compact.

1.2.10. X is ψ-compact $\Rightarrow$ X is μ-compact.

1.2.8. and 1.2.10. are due to Mandelker [9], 1.2.9. was proved by Riordan [R1].

2.0. In this section we shall present the only new positive implication which holds between these properties. We need the following definition:

2.0.1. X is said to be a weak-cb-space, if, for every positive, normal lowersemicontinuous function g defined on X, there exists a function $f \in C(X)$ such that $0 < f(x) \not\leq g(x)$ for all $x \in X$.

Weak cb-spaces were defined by Mack and Johnson [8, 3.1] where the first of the following two results is given:

2.0.2. If X is pseudocompact, then X is a weak cb-space.

2.0.3. If X is almost realcompact and a weak cb-space, then X is realcompact [2, 1.2].

We are now ready to prove:

2.1.1. X is almost realcompact $\Longrightarrow$ X is ψ-compact. To prove this statement, it suffices to show that $M^{\beta X-\upsilon X} \subseteq$ $\subseteq O^{\beta X-X}$, or, in view of 1.2.6. and 1.2.7., equivalently, that any $f \in C(X)$ with pseudocompact support has compact support. Let $f \in C(X)$ with $S_X(f)$ be pseudocompact. Since $S_X(f)$ is closed, it is almost realcompact, and since it is pseudocompact, it is also a weak-cb-space. Thus by the above-noted result, $S_X(f)$ is realcompact and pseudocompact, and hence compact by [4, 5.9].

3. Counterexamples. In this section we shall consider five spaces which provide counterexamples to show that no other logical relations hold between the properties under consideration.

3.1. Our first example is the space M, constructed by Mrowka [10], which is the union of two closed, realcompact subspaces M_1 and M_2, but is not realcompact. It is well-known that M is almost realcompact: the identity mapping from the topological sum $M_1 \cup M_2$ to M is perfect, and the domain is realcompact. By the remarks of section 1.2, the perfect image of a realcompact space must be realcompact. It follows by 1.2.2. and 1.2.3. that M is also a-realcompact and c-realcompact. We shall now show that M is not λ-compact:

3.1.1. If X is λ-compact, and if $f \in M^{\upsilon X-X}$, then $S_X(f)$ is realcompact.

To prove this, note that by hypothesis we have $f \in M^{\upsilon X-X} = O^{\upsilon X-X}$. Hence $cl_{\upsilon X} S_X(f) = S_X(f)$ by 1.2.7. and so $S_X(f)$ is a closed subspace of the realcompact space υX and thus realcompact.

3.1.2. M is not λ-compact.

The subsets M_1 and M_2 of M each contains a copy of the real line L and the points in the upper half of the Euclidean plane with rational coordinates, topologized by letting neighborhoods of points p of L to be of the form $\{p\} \cup D$ where D is the interior of a disc tangent at p. Thus we may agree to denote the points of $M-M_2$ by their Euclidean coordinates (x,y), where $y > 0$, and the points of $M-M_1$ likewise, with $y < 0$. M is constructed from the topological sum of M_1 and M_2 by choosing a suitable identification of the points of $L \subseteq M_1$ with those of $L \subseteq M_2$. Thus the points of $M_1 \cap M_2$ could each be labelled (x,0) using two distinct values of x. Using this notation, it follows from the construction of M that the following function is continuous on M:

$$f(m) = \begin{cases} y \text{ if } m = (x,y) \notin M_1 \cap M_2 \\ 0 \text{ if } m \in M_1 \cap M_2 \end{cases}.$$

The cozero set of this function is the discrete (hence realcompact) space $M - (M_1 \cap M_2)$, while the support of this function is the non-realcompact space M. Hence M is not λ-compact.

We conclude our discussion of the space M by noting that since M is ψ-compact, hence also η-compact and μ-compact by (1.2.1), the question of Riordan [11] on the equivalence of λ- and μ-compactness is settled in the negative.

3.2. In this section we consider a space constructed by Isbell, and described in [4, 5I]. Let E be a maximal family of infinite subsets of the space N of natural numbers, such that the intersection of any two is finite. Let D be a discrete set indexed by the members of E. Topologize $\Psi =$ $= N \cup D$, by defining, for each $d_e \in D$, a cofinite subset of e as a neighborhood of d_e. The points of N are discrete. It can be shown, as in [4, 5I], that Ψ is pseudocompact and locally compact, but not compact.

3.2.1. Let X be a pseudocompact, completely regular Hausdorff space. If X has any of the following properties, then X is compact:

(a) almost realcompactness

(b) c-realcompactness

(c) ψ-compactness

(d) η-compactness

Since a pseudocompact, realcompact space is compact, the (a) and (b) parts follow from the results of [2]. (c) and (d) follow from [6, 5.2 and 6.2].

3.2.2. ψ is not almost realcompact, c-realcompact, ψ-compact, nor η-compact.

3.2.3. ψ is not λ-compact.

To prove this, we need the following notation:

3.2.4. $C_{\infty}(X)$ denotes the family of all functions f in

C(X) for which the set $\{x \in X \mid |f(x)| \geq 1/n\}$ is compact for every $n \in N$.

3.2.5. $M^{\beta X - X} = C_{\infty}(X) \cap M^{\beta X - \upsilon X}$ [6, 3.2].

3.2.6. If X is not countably compact, $O^{\beta X - X} \neq C_{\infty}(X)$ [4, 7G2].

The result that Ψ is not λ-compact is now obvious: since Ψ is pseudocompact, $\beta\Psi - \upsilon\Psi = \emptyset$, hence $M^{\upsilon\Psi - \Psi} = M^{\beta\Psi - \Psi} = C_{\infty}(\Psi)$, and $O^{\upsilon\Psi - \Psi} = O^{\beta\Psi - \Psi}$. If Ψ were λ-compact, then we would have contradicted 3.2.6.

3.2.7. Ψ is a-realcompact.

In order to prove 3.2.7, let $\mathcal{U}$ be a closed ultrafilter on Ψ with the countable intersection property. Denote by Ψ_d the underlying set of Ψ, with the discrete topology, and observe that Ψ_d, being of non-measurable cardinal, is realcompact. $\mathcal{U}$ is a z-filter on Ψ_d, hence is contained in some z-ultrafilter $\mathcal{X}$. Let $\{A_n \mid n \in N\}$ be any countable subfamily of $\mathcal{X}$. By [4, 5I4], each subset of Ψ is a G_δ-subset, so that $A_n = \cap\{F_{n,i} \mid i \in N\}$, for each $n \in N$, where each $F_{n,i}$ is a closed subset of Ψ. Since, for each $n \in N$, $A_n \in Z$, it meets each member of $\mathcal{U}$, and hence F_{n,i_n}, for at least one $i_n \in N$, meets each member of $\mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $F_{n,i_n} \in \mathcal{U}$, and since $\mathcal{U}$ has the countable intersection property, there is a point $p \in \cap\{F_{n,i_n} \mid n \in N\} = \cap\{A_n \mid n \in N\} \neq \emptyset$. Thus $\mathcal{X}$ has the countable intersection property, and hence converges to a point of the (realcompact) space Ψ_d. But then this point must be an adherent point of $\mathcal{U}$, and hence Ψ is a-realcompact.

3.3. In this section we consider the Tychonoff Corkscrew, S, constructed by Johnson and Mandelker [6, 7.3]:

Let T be the Tychonoff plank, and let $A = W^* \times \{\omega\}$ and $B = \{\omega_1\} \times N^*$ denote the top edge and right edge of T^*. Let S^* denote the space obtained from $T^* \times N$ by identifying $A \times \{2n-1\}$ with $A \times \{2n\}$ and identifying $B \times \{2n\}$ with $B \times \times \{2n+1\}$. Let t denote the corner point (ω_1, ω, n) of S^*. Let $S = S^* - \{t\}$.

This space is known to be η-compact, but not ψ-compact and not μ-compact. Also, $\nu S = S^*$. It follows that S is not λ-compact.

S is not a-realcompact, since it contains a closed pseudocompact subspace $\{(\alpha, \beta, i) \in S \mid i=1\}$. If this subspace were a-realcompact, it would have to be compact, and this is false.

We need the following result [5, 2.4] to show that S is c-realcompact: The following two statements are equivalent:

3.3.1. νX is the smallest c-realcompact subspace of βX containg X.

3.3.2. For every decreasing sequence $\{A_n\}_{n\in N}$ of regularly closed subsets of X, $cl_{\nu X}(\cap\{A_n \mid n \in N\}) = \cap\{cl_{\nu X} A_n \mid n \in N\}$.

Since $\nu S = S \cup \{t\}$, we shall show that S is c-realcompact by showing that S must contain a strictly smaller c-realcompact subspace. Indeed, consider the following decreasing sequence of regularly closed subsets of S: For each $n \in N$, let $B_n = \{(\alpha, \beta, i) \in S \mid i > n\}$. It can be verified that $t \in \cap\{cl_{\nu X} B_n \mid n \in N\}$, while $cl_{\nu X}(\cap\{B_n \mid n \in N\}) = cl_{\nu X}(\emptyset) = \emptyset$.

3.4. We consider next the non-realcompact P-space P constructed in [4, 9L]. It is the subspace of the space $W(\omega_2)$ of all ordinals less than ω_2 obtained by deleting all non-isolated points having a countable base. We denote it by P. Recall the following definition:

3.4.1. A completely regular space X is called a P-space if, for every $p \in \beta X$, $O^p = M^p$.

It follows that P is ψ-compact and λ-compact. Since a pseudocompact support in P is itself a P-space, it is compact. This means that P (or any P-space) is ψ-compact, and thus also η-compact. In what follows we need the concept of a cb-space [7],[11].

3.4.2. A space X is a <u>cb-space</u>, if for every positive, lowersemicontinous function g defined on X, there exists $f \in C(X)$ such that $0 < f(x) \leqq g(x)$ for all $x \in X$.
It follows from Dykes [2.1.10, 3.1], that a cb-space which is almost realcompact, or a-realcompact, or c-realcompact must be, in fact, realcompact. We shall show by proving the following result that P has none of these properties:

3.4.3. P is a cb-space.

Since $W(\omega_2)$ is normal and cb, [4, 5I1(b)] by a result of Mack [7, Theorem 6], it suffices to show that P is an F_σ subset. Indeed, for each $\alpha \in W(\omega_2) - P$, let $\{V_n(\alpha) \mid n \in N\}$ be a countable base at α. For each $n \in N$, let $F_n = W(\omega_2) - \cup\{V_n(\alpha) \mid \alpha \in W(\omega_2) - P\}$. Each F_n is closed in $W(\omega_2)$, and $P = \cup\{F_n \mid n \in N\}$, an F_σ -subset.

3.5. Our final example is the space W of countable ordinals. This space is normal and countably compact [4, 5.12], hence a cb-space. It follows as above that W is not almost realcompact, a-realcompact, or c-realcompact. Johnson and Mandelker [6, 7.2] note that W is μ-compact, but not ψ-compact, and hence not η-compact. Since by pseudocompactness, $\beta W = \nu W$, and since also W is μ-compact W must be λ-compact as well.

4.0. The only question left open is whether every c-realcompact space must be η-compact. From 1.2.8. and 2.1.1. a c-realcompact which is not η-compact cannot be almost realcompact. Examples of c-realcompact spaces which are not almost realcompact exist. One such is the space H considered by Mack and Johnson in [8] p. 240-41. To describe it we start with the space T^* where

$$T^* = \{(\sigma,\tau) \in W^*(\omega_1) \times W^*(\omega_1) : \sigma \leq \tau\}$$

and let A and B denote respectively the top edge and the diagonal of T. Let H^* be the space obtained from $T^* \times N$ by identifying $A \times \{2n-1\}$ with $A \times \{2n\}$ and $B \times \{2n\}$ with $B \times \{2n+1\}$. Finally let w denote the cornerpoint (ω_1, ω_1) of H^*. Then $H = H^* - \{w\}$. By [8, p. 649] this space H is c-realcompact but not almost realcompact. One can show, as in 3.3, that it is not a-realcompact either. However it turns out that H is η-compact. This follows from the following result of Johnson-Mandelker [6, 6.4]:

4.0.1. Let ηX denote the smallest η-compact subspace of βX which contains X. Then $\eta X = X \cup \mathrm{int}_{\beta X} \nu X$. Since $H = = H^* = H \cup \{w\}$ is not locally compact [8, 241] but H is locally compact, we have $H = \mathrm{int}_{\nu H} H$ and hence $H = \eta H$. Thus H is η-compact.

Further, since w is a P-point of $W^*(\omega_1)\times W^*(\omega_1)$, w is a P-point of βH (P-point property is preserved by quotients). Thus $M^{\upsilon X-X} = M^w = O^w = M^{\upsilon X-X}$ and so H is λ-compact. Hence H is also μ-compact and ψ-compact.

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