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A REMARK CONCERNING COMMUTATIVITY MODULO RADICAL IN BANACH ALGEBRAS

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Abstract: Let a be a fixed element of a Banach algebra A, n a natural number. The following conditions are equivalent

1^o for each strictly irreducible representation T of A, there exists a scalar λ_T such that $T((a-\lambda_T)^n)=0$

2^o $(D_a^n x)^n \in \operatorname{Rad} A$ for each $x \in A$

(D_a denotes the commutator operator on A, i.e. $D_a x = ax - xa$)

3^o $|D_a^n x|_\sigma = 0$ for each $x \in A$.

Key words: Banach algebra, radical, strictly irreducible representation.

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In a recent paper [1] V. Pták has shown that, for an element a of a Banach algebra A with a unit 1, the following conditions are equivalent:

1^o for each strictly irreducible representation T of A there exists a scalar λ_T such that

$$T((a-\lambda_T)^2) = 0$$

2^o $[[x,a],a]^2 \in \operatorname{Rad} A$ for each $x \in A$

3^o $|[[x,a],a]|_\sigma = 0$ for each $x \in A$.

Here $[x,y] = xy - yx$ and $|x|_\sigma$ denotes the spectral radius of x.

These conditions are related to the treatment of a "weaker" commutativity in Banach algebras. We intended to show that they have an appropriate analogue for higher powers as well.

For an $a \in A$, denote by D_a the commutator operator on A, i.e. $D_a x = ax - xa$. We shall use the formula for the n-th iteration:

$$D_a^n x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^k x a^{n-k}$$

<u>Proposition</u>. Let a be a fixed element of a Banach algebra A, T a strictly irreducible representation of A, n a natural number. Then the following conditions are equivalent:

1^o there exists a scalar λ_T such that $T((a - \lambda_T)^n) = 0$

2^o $T(D_a^n x)^n = 0$ for each $x \in A$

3^o $|T(D_a^n x)|_\sigma = 0$ for each $x \in A$.

Proof. Obviously $D_a x = D_{a-\lambda} x$ for each scalar λ. Then, for each representation T of A and $b \in A$, we have

$$T((D_b^n x))^n = (T(D_b^n x))^n = \left[\sum_{k=0}^{n} \binom{n}{k} (-1)^k T(b)^k\ T(x)\ T(b)^{n-k}\right]^n$$

Apart from scalar coefficients, each summand of the last expression is of the type

$$T(b)^{j_1}\ T(x)\ T(b)^{n-j_1+j_2}\ T(x)\ T(b)^{n-j_2+j_3}\ \ldots\ldots\ T(x)\ T(b)^{n-j_n}$$

with $0 \leq j_i \leq n$ arbitrary. As it is impossible to have $j_1 < n,\ -j_1 + j_2 < 0, \ldots, -j_{n-1} + j_n < 0,\ j_n > 0$, each summand contains $T(b)^k$ with $k \geq n$.

Now, assume 1^o and set $b = a - \lambda_T$ so that $T(b)^n = 0$, and consequently $T(D_a^n x)^n = T(D_b^n x)^n = 0$ as well.

The implication $2^o \longrightarrow 3^o$ is trivial. To prove $3^o \longrightarrow 1^o$ we shall apply the Jacobson density theorem. Assume 3^o and consider a fixed strictly irreducible representation T of A into L(X), the algebra of all linear operators on a vector space X.

The strict irreducibility of T enables us to endow X by a norm in which X becomes a Banach space and all $T(a)$ $(a \in A)$ are bounded (for example in [2]).

First we shall show that there exists a polynomial p of degree not exceeding n such that $p(T(a)) = 0$ and finally that it has only one root. Suppose not. It follows that $1, T(a), \dots$ $\dots, T(a)^n$ are linearly independent so that there exists a $u \in X$ such that vectors $u, T(a)\, u, \dots, T(a)^n\, u$ are also linearly independent. According to the density theorem [2] there exists an $x \in A$, for which

$$\begin{aligned} &T(x)\, u = 0 \\ &T(x)\, T(a)\, u = 0 \\ &\quad \vdots \\ &T(x)\, T(a)^{n-1}\, u = 0 \\ &T(x)\, T(a)^n\, u = u. \end{aligned}$$

It follows that $T(D_a^n\, x)u = u$ whence $|T(D_a^n\, x)|_\sigma \geq 1$ which is a contradiction to 3^o. Let p be a polynomial of minimal degree for which $p_T(T(a)) = 0$. Suppose λ_1, λ_2 are two different roots of p_T. There exist non-zero vectors $u_1, u_2 \in X$ such that $T(a)\, u_1 =$ $= \lambda_1\, u_1$, $T(a)\, u_2 = \lambda_2\, u_2$. Again, there exists an $x \in A$ such that $T(x)\, u_1 = u_2$, $T(x)\, u_2 = (-1)^n\, u_1$. It follows that

$$\begin{aligned} &T(D_a^n\, x)\,(u_1 + u_2) = \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left\{ T(a)^k T(x) T(a)^{n-k} u_1 + T(a)^k T(x) T(a)^{n-k} u_2 \right\} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left\{ \lambda_1^{n-k} \lambda_2^k\, u_2 + \lambda_1^k\, \lambda_2^{n-k} (-1)^n u_1 \right\} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_1^{n-k}\, \lambda_2^k\, u_2 + \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \lambda_1^{n-j}\, \lambda_2^j\, u_1 \\ &= (\lambda_1 - \lambda_2)^n\, (u_1 + u_2), \end{aligned}$$

whence $|T(D_a^n x)|_\sigma \geq |\lambda_1 - \lambda_2|$ which is again a contradiction to 3°.

The proof is complete.

The radical being the intersection of kernels of all strictly irreducible representations we obtain also the following

Corollary. Under the same assumptions as in the Proposition, the following conditions are equivalent:

1° for each strictly irreducible representation T of A, there exists a scalar λ_T such that $T((a - \lambda_T)^n) = 0$

2° $(D_a^n x)^n \in \mathrm{Rad}\, A$ for each $x \in A$

3° $|D_a^n x|_\sigma = 0$ for each $x \in A$.

References

[1] V. PTÁK: Commutators in Banach algebras, Proc. Edinburgh Math. Soc. 22(1979), 207-211.

[2] A.M. SINCLAIR: Automatic continuity of linear operators, London Math. Soc. Lecture Notes Series 21, Cambridge Univ. Press, Cambridge 1976.

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