Commentationes Mathematicae Universitatis Carolinae

Jana Jurečková

Tail-behaviour of location estimators in non-regular cases

Commentationes Mathematicae Universitatis Carolinae, Vol. 22 (1981), No. 2, 365--375

Persistent URL: http://dml.cz/dmlcz/106083

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

22,2 (1981)

TAIL-BEHAVIOUR OF LOCATION ESTIMATORS IN NON-REGULAR CASES

Jana JUREČKOVÁ

Abstract: Let $X_1,\dots,X_n$ be a sample from a population with the density $f(x-\theta)$ such that $f(x)=0$ for $x\notin(-a,a)$, $a>0$. It is proved that the probabilities $P_\theta(T_n-\theta<-a+\delta)$, $P_\theta(T_n-\theta)>a-\delta)$ with T_n being a translation-equivariant estimator of θ tend to 0 as $\delta\downarrow 0$ at most n-times faster than $F(-a+\delta)$ and $1-F(a-\delta)$, respectively. It is proved that the upper bounds are attained by every L-estimator T_n which puts positive weights on the extreme observations while the upper bound cannot be attained if the extreme observations are trimmed-off. It among others means that, in the case of distribution with compact support, the sample mean dominates the sample median.

Key words: Translation-equivariant estimator, L-estimator, distribution with compact support.

Classification: 62F11, 62G05

---

1. Introduction. Let $X_1,X_2,\dots$ be a sequence of independent random variables, identically distributed according to an absolutely continuous distribution function $F(x-\theta)$ with the density $f(x-\theta)$ such that

(1.1) $f(x)>0$ for $-a<x<a$, $a>0$

$f(x)=0$ for $x\leq -a$, $x\geq a$,

(1.2)
$$\lim_{\delta\downarrow 0}\delta^{-\alpha}F(-a+\delta)=A$$
$$\lim_{\delta\downarrow 0}\delta^{-\beta}\,1-F(a-\delta))=B$$

and

(1.3)
$$\lim_{\delta\downarrow 0} \delta^{1-\alpha} f(-a+\delta) = A'$$
$$\lim_{\delta\downarrow 0} \delta^{1-\beta} f(a-\delta) = B'$$

where α, β are finite positive constants and A, A', B, B' are finite positive numbers. The problem is that of estimating the location parameter Θ.

The asymptotic theory of estimation of location of the distribution with the compact support was developed by Akahira [1],[2],[3]. He dealt with the existence of consistent estimator of Θ, with the rate of the consistency and with the asymptotic distribution of the estimators. Under the assumption that f is twice differentiable, satisfies (1.1),(1.3) and

(1.4)
$$\lim_{\delta\downarrow 0} \delta^{2-\alpha} |f'(-a+\delta)| = A''$$
$$\lim_{\delta\downarrow 0} \delta^{2-\beta} |f'(a-\delta)| = B'', \quad 0 < A'', B'' < \infty$$

and if $f''(x)$ is bounded in the case $\gamma = \min(\alpha, \beta) \geq 2$, Akahira [1] proved the existence of c_n-consistent estimator of Θ with c_n depending on γ. While $c_n = n^{1/2}$ if $\gamma > 2$ and $c_n = (n.\log n)^{1/2}$ if $\gamma = 2$ and the corresponding c_n-consistent estimator is e.g. the maximum likelihood estimator, $c_n = n^{1/\gamma}$ if $0 < \gamma < 2$ and one of the possible consistent estimators is

(1.5) $$T_n = \frac{1}{2}\left(X_n^{(1)} + X_n^{(n)}\right)$$

where $X_n^{(1)} \leq \ldots \leq X_n^{(n)}$ are the order statistics corresponding to $X_1, \ldots, X_n$.

Moreover, Akahira proved in [3] that the statistic

$(X_n^{(1)}, X_n^{(n)})$ is asymptotically sufficient as $n \to \infty$ in the sense of LeCam [10], whatever is the value of $\gamma > 0$.

Let us restrict attention to the translation-equivariant estimators, i.e. to the estimators T_n satisfying (2.1) below and among them to the estimators satisfying the natural condition $X_n^{(1)} \le T_n \le X_n^{(n)}$.
We shall consider the finite-sample behaviour of the estimators; more precisely, the behaviour of the probabilities

(1.6) $P_\theta(T_n - \theta < -a + \delta)$, $P_\theta(T_n - \theta > a - \delta)$

for small values of $\delta > 0$. These probabilities tend to 0 as $\delta \downarrow 0$; we expect from a good estimator T_n that the probabilities in (1.6) tend to 0 as fast as possible.

More authors have considered similar measure of performance of estimators of location in the case that the underlying distribution is extended over all real line (Bahadur [4] and [5], Fu [6], Sievers [11] have considered the case $n \to \infty$, Jurečková [7],[9] has considered the tail-behaviour of estimators for a fixed n). The present paper is an extension of the author's results of [7] and [9] to the non-regular case of distributions with compact support. Some of the present results are analogous to those being valid in the regular case while other results are quite different. It turns out that the statistics $(X_n^{(1)}, X_n^{(n)})$, being proved by Akahira as asymprotically sufficient, play a fundamental role for the distributions extended over a bounded interval, regardless of the values of $\gamma = \min(\alpha, \beta)$.

We shall show that the rate of convergence of probabilities in (1.6) to 0 is at most n-times faster than the rate of convergence of $F(-a+\delta)$ and $(1-F(a-\delta))$ to 0, respectively, as $\delta \downarrow 0$. Similarly as in Jurečková [7], we shall prove that trimming-off the extreme observations restricts the scope of possible rates of convergence for an L-estimator of Θ and the convergence is $[(n+1)/2]$ -times faster than that of $F(-a+\delta)$ or $(1-F(a-\delta))$ in the case of the sample median (if $n=2k+1$).

On the other hand, unlike in the regular case, we shall show that the estimator (1.5), or more generally,

$$(1.7) \quad T_n = \lambda X_n^{(1)} + (1-\lambda)\, X_n^{(n)}, \; 0 < \lambda < 1$$

attains the upper bound in the rate of convergence of (1.6). We shall even prove that the same property has the sample mean and more generally, that the same property has every L-estimator $T_n = \sum_{i=1}^{n} c_i \; X_n^{(i)}$ such that $c_1 > 0$, $c_n > 0$, whatever are the values α, β. It among others implies that the sample mean dominates the sample median with respect to the tail-behaviour in the non-regular cases.

The lower and upper bounds on the rate of convergence are derived in Section 2. Section 3 then investigates the tail-behaviour of L-estimators of Θ.

2. Lower and upper bounds on the rate of convergence.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, identically distributed according to the distribution function $F(x-\Theta)$ which has the density $f(x-\Theta)$ such that F and f satisfy (1.1) - (1.3). Let $T_n = T_n(X_1,\ldots,X_n)$ be an estimator of Θ based on $X_1,\ldots,X_n$. We shall restrict our considerations to the estimators which are translation-equivariant, i.e. which satisfy

(2.1) $T_n(X_1+c,\dots,X_n+c) = T_n(X_1,\dots,X_n) + c,\ c \in R^1$

and moreover, which are such that

(2.2) $X_n^{(1)} \leq T_n \leq X_n^{(n)}$

where $X_n^{(1)} \leq \dots \leq X_n^{(n)}$ are the order statistics of $(X_1,\dots$ $\dots,X_n)$. Denote

$$(2.3)\qquad B^-(T_n,\delta) = \frac{-\log P_\theta(T_n-\theta < -a+\delta)}{-\log F(-a+\delta)}$$

and

$$(2.4)\qquad B^+(T_n,\delta) = \frac{-\log P_\theta(T_n-\theta > a-\delta)}{-\log(1-F(a-\delta))},\quad 0 < \delta < 2a.$$

It is desirable to find an estimator T_n for which the probabilities

(2.5) $P_\theta(T_n < -a+\delta),\ P_\theta(T_n-\theta) > a-\delta)$

tend to 0 as $\delta \downarrow 0$ as fast as possible. The following theorem shows that the rates of convergence in which $F(-a+\delta)$ and $1-F(a-\delta)$ tend to 0, respectively, provide a natural upper and lower bounds on the rate of convergence of probabilities in (2.5). Analogous bounds appeared in the regular case (see [9]).

Theorem 2.1. Let $X_1, X_2,\dots$ be independent random variables, identically distributed according to an absolutely continuous distribution function $F(x-\theta)$ with the density $f(x-\theta)$ such that F and f satisfy (1.1) - (1.3). Then, for every translation-equivariant estimator $T_n = T_n(X_1,\dots,X_n)$ of θ satisfying (2.2), it holds

(2.6) $\quad 1 \leq \varliminf_{\delta \downarrow 0} B^-(T_n, \delta) \leq \varlimsup_{\delta \downarrow 0} B^-(T_n, \delta) \leq n$

and

(2.7) $\quad 1 \leq \varliminf_{\delta \downarrow 0} B^+(T_n, \delta) \leq \varlimsup_{\delta \downarrow 0} B^+(T_n, \delta) \leq n.$

Proof. It holds

$$P_\theta(T_n - \theta < -a+\delta) = P_0(T_n < -a+\delta) \leq P_0(X^{(1)} < -a+\delta)$$

$$= 1 - (1-F(-a+\delta))^n = F(-a+\delta) \sum_{j=0}^{n-1} (1-F(-a+\delta))^j \leq$$

$$\leq nF(-a+\delta)$$

so that

$$\varliminf_{\delta \downarrow 0} B^-(T_n, \delta) \geq 1.$$

Similarly,

$$P_0(T_n < -a+\delta) \geq P_0(X^{(n)} < -a+\delta) = (F(-a+\delta))^n,$$

thus $\varlimsup_{\delta \downarrow 0} B^-(T_n, \delta) \leq n$. The proof for $B^+(T_n, \delta)$ is analogous.

3. Tail-behaviour of L-estimators. Taking the lower and upper bounds of Section 2 into account, we are interested in the behaviour of various estimators from this point of view. A broad class of estimators satisfying (2.1) and (2.2) is that of L-estimators of the form

(3.1) $\quad T_n = \sum_{i=1}^{n} c_i X_n^{(i)}$

with $c_i \geq 0$, $i=1,\dots,n$; $\sum_{i=1}^{n} c_i = 1$. This class covers the sample mean, the sample median as well as the estimators (1.7). The following theorem shows that trimming-off $X_n^{(1)},\dots,X_n^{(k_1)}$ increases the lower bound in (2.6) by k_1 and decreases the upper bound in (2.7) by k_1; an analogous effect provides trimming-off $X_n^{(n-k_2+1)},\dots,X_n^{(n)}$.

Theorem 3.1. Let $T_n = \sum_{i=1}^{n} c_i X_n^{(i)}$ be an L-estimator of

Θ based on $X_1,\dots,X_n$ and let the distribution of $X_i - \Theta$ satisfy (1.1) - (1.3), $i=1,\dots,n$. Put $c_0 = c_{n+1} = 0$. Then, if $c_i = 0$ for $0 \leq i \leq k_1$ and for $n-k_2+1 \leq i \leq n+1$, where $0 \leq k_1 + k_2 < n$, it holds

(3.2) $$k_1+1 \leq \underline{\lim}_{\delta \downarrow 0} B^-(T_n,\delta) \leq \overline{\lim}_{\delta\downarrow 0} B^-(T_n,\delta) \leq n-k_2$$

and

(3.3) $$k_2+1 \leq \underline{\lim}_{\delta \downarrow 0} B^+(T_n,\delta) \leq \overline{\lim}_{\delta\downarrow 0} B^+(T_n,\delta) \leq n-k_1 .$$

Proof. It follows from the assumptions of the theorem that $X_n^{(k_1+1)} \leq T_n \leq X_n^{(n-k_2)}$. Then

$$P_0(T_n < -a+\delta) \leq P_0(X_n^{(k_1+1)} < -a+\delta) \leq \frac{n}{k_1+1} \binom{n-1}{k_1} (F(-a+\delta))^{k_1+1}$$

which implies the first inequality in (3.2). The remaining inequalities are proved analogously.

Corollary. Let T_n be the median of the sample $X_1,\dots,X_n$ from a distribution $F(x-\Theta)$ satisfying (1.1) - (1.3). Then

(3.4) $$\frac{n}{2} \leq \underline{\lim}_{\delta \downarrow 0} B^{\mp}(T_n,\delta) \leq \overline{\lim}_{\delta\downarrow 0} B^{\mp}(T_n,\delta) \leq \frac{n}{2} + 1 \text{ for } n \text{ even}$$

and

(3.5) $$\lim_{\delta \downarrow 0} B^-(T_n,\delta) = \lim_{\delta \downarrow 0} B^+(T_n,\delta) = \frac{n+1}{2} \text{ for } n \text{ odd}.$$

The behaviour of L-estimators described in Theorem 3.1 and its corollary is quite analogous as in the regular case (see [7] and [9]). The situation is quite different in the case of L-estimators which put positive weights on the extreme observations, i.e. for which $c_1 > 0$, $c_n > 0$. The following theorem states that any such L-estimator attains the upper bounds both in (2.6) and (2.7), whatever are the values α, β in (1.2) and (1.3). This among others implies that, in the case of the sample from a distribution with the compact support, such estimators as the sample mean and the estimators of the type (1.7) have more favourable tail-behaviour than the sample median. The results may be surprising but they are consistent with Akahira's result on the asymptotic sufficiency of $(X_n^{(1)}, X_n^{(n)})$.

Theorem 3.2. Let $X_1, \dots, X_n$ be independent random variables, identically distributed according to the distribution function $F(x-\theta)$ such that F and its density f satisfy (1.1) - (1.3). Let $T_n = \sum_{i=1}^{n} c_i X_n^{(i)}$ be an L-estimator of θ such that

(3.6) $c_1 > 0, \; c_n > 0.$

Then it holds

(3.7) $$\lim_{\delta \downarrow 0} B^-(T_n, \delta) = \lim_{\delta \downarrow 0} B^+(T_n, \delta) = n.$$

The theorem will be proved with the aid of the following lemma.

Lemma 3.1. Let T_n be the estimator of the form

(3.8) $$T_n = \lambda X_n^{(1)} + (1-\lambda) X_n^{(n)}, \quad 0 < \lambda < 1.$$

Then, under the assumptions (1.1) - (1.3), it holds

(3.9) $$\lim_{\delta \downarrow 0} B^-(T_n, \delta) = \lim_{\delta \downarrow 0} B^+(T_n, \delta) = n.$$

Proof of Lemma 3.1. From the well-known joint density of $X_n^{(1)}$ and $X_n^{(n)}$, we could easily derive the density of T_n; it has the form

$$
(3.10)\quad g(t) = \begin{cases} \dfrac{n(n-1)}{1-\lambda} \displaystyle\int_{-a}^{t} \left[F\left(\frac{t-\lambda u}{1-\lambda}\right) - F(u)\right]^{n-2} f\left(\frac{t-\lambda u}{1-\lambda}\right) f(u)\, du & \ldots\ -a < t < a(1-2\lambda) \\ \dfrac{n(n-1)}{\lambda} \displaystyle\int_{t}^{a} \left[F(u) - F\left(\frac{t-(1-\lambda)u}{\lambda}\right)\right]^{n-2} f\left(\frac{t-(1-\lambda)u}{\lambda}\right) f(u) & \ldots\ a(1-2\lambda) < t < a \\ 0 & \ldots\ \text{otherwise} \end{cases}
$$

so that, for $0 < \delta < 2(1-\lambda)a$,

$$
P_0(T_n < -a+\delta) = n \int_{-a}^{-a+\delta} \left[F\left(\frac{-a+\delta-\lambda u}{1-\lambda}\right) - F(u)\right]^{n-1} f(u)\, du
$$

(3.11)

$$
\leq n \left[F\left(-a + \frac{\delta}{1-\lambda}\right)\right]^{n-1} . F(-a+\delta).
$$

It follows from (1.2) that, to any >0, there exists a $\delta_0 > 0$ such that

$$
(3.12)\quad (A-\varepsilon)\delta^{\alpha} \leq F(-a+\delta) < F\left(-a+\frac{\delta}{1-\lambda}\right) \leq (A+\varepsilon)(1-\lambda)^{-\alpha}\delta^{\alpha}
$$

holds for $\delta \in (0, \delta_0)$, so that

$$
(3.13)\qquad B^{-}(T_n, \delta) \geq \frac{-\log[n(A+\varepsilon)^{n}(1-\lambda)^{-n\alpha}] - n\alpha \cdot \log \delta}{-\log(A-\varepsilon) - \alpha \cdot \log \delta}
$$

holds for $0 < \delta < \delta_0$ and this implies that

$$
(3.14)\quad \underline{\lim}_{\delta \downarrow 0} B^{-}(T_n, \delta) \geq n.
$$

If we put $Y_i = -X_i$, $i=1,\ldots,n$ and

$$
(3.15)\quad T_n' = (1-\lambda)\, Y_n^{(1)} + \lambda Y_n^{(n)} = -T_n
$$

we get quite analogously that

$$
(3.16)\quad \underline{\lim}_{\delta \downarrow 0} B^{+}(T_n, \delta) \geq n.
$$

The lemma then follows from (3.14), (3.16) and from Theorem 3.1.

Proof of Theorem 3.2. Let $T_n = \sum_{i=1}^{m} c_i X_n^{(i)}$ be an L-estimator such that $c_1 > 0$, $c_n > 0$. Then

(3.17) $T_n^{(1)} \leq T_n \leq T_n^{(2)}$

where

(3.18) $T_n^{(j)} = \lambda_j X_n^{(1)} + (1-\lambda_j)\, X_n^{(n)}$, $j=1,2$

and

(3.19) $\lambda_1 = \sum_{i=1}^{m-1} c_i$, $\lambda_2 = c_1$.

Then $0 < \lambda_j < 1$, $j=1,2$ and it follows from (3.17) that

$$P_0(T_n < -a+\delta) \leq P_0(T_n^{(1)} < -a+\delta)$$
(3.20)
$$P_0(T_n > a-\delta) \leq P_0(T_n^{(2)} > a-\delta).$$

Theorem 3.2 then follows from Lemma 3.1 and from Theorem 3.1.

References

[1] M. AKAHIRA: Asymptotic theory for estimation of location in non-regular case, I: Order of convergence of consistent estimators, Rep. Stat. Appl. Res., JUSE 22(1975), 8-26.

[2] M. AKAHIRA: Asymptotic theory for estimation of location in non-regular cases, II: Bounds of asymptotic distributions of consistent estimators, Rep. Stat. Appl. Res., JUSE 22(1975), 99-115.

[3] M. AKAHIRA: A remark on asymptotic sufficiency of statistics in non-regular case, Rep. Univ. Electro-Comm. 27(1976), 125-128.

[4] R.R. BAHADUR: Rates of convergence of estimates and test statistics, Ann. Math. Statist. 38(1967), 303-324.

[5] R.R. BAHADUR: Some Limit Theorems in Statistics, SIAM, Philadelphia (1971).

[6] J.C. FU: The rate of convergence of point estimators, Ann. Statist. 3(1975), 234-240.

[7] J. JUREČKOVÁ: Finite-sample comparison of L-estimators of location, Comment. Math. Univ. Carolinae 20 (1979), 509-518.

[8] J. JUREČKOVÁ: Rate of consistency of one-sample tests of location, Journ. Statist. Planning and Inference 4(1980), 249-257.

[9] J. JUREČKOVÁ: Tail-behavior of location estimators, Ann. Statist. 9(1981), No 3.

[10] L. LeCAM: On the asymptotic theory of estimation and testing hypotheses, Proc. 3rd Berkeley Symp. I(1956) 129-156.

[11] G.L. SIEVERS: Estimation of location: A large deviation comparison, Ann. Statist. 6(1978), 610-618.

Matematicko-fyzikální fakulta

Universita Karlova

Sokolovská 83, 186 00 Praha 8

Československo

(Oblatum 22.1. 1981)