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FINITE GRAPHS AND DIGRAPHS WHICH ARE NOT RECONSTRUCTIBLE FROM THEIR CARDINALITY RESTRICTED SUBGRAPHS

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Abstract: For every $n>1$ we construct two non-isomorphic graphs with $2n$ vertices having the same collection of n-vertex subgraphs. The constructions are given also for the case of disconnected, of unicyclic graphs and for trees. Finally the construction is modified so as to give examples of two non-isomorphic graphs with $3k+9$ vertices having the same collection of $2k$-vertex subgraphs.

Key words: Finite directed graphs, finite undirected graphs, Ulam conjecture.

Classification: 05C60

A digraph is a couple $G = \langle V(G), E(G)\rangle$ where $V(G)$ is a set and $E(G)$ an irreflexive binary relation on $V(G)$. If the relation $E(G)$ is symmetric (i.e., $E(G) = E(G)^{-1}$) then G is called a graph. For every digraph G its symmetrization sym $G = \langle V(G), E(G)\cup E(G)^{-1}\rangle$ is defined. For every subset Y of $V(G)$ we denote $G/Y = \langle Y, E(G)\cap Y^2\rangle$. Now, for every natural k we define $U_k(G) = \{G/Y; Y\subseteq V(G), \text{card } Y = k\}$. We write $G_1 \overset{k}{\sim} G_2$ if there is a one-to-one correspondence $\Phi: U_k(G_1)\longrightarrow U_k(G_2)$ such that for every $H\in U_k(G_1)$ $H\cong\Phi(H)$ holds.

Proposition 1. For every $n>1$ there are two digraphs G_1, G_2 such that card $V(G_1)$ = card $V(G_2) = 2n$, $G_1\not\cong G_2$ and $G_1 \overset{n}{\sim} G_2$.

Proof: Let $X = \{x,y\}\cup\{a_i;\ i=1,\dots,n-1\}\cup\{b_i;\ i=1,\dots$

$\ldots,n-1\}$ be a set having $2n$ elements. We define the binary relation $R = \{\langle a_i, a_{i+1}\rangle;\ i=1,\ldots,n-2\} \cup \{\langle b_i, b_{i+1}\rangle;\ i=1,\ldots$ $\ldots,n-2\}$.

Further let $R_1 = R \cup \{\langle x, a_1\rangle, \langle a_{n-1}, y\rangle\}$, $R_2 = R \cup \{\langle x, a_1\rangle,$ $\langle b_{n-1}, y\rangle\}$. We have two disconnected digraphs $G_1 = \langle X, R_1\rangle$, $G_2 = \langle X, R_2\rangle$. The digraph G_1 has two components with $n+1$ and $n-1$ vertices, the digraph G_2 has two components both with n vertices. Clearly G_1, G_2 are non-isomorphic.

Now, for every $i < n$ we define the set $Q(i) = \{a_j;\ j=i,\ldots$ $\ldots,n-1\} \cup \{b_j;\ j=i,\ldots,n-1\} \cup \{y\}$ and the set $\overline{Q}(i) = X - Q(i)$. For every $i=1,\ldots,n-1$ we define the bijection $f_i : X \longrightarrow X$ as follows:

$f_i(x) = x$, $f_i(y) = y$,

$f_i(a_j) = a_j$, $f_i(b_j) = b_j$ for every $j < i$,

$f_i(a_j) = b_j$, $f_i(b_j) = a_j$ for every $j \geqq i$.

The restrictions of f_i give isomorphisms $G_1/Q(i) \cong G_2/Q(i)$, $G_1/\overline{Q}(i) \cong G_2/\overline{Q}(i)$. Finally, we describe the mapping $\Phi :$ $U_n(G_1) \longrightarrow U_n(G_2)$. If Y is a subset of X having n elements then

(a) $\Phi(G_1/Y) = G_2/f_1(Y)$ for the case $x \notin Y$,

(b) $\Phi(G_1/Y) = G_2/Y$ for the case $x \in Y$, $y \notin Y$,

(c) $\Phi(G_1/Y) = G_2/f_k(Y)$ where $k = \min\{i;\ a_i \notin Y,\ b_i \notin Y\}$ for the case $x \in Y$, $y \in Y$.

The existence of the number $k \leqq n-1$ in (c) follows from the conditions $\operatorname{card} Y = n$, $x \in Y$, $y \in Y$.

Further the mapping $\Psi : U_n(G_2) \longrightarrow U_n(G_1)$ can be defined, if we substitute the symbols Ψ, G_2, G_1 for the symbols Φ, G_1, G_2 in (a),(b),(c). It follows immediately from the definitions of f_i, Φ, Ψ that for every Y having n elements it holds:

$\Phi(G_1/Y) \cong G_1/Y$ and $\Psi(G_2/Y) \cong G_2/Y$. Using the fact that for every $i=1,\dots,n-1$ $f_i \circ f_i$ = identity, we find $\Phi \circ \Psi$ = identity, $\Psi \circ \Phi$ = identity. Thus Φ is a bijection.

The analogous results can be obtained for undirected graphs applying the operation sym to G_1, G_2 from Proposition 1. The above described technique of the construction of Φ can be used to prove some similar statements.

Proposition 2. For every $n \geqq 2$ there are two oriented trees (and also two trees) T_1, T_2 such that card $V(T_1) =$ $=$ card $V(T_2) = 2n$, $T_1 \not\cong T_2$, $T_1 \overset{n}{\sim} T_2$.

Outline of proof: Let $X = \{a_i;\ i=1,\dots,2n-1\} \cup \{x\}$ be a set having 2n elements, let $R = \{\langle a_i, a_{i+1}\rangle;\ i=1,\dots,2n-2\}$. Further, $R_1 = R \cup \{\langle x, a_{n-1}\rangle\}$, $R_2 = R \cup \{\langle x, a_n\rangle\}$ and we take $T_1 =$ $= \langle X, R_1\rangle$, $T_2 = \langle X, R_2\rangle$. For the case of trees we take sym T_1, sym T_2.

Proposition 3. For every $n > 2$ there are two unicyclic digraphs (and also graphs) C_1, C_2 such that card $V(C_1) =$ $=$ card $V(C_2) = 2n$, $C_1 \not\cong C_2$, $C_1 \overset{n}{\sim} C_2$.

Outline of proof: Let $X = \{a_i;\ i=1,\dots,n-1\} \cup \{b_i;\ i=$ $=1,\dots,n-1\} \cup \{x,y\}$ be a set having 2n elements, let $R =$ $= \{\langle a_i, a_{i+1}\rangle;\ i=1,\dots,n-2\} \cup \{\langle b_{i+1}, b_i\rangle;\ i=1,\dots,n-2\} \cup$ $\cup \{\langle b_1, a_1\rangle, \langle a_{n-1}, b_{n-1}\rangle\}$. Further, $R_1 = R \cup \{\langle x, a_1\rangle, \langle a_{n-1}, y\rangle\}$, $R_2 = R \cup \{\langle x, b_1\rangle, \langle a_{n-1}, y\rangle\}$. We take $C_1 = \langle X, R_1\rangle$, $C_2 =$ $= \langle X, R_2\rangle$ and for the case of undirected graphs sym C_1, sym C_2.

Proposition 4. For every $k \geqq 2$ there are two graphs G_1, G_2 such that card $V(G_1)$=card $V(G_2) = 3k+9$ & $G_1 \not\cong G_2$ & $G_1 \overset{2k}{\sim} G_2$.

Proof. Let $k \geqq 2$ be given. Let $M = \{a,b,c,1,2,3,11,12,13\}$

be a set having 9 elements and let N be the set of all natural numbers. We define $R = \{\{a,2\},\{2,b\},\{b,3\},\{3,c\},\{c,1\},\{1,a\},\{a,12\},\{12,2\},\{b,13\},\{13,3\},\{c,11\},\{11,1\}\}$. We denote $a_0 = a$, $b_0 = b$, $c_0 = c$ and for every $i \in N$ $a_i = \langle a,i\rangle$, $b_i = \langle b,i\rangle$, $c_i = \langle c,i\rangle$.

Now, for every $m \in N$ we define $A_m = \{a_i \mid 0 \leqq i \leqq m\}$, $B_m = \{b_i \mid 0 \leqq i \leqq m\}$, $C_m = \{c_i \mid 0 \leqq i \leqq m\}$, $R_m^A = \{\{a_i, a_{i+1}\} \mid 0 \leqq i < m\}$, $R_m^B = \{\{b_i, b_{i+1}\} \mid 0 \leqq i < m\}$, $R_m^C = \{\{c_i, c_{i+1}\} \mid 0 \leqq i < m\}$. The graphs G_1, G_2 are defined as follows (see Fig. 1 for the case $k=5$):

$$G_1 = (M \cup A_{k-1} \cup B_k \cup C_{k+1},\ R \cup R_{k-1}^A \cup R_k^B \cup R_{k+1}^C),$$

$$G_2 = (M \cup A_{k+1} \cup B_k \cup C_{k-1},\ R \cup R_{k+1}^A \cup R_k^B \cup R_{k-1}^C).$$

Clearly, $\text{card } V(G_1) = \text{card } V(G_2) = 9+(k-1)+k+(k+1) = 3k+9$.

First, we prove that G_1, G_2 are non-isomorphic.
Let us suppose $\varphi : G_1 \longrightarrow G_2$ is an isomorphism. Since φ preserves degrees of vertices, it holds $\varphi(\{a,b,c\}) = \{a,b,c\}$, $\varphi(\{1,2,3\}) = \{1,2,3\}$, $\varphi(\{a_{k-1}, b_k, c_{k+1}\}) = \{a_{k+1}, b_k, c_{k-1}\}$. Thus, $\varphi(b_i) = b_i$ for every $i \leqq k$, $\varphi(a_i) = c_i$ for every $i \leqq k-1$, $\varphi(c_i) = a_i$ for every $i \leqq k+1$. Further, we have $\varphi(2) = 3$, which yields $\{\varphi(12),3\}$ and $\{\varphi(12),c\}$ are edges in G_2. No such a $\varphi(12)$ exists in G_2.

Secondly, we prove $G_1 \overset{2k}{\sim} G_2$.
Let us denote $S_1 = \{G_1/Z \mid |Z| = 2k\}$, $S_2 = \{G_2/T \mid |T| = 2k\}$. We are going to describe the bijection $\Phi : S_1 \longrightarrow S_2$ such that for every $G \in S_1$ $\Phi(G) \simeq G$.

We define the set $D = A_{k-1} \cup B_{k-1} \cup C_{k-1}$ and isomorphisms $\varphi_1, \varphi_2, \varphi_3, \Psi$.

$\varphi_1 = id_{D\cup\{b_k\}} : G_1/D \cup \{b_k\} \rightarrow G_2/D \cup \{b_k\}$,

$\varphi_2 : G_1/D \cup \{c_k, c_{k+1}\} \rightarrow G_2/D \cup \{a_k, a_{k+1}\}$, where $\varphi_2(1) = 2$, $\varphi_2(2) = 3$, $\varphi_2(3) = 1$, $\varphi_2(11) = 12$, $\varphi_2(12) = 13$, $\varphi_2(13) = 11$, $\varphi_2(a_i) = b_i$, $\varphi_2(b_i) = c_i$, $\varphi_2(c_i) = a_i$.

$\varphi_3 : G_1/D \rightarrow G_2/D$, where $\varphi_3(x) = \varphi_2^{-1}(x)$ for every $x \in D$.

$\psi : G_1/A_{k-1} \cup B_k \cup C_{k+1} \rightarrow G_2/A_{k+1} \cup B_k \cup C_{k-1}$, where $\psi(a_i) = c_i$, $\psi(b_i) = b_i$, $\psi(c_i) = a_i$.

Let $G = G_1/Z \in S_1$.

I. If $Z \cap \{b_k, c_k, c_{k+1}\} = \emptyset$ then $\Phi(C) = G_2/\varphi_1(Z) = G_2/Z$.

II. If $Z \cap \{b_k, c_k, c_{k+1}\} \neq \emptyset$ then we discuss 4 conditions:

(α) $(\exists i)(Z \cap \{a_i, b_i, c_i\} = \emptyset)$

(β) $(\exists i)(Z \cap \{a_i, b_i, c_i\} = \{a_i\})$

(γ) $(\exists i)(Z \cap \{a_i, b_i, c_i\} = \{b_i\})$

(δ) $(\exists i)(Z \cap \{a_i, b_i, c_i\} = \{c_i\})$.

1) Z satisfies (α). We define $i_o = \min \{i \mid Z \cap \{a_i, b_i, c_i\} = \emptyset\}$ and $Z_1 = Z \cap (\{a_i \mid i > i_o\} \cup \{b_i \mid i > i_o\} \cup \{c_i \mid i > i_o\})$, $Z_2 = Z - Z_1$.

Then $\Phi(G) = G_2/\varphi_1(Z_2) \cup \psi(Z_1)$.

2) Z does not satisfy (α) and Z satisfies (β). We define $i_o = \min \{i \mid Z \cap \{a_i, b_i, c_i\} = \{a_i\}\}$ and $Z_1 = Z \cap (\{b_i \mid i > i_o\} \cup \{c_i \mid i > i_o\})$, $Z_2 = Z - Z_1$.

Then $\Phi(G) = G_2/\varphi_3(Z_2) \cup \psi(Z_1)$.

3) Z does not satisfy (α),(β) and Z satisfies (γ). We define $i_o = \min \{i \mid Z \cap \{a_i, b_i, c_i\} = \{b_i\}\}$ and $Z_1 = Z \cap (\{a_i \mid i > i_o\} \cup \{c_i \mid i > i_o\})$, $Z_2 = Z - Z_1$.

Then $\Phi(G) = G_2/\varphi_1(Z_2) \cup \psi(Z_1)$.

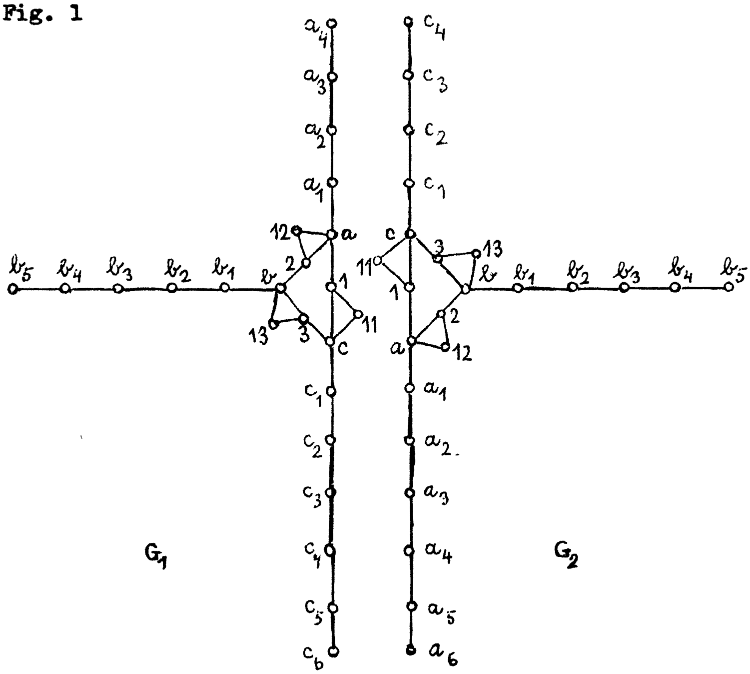
4) Z does not satisfy (α),(β),(γ) and Z satisfies (δ).

We define $i_o = \min\{i \mid Z \cap \{a_i, b_i, c_i\} = \{c_i\}\}$ and $Z_1 = Z \cap$ $\cap (\{a_i \mid i > i_o\} \cup \{c_i \mid i > i_o\})$, $Z_2 = Z - Z_1$.
Then $\Phi(G) = G_2/\varphi_2(Z_2) \cup \psi(Z_1)$.

5) Let us suppose Z does not satisfy $(\alpha), (\beta), (\gamma), (\delta)$. Then for every i, $0 \leqq i \leqq k-1$, $|Z \cap \{a_i, b_i, c_i\}| \geqq 2$. Thus, $|Z| \geqq 2k+1$. $G_1/Z \notin S_1$.

It can be easily shown (using the method of discussion again) that for every $G_2/T \in S_2$ there exists $G_1/Z \in S_1$ such that $G_2/T = \Phi(G_1/Z)$. So, Φ is fully determined and has all the needed properties.

Fig. 1



R e f e r e n c e s

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