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# ON EXTENDING TRANSITIVE HOMEOMORPHISMS FROM THE CANTOR SET TO THE PRODUCT OF TWO CANTOR SETS

Ryszard FRANKIEWICZ and Andrzej GUTEK

Abstract: We prove the following theorem: Let f be a transitive homeomorphism from the Cantor set C onto itself. Then there exists a homeomorphism g from C onto itself such that g(x) = x for some point x of C and the homeomorphism $g\times f: C\times C \to C\times C$ is transitive. More precisely, if the semiorbit $\{f^n(y): n=1,2,\dots\}$ is dense in C, then the homeomorphism g can be defined in such a way that for some point z of C the semiorbit $\{\langle g^n(z), f^n(y)\rangle: n=1,2,\dots\}$ is dense in $C\times C$.

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Classification: 54C10, 54C20.

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There is a number of papers in which possibilities of extending homeomorphisms are investigated. One can list papers of Knaster and Reichbach [3], J. Pollard [7], R.S. Pierce [6], J.W. Baker [1] and J. van Mill [4]. A possibility of extension to a transitive homeomorphism is studied in [2].

Let us remind that a homeomorphism h from the space X onto itself is said to be transitive if and only if there exists a point x whose orbit $\{h^n(x): n \text{ is an integer}\}$ is dense in X. We shall use the following property of transitive homeomorphisms:

Lemma (Oxtoby [5], p. 70). Let X be a complete, separable metric space without isolated points, and let h be a transitive homeomorphism from X onto itself. Then those points whose positive semiorbit is dense constitute a residual set in X.

A set is said to be <u>residual</u> iff it is the complement of a set of first category. A <u>positive semiorbit</u> of a point x is the set $\{h^n(x): n=1,2,\dots\}$.

<u>Theorem</u>. Let f be a transitive homeomorphism from the Cantor set C onto itself. Then there exists a homeomorphism g from C onto itself such that $g(x) = x$ for some point x of C and the homeomorphism $g\times f: C\times C \longrightarrow C\times C$ is transitive. More precisely, if the semiorbit $\{f^n(y): n=1,2,\dots\}$ is dense in C, then the homeomorphism g can be defined in such a way that for some point z of C the semiorbit $\{\langle g^n(z), f^n(y)\rangle: n=1,2,\dots\}$ is dense in $C\times C$.

<u>Proof</u>. Let us assume that the Cantor set C is given by the usual ternary expansion, and let $\mathcal{B}$ denote the basis defined by this expansion, i.e. it is a family of closed-open subsets of C, and if $I,J \in \mathcal{B}$, then $I\subseteq J$ or $J\subseteq I$ or $I\cap J = \emptyset$ and diam $I = 3^i\cdot$ diam $J$ for some integer i.

For every two subsets A and B of C put $A<B$ if and only if $a<b$ for every $a\in A$ and $b\in B$. For every set B of the basis $\mathcal{B}$ and for every positive integer k consider a partition $\{B(m,k): m=1,\dots,k\}$ of B into k disjoint subsets belonging to $\mathcal{B}$ diameters of which are less or equal to $k^{-1}\cdot$ diam B and such that $B(m,k)< B(p,k)$ for $m< p\leq k$. If k is equal to $2^j$ for some positive integer j, then we require diameters of any two such subsets to be equal one to another.

Let $B_1, B_2, \ldots, B_n, \ldots$ be defined by $B_n = [\frac{2}{3^n} \frac{1}{3^{n-1}}] \cap C$. Let the semiorbit $\{f^n(y): n=1,2,\ldots\}$ of a point y be dense in C. Such a point exists in virtue of the lemma.

The homeomorphism g is defined by induction.

The first step. $g_1$ is the linear and order preserving mapping from $B_k(1,2)$ onto $B_1(2,2)$.

Put z equal to $\frac{2}{3}$ and consider the point $\langle \frac{2}{3}, y \rangle$.

Thus we have obtained a chain consisting of the sets $A_1^1 = B_1(1,2)$ and $A_2^1 = B_1(2,2)$, and $g_1(A_1^1(m,2^k)) = A_2^1(m,2^k)$ for every positive integer k and for $m=1,\ldots,2^k$.

The n-th step ($n \geq 2$). Suppose we have constructed a chain $A_1^{n-1},\ldots,A_k^{n-1}$ of closed-open segments of C such that $\cup\{A_j^{n-1}: j=1,\ldots,k\} = \cup\{B_j: j=1,\ldots,n-1\}$ and $A_1^{n-1}, A_k^{n-1} \subseteq B_{n-1}$, and a function $g_{n-1}$ defined on $\cup\{A_j^{n-1}: j=1,\ldots,k-1\}$ such that $g_{n-1}|A_j^{n-1}$ is linear and order preserving mapping from $A_j^{n-1}$ onto $A_{j+1}^{n-1}$ for $j=1,\ldots,k-1$. For each segment $A_j^{n-1}$ let us consider cartesian products $P_j^{n-1}(t,i) = A_j^{n-1} \times C(t,2^{n-i})$, where $t=1,\ldots,2^{n-i}$, and i is such a positive integer that $A_j^{n-1} \subseteq B_i$. Let $N_n$ be the number of sets $P_j^{n-1}(t,i)$, where $i=1,\ldots,n-1$ and $j=1,\ldots,k$ and $t=1,\ldots,2^{n-i}$. Order these sets, putting the first one this set, which contains the point $\langle \frac{2}{3}, y \rangle$. Denote the s-th set in this ordering by $P_s$, and put $n^+(P_s) = k-j$ and $n^-(P_s) = j$ if and only if $P_s = P_j^{n-1}(t,i)$ for some j, t and i. We define for each set $P_s$ numbers $n_s$ and $\bar{m}_s$. We put $n_1 = 0$ and $\bar{m}_1 = 1$. Suppose we have defined $n_r$ and $\bar{m}_r$ for $r < s$. We put $n_s$ to be such a positive integer that $n_s - n_{s-1} - n^-(P_s) - n^+(P_{s-1}) = \bar{m}_s$ is a positive integer, and such that

$f^{n_s}(y) \in C(t,2^{n-i})$, where t and i are such that $P_s = A_j^{n-1} \times$ $\times C(t,2^{n-i})$. Such a positive integer $n_s$ exists, because the positive semiorbit of the point y is dense in C.

Put $M_n = \overline{m}_1 + \overline{m}_2 + \dots + \overline{m}_{N_n} + 1$. Let us consider a partition $\{B_n(m,M_n): m=1,\dots,M_n\}$ of $B_n$ into $M_n$ disjoint subsets belonging to $\mathcal{B}$ diameters of which are less or equal to $M_n^{-1} \cdot$ $\cdot \operatorname{diam} B_n$. Let the sets $A_1^{n-1}(s,N_n)$, where $s=1,\dots,N_n$, constitute the similar partition of $A_1^{n-1}$ into $N_n$ disjoint and non-void subsets. This partition induces partitions of the sets $A_j^{n-1}$, where $j=2,\dots,k$, if we put $A_j^{n-1}(s,N_n) =$ $= {g_{n-1}}^{j-1}(A_j^{n-1}(s,N_n))$.

Observe that if $\frac{2}{3} \in A_j^{n-1}$, then $\frac{2}{3} \in A_j^{n-1}(1,N_n)$. Let $m_s = \overline{m}_1 + \dots$ $\dots + \overline{m}_s$ for $s=1,\dots,N_n$.

We define $g_n$ as follows:

$g_n | B_n(m,M_n)$ is a linear and order preserving mapping from $B_n(m,M_n)$ onto $A_1^{n-1}(s,N_n)$ iff $m = m_s$ for some s, and onto $B_n(m+1, N_n)$ otherwise,

$g_n | A_k^{n-1}(s,N_n)$ is a linear and order preserving mapping from $A_k^{n-1}(s,N_n)$ onto $B_n(m_{s+1},M_n)$, where $m_{N_n+1} = M_n$ and

$g_n | \bigcup\{B_j: j=1,\dots,n-1\} \setminus A_k^{n-1} = g_{n-1}$.

Put $A_1^n = B_n(1,M_n)$ and $A_{j+1}^n = g_n(A_j^n)$ for $j=1,\dots,M_n+k\, N_n-1$.

Thus, $g_n$ is defined on $\bigcup\{B_j: j=1,\dots,n\} \setminus B_n(M_n,M_n)$ and is continuous and one-to-one. Let us observe that $(g_n \times f)^{n_s} \langle \frac{2}{3},y \rangle \in P_s$ for $s=1,\dots,N_n$, and $A_1^n, A_{M_n+k \cdot N_n}^n \subseteq B_n$. The mapping $g: C \longrightarrow C$ defined by $g|B_n = g_{n+1}|B_n$ and $g(0) = 0$ is a homeomorphism from C onto itself, and the positive semiorbit of the point $\langle \frac{2}{3}, y \rangle$ **is dense in** $C \times C$.

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