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A TERNARY VARIETY GENERATED BY LATTICES
William H. CORNISH

Abstract: The class of all subreducts of lattices, with respect to the ternary lattice-polynomial $s(x,y,z) = x \wedge (y \vee z)$, is a 4-based variety. Within its lattice of subvarieties, the subvariety of distributive supremum algebras is atomic and needs 4 variables in any equational description.

Key words: Subreduct, lattice, nearlattice, distributivity.

Classification: 03C05, 06A12, 06D99, 08B10

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0. **Introduction.** Consider the variety of algebras $(A;s)$ of type $\langle 3 \rangle$ that satisfy the following identities

(S1) $s(x,x,y) = x$,

(S2) $s(x,y,y) = s(y,x,x)$,

(S3) $s(x,y,z) = s(x,z,y)$,

(S4) $w \wedge s(x,y,z) = s(w \wedge x,y,z)$,

(S5) $s(x,s(y,v,w), s(z,v,w)) \leq s(x,v,w)$,

where $\wedge$ is the derived operation given by

(D1) $x \wedge y = s(x,y,y)$,

and $\leq$ is the derived relation defined by

(D2) $x \leq y$ if and only if $x = x \wedge y$.

Then, we show that an algebra is in this variety if and only if it is a subalgebra of a reduct $(L;s)$, where $(L; \wedge, \vee)$ is

a lattice and $s(x,y,z) = x \wedge (y \vee z)$. The variety generated a
single such lattice-reduct $(L;s)$ has already been studied
by Baker [1]; he showed that the variety is congruence-4-
distributive but not 3-distributive. Of course, his result
extends to our larger variety. However, this can be obtain-
ed from Hickman [2], as each reduct $(A;j)$ of an algebra $(A;s)$,
where j is the derived ternary operation given by

    (D3) $j(x,y,z) = s(y,x \wedge y,y \wedge z)$,

is a join algebra. We demonstrate this by showing that each
algebra is a nearlattice wherein $j(x,y,z) = (x \wedge y) \vee (y \wedge z)$.
Baker's result has already been exploited by Berman [2].

    In the presence of (S1) - (S4), (S5) is implied by

    (S6) $w \wedge s(x,y,z) = s(x,w \wedge y,w \wedge z)$.

The subvariety defined by the five identities (S1) - (S4) and
(S6), consists of subalgebras of reducts of distributive lat-
tices and so is the variety generated by $(2;s)$, where $(2; \wedge , \vee)$
is the two element lattice. This variety was described by
Lyndon [8; Theorem 3] in 1951, using nine identities.

    1. <u>Subreducts</u>. Before proceeding to the main results,
we would like to make some remarks on <u>subreducts</u>, that is
subalgebras of reducts, of the algebras in a variety.

    Let $\underline{V}$ be a variety of finitary algebras, p be an n-ary
$\underline{V}$-polynomial, and $\underline{V}_p$ be the class of all subalgebras of al-
gebras having the form $(A;p)$, where A is the underlying set
of a $\underline{V}$-algebra. Then, $\underline{V}_p$ is a class of algebras of type $\langle n \rangle$,
but it is not necessarily a variety. For example, if $\underline{V}$ is
the variety of Abelian groups and p is the group-product,
then $\underline{V}_p$ is the class of all commutative cancellative semi-

groups and the cancellation law can fail in homomorphic i-
mages of such semigroups. In general, $\underline{V}_p$ is closed under
the formation of products and subalgebras. A sufficient
condition to ensure closure under homomorphic images is as
follows.

Suppose p is <u>congruence-determining</u> in the sense that
the $\underline{V}$-congruences on a $\underline{V}$-algebra are precisely the $\underline{V}_p$-cong-
ruences on its $\underline{V}_p$-reduct. Also, assume that $\underline{V}_p$ enjoys the
Congruence Extension Property. Then, it is immediate that
$\underline{V}_p$ is closed under homomorphic images and so is a variety.
In the above case concerning Abelian groups, p is congruen-
ce-determining but the Congruence Extension Property fails
for commutative cancellative semigroups. An instance where
these conditions hold is supplied by the case of $\underline{V}$ being
the variety of Boolean algebras $(A; \wedge, \vee, ', 0, 1)$ and p being
the binary polynomial $p(x,y) = x^*y = x \wedge y'$. Here, $\underline{V}_p$ is the
variety of implicative BCK-algebras; for an account, see [3]
and especially Corollary 1.5, therein. Thus, $\underline{V}_p$ has the Con-
gruence Extension Property; in fact, any variety of BCK-al-
gebras has the Congruence Extension Property, [5; Theorem
3.1]. It is not hard to show that $p(x,y) = x^*y$ is congruen-
ce-determining on a Boolean algebra, or alternatively, this
can be deduced from [4, Corollary 3.13].

However, each of these two conditions is not necessary
for $\underline{V}_p$ to be a variety. For instance, when $\underline{V}$ is the variety
of (distributive) lattices $(A; \wedge, \vee)$ and p is given by
$p(x,y) = x \wedge y$, $\underline{V}_p$ is the variety of semilattices; here, $\underline{V}_p$
has the Congruence Extension Property but p is not congruence

-determining. On the other hand, the main result of this paper, as described in Section 0, supplies an example where the Congruence Extension Property fails (because it fails in any variety of non-distributive lattices) and p, that is s, is congruence-determining. Actually, Baker commented on this property of s in [1; Note 1, p. 143]; this can also be seen from Hickman [7; Proposition 2.2], since (D3) of Section 0 gives Hickman's j as a polynomial in terms of s. Another such example is provided by Hickman's join algebras, see [7; Proposition 1.1(iii), Theorem 2.1]. Thus, the main result of this note has a significance with respect to a model-theoretic problem.

2. **Main results.** A partially ordered set is said to have the <u>upper bound property</u> if any two elements possess a supremum, whenever they have a common upper bound. A <u>nearlattice</u> is a lower semilattice with the upper bound property. In any nearlattice $(A; \wedge, \doteq)$, $j(x,y,z) = (x \wedge y) \vee (y \wedge z)$ is defined on the whole of A. The resulting ternary algebras $(A; j)$ are equationally definable and are called <u>join algebras</u>. Actually, in [7; Theorem 2.1, Proposition 2.4], Hickman showed that the variety of join algebras and their homomorphisms is isomorphic to the category of nearlattices and their nearlattice-homomorphisms; the isomorphism commutes with the forgetful functors to the category of sets. We will give no further details; additional information on nearlattices and their congruences etc. can be found in [4; Section 3]. It should be mentioned, however, that a lower semilattice is a

nearlattice if and only if the new semilattice, that results from the addition of a largest element, is really a lattice. Hence, the variety of join algebras is the variety $\underline{V}_p$, when $\underline{V}$ is the variety of lattices and p is j.

We make use of the notation of Section O. A ternary algebra (A;s), satisfying (S1) - (S5), will be called a supremum algebra.

Lemma 2.1. Let (A;s) be a supremum algebra. Then, $(A;\wedge,\leqq)$ is a nearlattice. For any $a,b,c \in A$, $s(a,b,c) =$ $= a \wedge (b \vee c)$, when $b \vee c$ exists.

Proof. By (S1) and (D1), $a \wedge a = s(a,a,a) = a$. From (S2), $a \wedge b = b \wedge a$. From (S4), $a \wedge (b \wedge c) = a \wedge s(b,c,c) = s(a \wedge b,c,c) =$ $= (a \wedge b) \wedge c$. Thus, (D2) says that $(A;\wedge,\leqq)$ is a lower semilattice, wherein $\leqq$ is the induced partial order.

Suppose $b,c \leqq a$ and let $d = s(a,b,c)$. Due to (S4) and (S1), $b \wedge d = s(b \wedge a,b,c) = s(b,b,c) = b$, i.e. $b \leqq d$. Similarly, $c \leqq d$, and d is a common upper bound of b and c. Let e be another such bound. Then, $d = s(a,b,c) = s(a,b \wedge e,c \wedge e) =$ $= s(a,s(b,e,e), s(c,e,e)) \leqq s(a,e,e) = a \wedge e \leqq e$, by (S5). Hence, $d = b \vee c$.

When $b \vee c$ exists, the above reasoning shows that $b \vee c =$ $= s(b \vee c,b,c)$. Hence, $a \wedge (b \vee c) = a \wedge s(b \vee c,b,c) = s(a \wedge (b \vee c,),$ $b,c) = (b \vee c) \wedge s(a,b,c) \leqq s(a,b,c)$, due to two applications of (S4). But due to (S5), $s(a,b,c) = s(a,b \wedge (b \vee c),c \wedge (b \vee c)) =$ $= s(a,s(b,b \vee c,b \vee c), s(c,b \vee c,b \vee c)) \leqq s(a,b \vee c,b \vee c) =$ $= a \wedge (b \vee c)$. Hence, $s(a,b,c) = a \wedge (b \vee c)$.

Notice that the lemma implies that j, as defined by (D3), is given by $j(x,y,z) = (x \wedge y) \vee (y \wedge z)$ on the underlying near-

lattice of a supremum algebra. Hence, the reduct $(A;j)$ of a supremum algebra $(A;s)$ is a join algebra, and so, by Hickman [7], we obtain

Corollary 2.2. <u>The variety of supremum algebras is congruence-4-distributive, but not congruence-3-distributive.</u>

In order to establish the characterization of supremum algebras as subreducts, we need to introduce the appropriate ideal-theoretic concepts. An <u>s-ideal</u> of a supremum algebra $(A;s)$ is a non-empty subset K such that, for any $a \in A$ and $k_1, k_2 \in K$, $s(a, k_1, k_2) \in K$, When $a \leq k$ and $k \in K$, $a = s(a,k,k)$, and so $a \in K$. If $k_1, k_2 \in K$ and $k_1 \vee k_2$ exists, then Lemma 2.1 implies $k_1 \vee k_2 = s(k_1 \vee k_2, k_1, k_2)$ and so $k_1 \vee k_2 \in K$. Thus, an s-ideal is a <u>nearlattice-ideal</u> of the underlying nearlattice. Nevertheless, the true nature of s-ideals is a mystery to us, even though we can exploit them.

When K_1 and K_2 are s-ideals of a supremum algebra $(A;s)$, there are $k_1 \in K$ and $k_2 \in K$ and so, $k_1 \wedge k_2 = s(k_1 \wedge k_2, k_1, k_2) = = s(k_1 \wedge k_2, k_2, k_2)$, says that $k_1 \wedge k_2$ is in the set-intersection $K_1 \cap K_2$. It follows that $K_1 \cap K_2$ is an s-ideal. Also, let $T_0 = \{a \in A : a = s(a, k_1, k_2), k_1 \in K, k_2 \in K_2\}$ and $T_{n+1} = \{a \in A : :a = s(a, t_1, t_2), t_1, t_2 \in T_n\}$ for $n \geq 1$. Then, we have inductively defined the sequence $K_1, K_2 \subseteq T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq T_{n+1} \subseteq \ldots$. Moreover, it is not hard to show that $\cup \{T_n : n \geq 0\}$ is the smallest s-ideal containing both K_1 and K_2. Hence, when ordered by set-inclusion, the s-ideals of $(A;s)$ form a lattice, wherein the infimum and supremum of s-ideals K_1 and K_2 are given by $K_1 \cap K_2$ and $K_1 \vee K_2 = \cup\{T_n : n \geq 0\}$, respectively. Also, it is not hard to see that for any $b \in A$, $(b] = \{a \in A : a \leq b\}$

is the s-ideal generated by b. In general, the s-ideal gene-
rated by a non-empty subset B of A is $\cup\{S_n : n \geq 0\}$, where $S_0 =$
$= \{a \in A : a = s(a,b_1,b_2), b_1, b_2 \in B\}$, $S_{n+1} = \{a \in A : a = s(a,r_1,r_2),$
$r_1, r_2 \in S_n\}$, for $n \geq 1$.

Lemma 2.3. Let $(A;s)$ be a supremum algebra and $a, b, c \in A$.
Then, in the lattice of s-ideals $(b] \vee (c] = \{d \in A : d =$
$= s(d,b,c)\}$, and consequently $(a] \cap ((b] \vee (c]) = (s(a,b,c)]$.

Proof. Let $T = \{d \in A : d = s(d,b,c)\}$. Then, $b, c \in T$, by
(S1) and (S3). Also, if $d_1 \leq d$ and $d = s(d,b,c)$ then $d_1 =$
$= (d_1,b,c)$, due to (S4). In other words, T is hereditary.
Now, if $t_1, t_2 \in T$ and $e \in A$, then, because of (S5) and (S4),
$s(e,t_1,t_2) = s(e,s(t_1,b,c), s(t_2,b,c)) \leq s(e,b,c)$, and
$s(e,b,c) = s(e,b,c) \wedge s(e,b,c) = s(s(e,b,c) \wedge e,b,c) =$
$= s(s(e,b,c),b,c)$. Hence, $s(e,b,c) \in T$, and so $s(e,t_1,t_2) \in T$,
too. Thus, T is an s-ideal containing b and c. It immediately
follows that $T = (b] \vee (c]$.

The remaining assertion follows quickly from (S4), and
its consequence that $s(a,b,c) \leq a$.

Theorem 2.4. Let $\underline{L}$ be the variety of all lattices and
s be the ternary lattice-polynomial $s(x,y,z) = x \wedge (y \vee z)$.
Then, $\underline{L}_s$ is the variety of supremum algebras.

Proof. It is easily verified that each algebra in $\underline{L}_s$
satisfies (S1) - (S5). On the other hand, a supremum algebra
$(A;s)$ is in $\underline{L}_s$, as the map $a \longrightarrow (a]$ is a supremum algebra-em-
bedding of $(A;s)$ into the s-reduct of its lattice of s-ideals.

We now turn to distributivity. A nearlattice is distri-
butive, when the infimum distributes over existent finite su-

prema. This is equivalent to the associated join algebra sa-
tisfying the identity $w \wedge j(x,y,z) = j(w \wedge x, y, w \wedge z)$, where
$x \wedge y = j(x,y,x)$; see [7; Theorem 3.3]. More importantly, a
nearlattice is distributive if and only if either each ini-
tial segment is a distributive sublattice or the lattice of
nearlattice-ideals is distributive or the finitely generat-
ed nearlattice-ideals form a distributive lattice, when or-
dered by set-inclusion. The equivalence for these last two
conditions is contained in the proof of [6; Theorem 1.2].

Theorem 2.5. The following conditions on a ternary al-
gebra (A;s) are equivalent.

(i) (A;s) is a supremum algebra satisfying the identity
(S7) $s(x,y,z) = s(x, x \wedge y, x \wedge z)$.

(ii) (A;s) is a supremum algebra, whose lattice of s-
ideals is distributive.

(iii) (A;s) satisfies the identities (S1) - (S4) and
(S6), where $\wedge$ is defined by (D1).

Proof. (i) $\Rightarrow$ (ii). It follows from (i) and Lemma 2.1
that each initial segment of the underlying nearlattice is
a distributive lattice. It also follows from (i) and Lemma
2.1 that each nearlattice-ideal is an s-ideal. Thus, the two
concepts of ideal coincide here. As the nearlattice is dis-
tributive, (ii) follows.

(ii) $\Rightarrow$ (iii) holds because a distributive lattice sa-
tisfies (S6).

(iii) $\Rightarrow$ (i). Reasoning along much the same lines as
in the proof of Lemma 2.1, we see that (iii) implies that
$(A; \wedge, \leqslant)$ is a nearlattice. Also, $s(a,b,c) = b \vee c$, whenever a is

an upper bound of both b and c. Due to (S6), $s(x,y,z) =$
$= s(x \wedge x, y, z) = x \wedge s(x,y,z) = s(x, x \wedge y, x \wedge z)$. Hence, (S7)
holds and $s(x,y,z) = (x \wedge y) \vee (x \wedge z)$. Then, (S6) says that
the infimum distributes over such a supremum. Using these
observations, it is possible to express the left side of (S5)
as a supremum of infima, and so establish (S5).

Corollary 2.6. <u>Let</u> $\underline{D}$ <u>be the variety of distributive
lattices and</u> s <u>be the ternary</u> $\underline{D}$-<u>polynomial</u> $s(x,y,z) = x \wedge (y \vee z) =$
$= (x \wedge y) \vee (x \wedge z)$. <u>Then,</u> $\underline{D}_s$ <u>is the variety of all algebras
satisfying the conditions of Theorem 2.5.</u>

Call the algebras of Theorem 2.5, <u>distributive</u> supremum
algebras. The above results show that Hickman's distributive
join algebras and distributive supremum algebras are defini-
tionally equivalent. And, perhaps, it would be more natural
to describe join algebras by $(x \wedge y) \vee (x \wedge z)$ instead of j. Co-
rollary 2.6 ensures our remarks at the end of Section 0.
Hickman's equational base for the variety of distributive su-
premum algebras has 9 identities, as does Lyndon's. Ours has
5 identitities. However, both the variety of supremum algebras,
and its subvariety of distributive algebras, can be defined
by using at most 4 identities. This is because of Corollary
2.2, and Padmanabhan and Quackenbush [9; Theorem 1].

A related question is that of the minimum number of va-
riables needed in an equational base for one of these varie-
ties.

Theorem 2.7. <u>An equational base for either supremum al-
gebras or distributive supremum algebras need at least</u> 4 <u>va-</u>

riables. And 4 is sufficient for distributive supremum al-
gebras.

Proof. Suppose 3 variables are sufficient for either
of the varieties. Because of Theorem 2.5 and (S7), 3 vari-
ables suffice for distributive supremum algebras. Hence 3
variables suffice to equationally describe distributive
join algebras. Now consider the 5-element modular non-dis-
tributive lattice. The associated join algebra is not dis-
tributive, yet all of its 3-generated join subalgebras are
distributive. This gives the desired contradiction. The re-
maining assertion follows from Theorem 2.5(iii).

When looking for examples of supremum algebras, it is
important to observe that any hereditary subset of a latti-
ce is closed under s, and so becomes a supremum subalgebra.
We do not know whether all supremum algebras arise this way.
But distributive supremum algebras do! As observed in the
proof of Theorem 2.5, the s-ideals and the nearlattice-ide-
als coincide on a distributive supremum algebra. Moreover,
the finitely generated nearlattice-ideals form a lattice
when the nearlattice is distributive, c.f. [6; Theorem 1.2].
Then, due to the upper bound property, a distributive sup-
remum algebra is a hereditary subset of its lattice of fi-
nitely generated s-ideals.

Another feature of distributive supremum algebras is
that they are s-isomorphic if and only if they are order-i-
somorphic. In general, this is not the case. For let us con-
sider the 4-element nearlattice ., which has 3 atoms a, b,
and c, and smallest element O. It is a hereditary subset of

the following 4 lattices L_1, L_2, L_3, L_4. Here, L_1 is the Boo-
lean lattice with a,b, and c as atoms; the associated al-
gebra $A_1 = (A;s)$ is distributive, but not subdirectly irre-
ducible. Then, L_2 is the 5-element modular non-distributive
lattice; the algebra $A_2 = (A;s)$ is simple and not distribu-
tive. The lattice L_3 has 5 elements; the new elements are d
and e, $d = b \vee c$, $e = a \vee d = a \vee b \vee c$; the algebra $A_3 = (A;s)$
is subdirectly irreducible, but not simple. The fourth lat-
tice L_4 has 6 elements; the new elements are d,e,f, $d = a \vee b$,
$e = b \vee c$, $f = d \vee e$; the algebra $A_4 = (A;s)$ is also subdirect-
ly irreducible but not simple. Up to s-isomorphism, $A_1 - A_4$
are the only distinct supremum algebras which contain A as a
subnearlattice. Because of congruence-distributivity, A_2, A_3
and A_4 generate distinct varieties which cover the variety
of distributive supremum algebras. Thus, to us, the study of
the lattice of subvarieties of supremum algebras seems hope-
lessly difficult.

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