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A NOTE ON THE FINITE EXTENSIVITY PROPERTY

Jarmila FAUKNEROVÁ

Abstract: Let t, s be groupoid terms with $\ell(t) + \ell(s) \leqq 4$. Then the variety of groupoids satisfying $t \doteq s$ has the finite extensivity property.

Key words: Groupoid, variety, finite extensivity.

Classification: 08A30, 08B05

By [1], the variety Mod $(t \doteq s)$ is extensive for all groupoid terms s, t with $\ell(t) + \ell(s) \leqq 4$. This result is improved in [2] for $\ell(t) + \ell(s) \leqq 5$. In the present note, similar questions are treated for the class of finite groupoids.

1. A variety $\mathcal{V}$ of groupoids is said to have the finite extensivity property if for any two finite groupoids G, H $\in \mathcal{V}$ there exists a finite groupoid K $\in \mathcal{V}$ such that both G and H are isomorphic to subgroupoids of K. Clearly, $\mathcal{V}$ satisfies this property iff for every finite groupoid G $\in \mathcal{V}$ there exists a finite groupoid H $\in \mathcal{V}$ such that G is isomorphic to a subgroupoid of H and H contains at least one idempotent element.

Let t, s be groupoid terms. We denote by $\ell(t)$ the length of t, by var (t) the set of all variables occurring in t and by Mod $(t \doteq s)$ the variety of all groupoids satisfying the identity $t \doteq s$.

2. Throughout this section, let $\mathcal{T} = \mathrm{Mod}\ (x \doteq y \cdot xy)$. One may check easily that $\mathcal{T} = \mathrm{Mod}\ (x \doteq yx \cdot y)$ and every groupoid from $\mathcal{T}$ is a quasigroup.

2.1. <u>Proposition</u>. Let $G \in \mathcal{T}$ be a finite groupoid, card $G=m$. Suppose that G contains no idempotent element. Then:
(i) $m=3k$ for some $k \geqq 1$
(ii) If G is a subgroupoid of a groupoid $H \in \mathcal{T}$ such that H contains at least one idempotent then card $H \geqq 2m+1$.

Proof. (i) Put $f(a,b) = \{(a,b),(b,ab),(ab,a)\}$ for all $(a,b) \in G^2 = G \times G$. Since G contains no idempotent, $f(a,b)$ is a three-element subset of G^2. Moreover, if $(a,b),(c,d) \in G^2$ are such that $f(a,b) \cap f(c,d) \neq \emptyset$, then, using the fact that $G \in \mathcal{T}$, one may see easily that $f(a,b)=f(c,d)$. Consequently, m^2 is divisible by 3 and the rest is clear.

(ii) We can assume that H is finite. Since $G \neq H$ and H is a quasigroup, card $H \geqq 2m$. Suppose card $H=2m$ and define a relation r on H by $(a,b) \in r$ iff either $a,b \in G$ or $a,b \in H \setminus G$. Then r is a congruence of H and the corresponding factorgroupoid H/r is a two-element idempotent quasigroup, a contradiction.

2.2. <u>Proposition</u>. Let $G \in \mathcal{T}$ be finite groupoid, m=card G. Then there exists a finite groupoid $H \in \mathcal{T}$ such that card $H=2m+1$, G is a subgroupoid of H and every element of $H \setminus G$ is idempotent.

Proof. We can assume that $G = \{1,2,\dots,m\}$. Denote by $\circ$ the binary operation of the groupoid G and put $H=\{1,2,\dots,2m+1\}$. We shall define a binary operation $*$ on H in the following four steps:

(i) Let $a,b\in G$. Then $a*b=a\circ b$

(ii) Let $a,b\in H\setminus G$. Then $a=m+i$, $b=m+j$ for some $1\leqq i,\ j\leqq m+1$ and we put $a*b=j-i$ if $i<j$, $a*b=m+i$ $(=a)$ if $i=j$ and $a*b=$ $=j-i+m+1$ if $j<i$. Obviously, $a*b\in G$ for $a\neq b$ and $a*b=a*a=a$ for $a=b$.

(iii) Let $a\in H\setminus G$ and $b\in G$. By (ii), there exists a uniquely determined $c\in H\setminus G$ with $c*a=b$ and we put $a*b=c$.

(iv) Let $a\in G$ and $b\in H\setminus G$. By (ii), there exists a uniquely determined $c\in H\setminus G$ with $b*c=a$ and we put $a*b=c$.

We have defined the operation $*$. Moreover, $G(\circ)$ is a subgroupoid of $H(*)$ and every element of $H\setminus G=\{m+1,\dots,2m+1\}$ is idempotent. It remains to show that $H(*)\in\mathcal{T}$. For, let $a,b\in H$, $a*b=c$. The following cases can arise: .

(v) $a,b\in G$. Then $b*(a*b)=b\circ(a\circ b)=a$.

(vi) $a,b\in H\setminus G$, $a=b$. Then $b*(a*b)=a*(a*a)=a$ by (ii).

(vii) $a,b\in H\setminus G$, $a\neq b$. Then $c\in G$ by (ii) and $b*(a*b)=b*c=a$ by (iii).

(viii) $a\in G$, $b\in H\setminus G$. Then $b*(a*b)=b*c=a$ by (iv).

(ix) $a\in H\setminus G$, $b\in G$. Then $c*a=b$ by (iii) and $b*(a*b)=$ $=(c*a)*c=a$ by (iv).

2.3. Corollary. The variety $\mathcal{T}$ has the finite extensivity property.

2.4. Example. Let $G(+)=\{0,1,\dots,3k-1\}$ be the cyclic group of integers modulo $3k$, $k\geqq 1$. Put $a\circ b=-a-b+1$ for all $a,b\in G$. Then $G(\circ)\in\mathcal{T}$, $G(\circ)$ contains no idempotent element, $G(\circ)$ is commutative and card $G=3k$.

3. In this section, let $\mathcal{R}=$ Mod $(x\doteq yy\cdot xy)$. We have

$\mathcal{R}$ = Mod $(x \doteq y(x \cdot yy))$=Mod $(x \doteq (yy \cdot x)y)$= Mod $(x \doteq yx \cdot yy)$ and every groupoid from $\mathcal{R}$ is a quasigroup.

3.1. Proposition. Let $G \in \mathcal{R}$ be a finite groupoid card G=m. Suppose that G contains no idempotent element. Then:

(i) m is an even number,

(ii) If G is a subgroupoid of a groupoid $H \in \mathcal{R}$ such that H contains at least one idempotent then card $H \geq 2m+1$.

Proof. (i) Let $a \in G$ and b=aa. Then $a \neq b$ and $bb = aa \cdot aa = a$. The rest is clear.

(ii) We can proceed similarly as in the proof of 2.1 (ii).

3.2. Proposition. Let $G \in \mathcal{R}$ be a finite groupoid m=card G. Then there exists a finite groupoid $H \in \mathcal{R}$ such that card H=2m+1, G is a subgroupoid of H and H contains at least one idempotent element belonging to $H \setminus G$.

Proof. We can assume that $G = \{1,2,\dots,m\}$. Denote by $\circ$ the binary operation of G and put $H = \{1,2,\dots,2m+1\}$. We shall define an operation $*$ in H in the following four steps:

(i) Let $a,b \in G$. Then $a * b = a \circ b$.

(ii) Let $a,b \in H \setminus G$. Then $a=m+i$, $b=m+j$ for some $1 \leq i, j \leq m+1$ and we put $a * b = m+(i \circ i)$ if $i=j \leq m$, $a * b = 2m+1$ if $i=j=m+1$, $a * b = i \circ j$ if $i \neq j$ and $i,j \leq m$, $a * b = i \circ i$ if $i \leq m$ and $j=m+1$, $a * b = j \circ j$ if $i=m+1$ and $j \leq m$.

(iii) Let $a \in H \setminus G$, $b \in G$. Then $a=m+i$ for some $1 \leq i \leq m+1$ and we put $a * b = 2m+1$ if $i=b$, $a * b = m+(b \circ b)$ if $i=m+1$, $a * b = m+(i \circ b)$ if $i \neq b$ and $i \leq m$.

(iv) Let $a \in G$, $b \in H \setminus G$. Then $b=m+j$ for some $1 \leq j \leq m+1$ and we put $a * b = m+1$ if $j=a$, $a * b = m+(a \circ a)$ if $j=m+1$, $a * b = m+(a \circ j)$ if $j \neq a$ and $j \leq m$.

Clearly, $G(\circ)$ is a subgroupoid of $H(*)$, card H=2m+1 and the element $2m+1 \in H \setminus G$ is idempotent in $H(*)$. It remains to show that $H(*) \in \mathfrak{R}$. For, let $a,b \in H$. The following cases can arise:

(v) $a,b \in G$. Then $(b*b)*(a*b)=(b\circ b)\circ(a\circ b)=a$.

(vi) $a=b \in H \setminus G$, $a=m+i$, $1 \leq i \leq m$. Then $(b*b)*(a*b)=$
$=(m+(i\circ i))*(m+(i\circ i))=m+((i\circ i)\circ(i\circ i))=m+i=a$ by (ii).

(vii) $a=b=2m+1$. Then $(b*b)*(a*b)=a*a=a$ by (ii).

(viii) $a,b \in H \setminus G$, $a \neq b$, $a=m+i$, $b=m+j$, $1 \leq i, j \leq m$. Then
$(b*b)*(a*b)=(m+(j\circ j))*(i\circ j)=m+((j\circ j)\circ(i\circ j))=m+i=a$ by (ii) and (iii).

(ix) $a,b \in H \setminus G$, $a=m+i$, $1 \leq i \leq m$, $b=2m+1$. Then $(b*b)*(a*b)=$
$=(2m+1)*(i\circ i)=m+((i\circ i)\circ(i\circ i))=m+i=a$ by (ii) and (iii).

(x) $a,b \in H \setminus G$, $a=2m+1$, $b=m+j$, $1 \leq j \leq m$. Then $(b*b)*(a*b)=$
$=(m+(j\circ j))*(j\circ j)=2m+1=a$ by (ii) and (iii).

(xi) $a \in H \setminus G$, $b \in G$, $a=m+i$, $1 \leq i \leq m$, $b=i$. Then $(b*b)*(a*b)=$
$=(i\circ i)*(2m+1)=m+((i\circ i)\circ(i\circ i))=m+i=a$ by (i),(iii) and (iv).

(xii) $a \in H \setminus G$, $b \in G$, $a=2m+1$. Then $(b*b)*(a*b)=$
$=(b\circ b)*(m+(b\circ b))=2m+1=a$ by (i),(iii) and (iv).

(xiii) $a \in H \setminus G$, $b \in G$, $a=m+i$, $1 \leq i \leq m$, $i \neq b$. Then
$(b*b)*(a*b)=(b\circ b)*(m+(i\circ b))=m+((b\circ b)\circ(i\circ b))=m+i=a$ by (i),(iii) and (iv).

(xiv) $a \in G$, $b \in H \setminus G$, $b=m+j$, $1 \leq j \leq m$, $a=j$. Then $(b*b)*(a*b)=$
$=(m+(j\circ j))*(2m+1)=(j\circ j)\circ(j\circ j)=j=a$ by (ii) and (iv).

(xv) $a \in G$, $b \in H \setminus G$, $b=2m+1$. Then $(b*b)*(a*b)=(2m+1)*(m+$
$+(a\circ a))=(a\circ a)\circ(a\circ a)=a$ by (ii) and (iv).

(xvi) $a \in G$, $b \in H \setminus G$, $b=m+j$, $1 \leq j \leq m$, $a \neq j$. Then $(b*b)*(a*b)=$
$=(m+(j\circ j))+(m+(a\circ j))=m+((j\circ j)\circ(a\circ j))=a$ by (ii) and (iv).

3.3. <u>Corollary</u>. The variety $\mathfrak{R}$ has the finite extensivity property.

3.4. Example. Let F be a four-element field and $0,1 \neq a \in F$. Put $x \circ y = ax + a^{-1}y + 1$ for all $x,y \in F$. It is easy to check that $F(\circ) \in \mathcal{R}$ and $F(\circ)$ contains no idempotent.

4.1. Lemma. Let t be a groupoid term such that $x \notin var(t)$. Then Mod $(x \doteq t)$=Mod $(x \doteq y)$.

Proof. Obvious.

4.2. Lemma. The varieties Mod $(x \doteq x)$, Mod $(x \doteq xx)$, Mod $(x \doteq xy)$ and Mod $(x \doteq yx)$ have the finite extensivity property.

Proof. Obvious.

4.3. Lemma. Let t, s be two groupoid terms such that var (t)=var (s). Then the variety Mod $(t \doteq s)$ has the finite extensivity property.

Proof. Easy.

4.4. Lemma. The varieties Mod $(x \doteq x \cdot xy)$, Mod $(x \doteq x \cdot yx)$, Mod $(x \doteq x \cdot yy)$, Mod $(x \doteq y \cdot yx)$ have the finite extensivity property.

Proof. Let $G \in$ Mod $(x \doteq x \cdot xy)$, $e \notin G$, $H = G \cup \{e\}$, ae=a and ea=e for every $a \in H$. Obviously, $H \in$ Mod $(x \doteq x \cdot xy)$. The remaining cases are similar.

4.5. Lemma. The varieties Mod $(x \doteq x \cdot yz)$, Mod $(x \doteq y \cdot xz)$ and Mod $(x \doteq y \cdot zx)$=Mod $(x \doteq y \cdot xx)$ have the finite extensivity property.

Proof. (i) Let $G \in$ Mod $(x \doteq x \cdot yz)$ and $a \in G$. Then aa=aa aa.
(ii) Let $G \in$ Mod $(x \doteq y \cdot xz)$ and $a,b \in G$. Then a=a(a(bb·a))= =a·bb=b.

(iii) Let $G \in$ Mod ($x \doteq y \cdot xx$) and $a, b \in G$. Then $aa = b(aa \cdot aa) = ba$ and we see that Mod ($x \doteq y \cdot xx$)=Mod ($x \doteq y \cdot zx$). Now, let $e \notin G$, $H = G \cup \{e\}$, $ae = e$ and $ea = aa$ for every $a \in H$. Obviously, $H \in$ Mod ($x \doteq y \cdot xx$).

4.6. Lemma. The varieties Mod ($xx \doteq xy$)=Mod ($xy \doteq xz$) and Mod ($xy \doteq zx$)=Mod ($xy \doteq zu$) have the finite extensivity property.

Proof. Easy.

4.7. Theorem. Let t, s be groupoid terms such that $\ell(t) + \ell(s) \leq 4$. Then the variety Mod($t \doteq s$) has the finite extensivity property.

Proof. Apply 2.3, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6 (and their duals).

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