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ZERO-DIMENSIONAL OPEN MAPPINGS WHICH INCREASE DIMENSION

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Abstract: Using a notion of 0-soft rectangular diagram it is shown that any positive-dimensional compact of weight τ is an image of a 1-dimensional compact of the same weight under 0-soft mapping whose fibers are all homeomorphic to the Cantor cube D^{τ} of weight τ.

Key words: AE(0), 0-soft mapping, 0-soft diagram.

Classification: 54F45, 54C10, 54C55

Introduction. It is well-known that open mappings with zero-dimensional fibers can arbitrarily increase dimension. The first example of such mapping was described by Kolmogorov in [7]. In this example the domain is a 1-dimensional Peano continuum and the range is 2-dimensional. Then Keldyš [6] constructed a similar example where the range is a square I^2. The fact that Menger's universal curve can be mapped onto every Peano continuum by an open mapping whose fibers are all homeomorphic to the Cantor set was established by Wilson [12]. The most general result in this direction is the following important theorem of Pasynkov [10]:

Pasynkov's Theorem. Any positive-dimensional compact of weight τ is an image of a 1-dimensional (in the sense of dim) compact of the same weight under an open mapping with zero-dimensional fibers.

The purpose of this note is to prove a stronger result than the last one. Namely, the following:

<u>Theorem</u>. Any positive-dimensional compact of weight τ is an image of a 1-dimensional compact of the same weight under a 0-soft mapping whose fibers are all homeomorphic to the Cantor cube D^{τ} of weight τ.

Before getting down to the proof of this theorem, I'll recall some preliminary notions and results that will be needed.

<u>Some preliminary notions and results</u>. All the topological spaces considered will be compact (and Hausdorff), and the mappings - continuous. Under the dimension I mean the covering dimension dim. wX denotes the weight of a space X.

Recall that a compact X is an absolute extensor for zero-dimensional compacts ($X \in AE(0)$), if for any zero-dimensional compact B and for any its closed subset A any mapping $g: A \longrightarrow X$ can be extended over the whole B. Such spaces have been actively studied (see a recent survey of Ščepin [11]; note here that dimensional properties of AE(0)'s have been considered also in [4],[2]). A similar property of a mapping is 0-softness. Let us recall that a mapping $f: X \longrightarrow Y$ is said to be 0-soft if for any zero-dimensional compact B, any closed subset A of it, and any two mappings $g: A \longrightarrow X$ and $h: B \longrightarrow Y$ such that $fg = h/A$ there exists a mapping $k: B \longrightarrow X$ such that $fk = h$ and $k/A = g$.

From Ščepin's results it follows that every 0-soft mapping is open and surjective. Michael's selection theorem [9] implies that for mappings between metric compacts the converse is also true. On the other hand it is easy to see that a constant mapping of a compact X is 0-soft iff X is AE(0). So a simple example of an open but not 0-soft mapping is a constant mapping of any compact which is not AE(0). Thus the class of 0-soft mappings is strictly included in the class of open ones.

An inverse system $\{X_\alpha, p_\alpha^\beta, \lambda\}$ indexed by the ordinals less than some λ is said to be continuous if for all limit ordinals $\gamma < \lambda$, the natural mapping from X_γ to $\lim\{X_\alpha, p_\alpha^\beta, \gamma\}$ is a homeomorphism. If $S_1 = \{X_\alpha, p_\alpha^\beta, \lambda\}$ and $S_2 = \{Y_\alpha, q_\alpha^\beta, \lambda\}$ are inverse systems and for each $\alpha < \lambda$ $f_\alpha : X_\alpha \to Y_\alpha$ is a mapping such that all the rectangular diagrams that arise are commutative, then the system $\{f_\alpha : \alpha < \lambda\}$ is called a morphism between S_1 and S_2. Clearly, in this case, there exists a limit mapping $\lim f_\alpha : \lim S_1 \to \lim S_2$. I'll denote rectangular diagrams consisting of the compacts $X_{\alpha+1}$, $Y_{\alpha+1}$, X_α, Y_α and corresponding mappings $f_{\alpha+1}$, f_α, $p_\alpha^{\alpha+1}$, $q_\alpha^{\alpha+1}$ by $D(\alpha, \alpha+1)$.

Let us recall [11] also that for a commutative diagram

$$D = \begin{array}{ccc} X_2 & \xrightarrow{f_2} & Y_2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

the diagonal $p \Delta f_2$ (see [3]) of the mappings p and f_2 considered as a mapping of X_2 into the fibered product of X_1 and Y_2 with respect to f_1 and q (see [1]) is said to be a characteristic mapping of D.

Clearly, a diagram "is" a corresponding fibered product iff its characteristic mapping is a homeomorphism. Note also that a diagram is bicommutative [8] iff its characteristic mapping is surjective (see, for example, [11]).

<u>O-soft diagrams and the proof of the theorem.</u>

Definition. A rectangular commutative diagram is said to be O-soft iff its characteristic mapping is O-soft.

Lemma 1. Let D be a O-soft diagram. Then for any zero-dimensional compact B, any closed subset A of it, and any three mappings $h: B \longrightarrow Y_2$, $g: A \longrightarrow X_2$ and $k_1: B \longrightarrow X_1$ such that $f_1 k_1 = qh$, $f_2 g = h/A$ and $pg = k_1/A$ there exists a mapping $k_2: B \longrightarrow X_2$ such that $g = k_2/A$, $f_2 k_2 = h$ and $pk_2 = k_1$.

Proof. Let Z be a fibered product of X_1 and Y_2 with respect to f_1 and q; let $q': Z \longrightarrow X_1$ and $f_1': Z \longrightarrow Y_2$ be the canonical projections (see [1]). Clearly the diagonal $k_1 \triangle h$ maps B into Z. It is not hard to show that $(k_1 \triangle h)/A = rg$ where $r: X_2 \longrightarrow Z$ is a characteristic mapping of D. By our assumption r is O-soft. Consequently, there exists a mapping $k_2: B \longrightarrow X_2$ such that $rk_2 = k_1 \triangle h$ and $g = k_2/A$. It is easy to check that k_2 is the desired mapping. The lemma is thereby proved.

Lemma 2. Let $S_1 = \{X_\alpha, p_\alpha^\beta, \lambda\}$ and $S_2 = \{Y_\alpha, q_\alpha^\beta, \lambda\}$ be two well-ordered continuous inverse systems. Suppose that $\{f_\alpha, \alpha < \lambda\}$ is a morphism between S_1 and S_2 such that for each $\alpha < \lambda$ (λ is a limit number) diagrams $D(\alpha, \alpha+1)$ are O-soft. Then the limit mapping $f = \lim f_\alpha : \lim S_1 \longrightarrow \lim S_2$ is O-soft whenever f_0 is O-soft mapping.

Proof. Suppose that B is a zero-dimensional compact. A is a closed subset of B and $g: A \longrightarrow \lim S_1$ and $h: B \longrightarrow \lim S_2$ are mappings such that $fg = h/A$. Let us prove that there is an extension $k: B \longrightarrow \lim S_1$ of g such that $fk = h$. Set

$g_\alpha = p_\alpha \cdot g$ and $h_\alpha = q_\alpha \cdot h$ ($\alpha < \lambda$) where p_α and q_α are the limit projections of S_1 and S_2 respectively. For $\alpha = 0$ the mapping $f_0 : X_0 \longrightarrow Y_0$ is 0-soft by assumption. Consequently, there exists an extension $k_0 : B \longrightarrow X_0$ of the mapping $g_0 : A \longrightarrow X_0$ such that $f_0 k_0 = h_0$. Suppose now that for all $\alpha < \beta$ ($\beta < \lambda$) we have already constructed extensions $k_\alpha : B \longrightarrow X_\alpha$ of the mappings g_α in such a way that all the diagrams that arise are commutative (in particular, $f_\alpha k_\alpha = h_\alpha$). If the number β is a limit ordinal, then we set $k_\beta = \lim_{\alpha < \beta} k_\alpha : B \longrightarrow X_\beta$. Obviously, $k_\beta / A = g_\beta$, $f_\beta k_\beta = h_\beta$ and $p^\beta_\alpha k_\beta = k_\alpha$ ($\alpha < \beta$).

Suppose now that $\beta = \alpha + 1$. By assumption, the diagram $D(\alpha, \alpha+1)$ is 0-soft. Consequently, we can use lemma 1 and so, there exists an extension $k_{\alpha+1} : B \longrightarrow X_{\alpha+1}$ of $g_{\alpha+1}$ such that $f_{\alpha+1} k_{\alpha+1} = h_{\alpha+1}$ and $p^{\alpha+1}_\alpha k_{\alpha+1} = k_\alpha$.

Continuing the construction, we get a family $\{k_\alpha\}$ of extensions of the mappings g_α, $\alpha < \lambda$, and all the diagrams arising are commutative. Next, let $k = \lim k_\alpha : B \longrightarrow \lim S_1$. It is easy to check that k is the desired extension of g. Thus f is 0-soft. The lemma is proved.

Lemma 3. Any positive-dimensional compact Y of weight τ is an image of a 1-dimensional compact X of the same weight τ under 0-soft mapping with zero-dimensional fibers.

Proof (by induction on the weight of Y). Suppose that $wY = \aleph_0$, i.e., Y is metrizable. It follows from Pasynkov's theorem mentioned in the introduction that Y is an image of a 1-dimensional metrizable compact under an open mapping with zero-dimensional fibers. But every open mapping between metrizable compacts is 0-soft.

Suppose now that the lemma has been proved for all positive-dimensional compact spaces Y of weight $< \tau$, and let $wY = \tau$. As it is well-known, we can represent Y as a limit space of a continuous well-ordered inverse system $S_Y = \{Y_\alpha, q_\alpha^\beta, \lambda\}$ where all Y_α's are compacts with $wY_\alpha < \tau$. Without less of generality we can assume that each Y_α is positive-dimensional.

Now let us construct a continuous well-ordered inverse system $S_X = \{X_\alpha, p_\alpha^\beta, \lambda\}$ "parallel" to S_Y and a morphism $\{f_\alpha\} : S_X \longrightarrow S_Y$ such that $\lim S_X$ and $\lim f_\alpha$ will be the desired objects.

By the inductive hypothesis Y_0 is an image of a 1-dimensional compact X_0 with $wX_0 = wY_0$ under a 0-soft mapping f_0 with zero-dimensional fibers. Suppose now that for all $\alpha < \gamma$ ($\gamma < \lambda$) we have already constructed:

a) 1-dimensional compacts X_α with $wX_\alpha = wY_\alpha$;

b) 0-soft mappings $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ with zero-dimensional fibers;

c) mappings $p_\alpha^\beta : X_\beta \longrightarrow X_\alpha$ ($\alpha < \beta < \gamma$)

such that

1) all the diagrams $D(\alpha, \alpha+1)$ are 0-soft ($\alpha < \gamma$);

2) $p_\alpha^\beta = p_\alpha^\delta \cdot p_\delta^\beta$ for every α, δ and β with $\alpha < \delta < \beta < \gamma$.

Let us construct X_γ, f_γ and p_α^γ ($\alpha < \gamma$).

If the number γ is a limit ordinal, then we set $X_\gamma = \lim \{X_\alpha, p_\alpha^\beta, \gamma\}$ and $f_\gamma = \lim\limits_{\alpha < \gamma} f_\alpha$. Obviously, $\dim X_\gamma = 1$, $wX_\gamma = wY_\gamma$. It is easy to check (see [1]) that f_γ has a zero-dimensional fibers. By lemma 2, f_γ is 0-soft. Clearly, p_α^γ are the limit projections of an inverse system $\{X_\alpha, p_\alpha^\beta, \gamma\}$.

Suppose now that $\gamma = \alpha + 1$.

Consider a fibered product Z of X_α and $Y_{\alpha+1}$ with respect to f_α and $q_\alpha^{\alpha+1}$. Let $(q_\alpha^{\alpha+1})': Z \to X_\alpha$ and $f'_\alpha : Z \to Y_{\alpha+1}$ be the canonical projections. It follows from [1] that f'_α is a surjection with a zero-dimensional fiber. It is easy to see also that f'_α is 0-soft, in particular, open and hence $\dim Z > 0$ (recall that $Y_{\alpha+1}$ is positive-dimensional). Since all the mappings considered are perfect, $wZ \geq wY_{\alpha+1}$. On the other hand, since Z is a subspace of the topological product $X_\alpha \times Y_{\alpha+1}$, we have the following inequalities: $wZ \leq \max\{wX_\alpha, wY_{\alpha+1}\} = \max\{wY_\alpha, wY_{\alpha+1}\} \leq \leq wY_{\alpha+1}$. Thus $wZ = wY_{\alpha+1} < \tau$. Again by the inductive hypothesis Z is an image of a 1-dimensional compact $X_{\alpha+1}$ with $wX_{\alpha+1} = = wZ = wY_{\alpha+1}$ under 0-soft mapping h with zero-dimensional fibers. Set $f_{\alpha+1} = f'_\alpha \cdot h$ and $p_\alpha^{\alpha+1} = (q_\alpha^{\alpha+1})' \cdot h$, Obviously, $f_{\alpha+1}$ is 0-soft as a composition of 0-soft mappings and has zero-dimensional fibers as a composition of mappings with zero-dimensional fibers. Finally, let us note that the arised rectangular diagram is 0-soft since its characterostic mapping, as it is easy to see, coincides with 0-soft mapping h.

Continuing the construction, we set $X = \lim\{X_\alpha, p_\alpha^\beta, \lambda\}$ and $f = \lim f_\alpha$. Obviously, $\dim X = 1$, $wX = wY$ and f is surjective mapping with zero-dimensional fibers. By lemma 2, f is 0-soft. The lemma is thereby proved.

Now I give a proof of the theorem stated in the introduction.

Proof. Let Y be a positive-dimensional compact of weight τ. By lemma 3, Y is an image of a 1-dimensional compact Z with $wZ = \tau$ under 0-soft mapping h with zero-dimensional fibers. Set $X = Z \times D^\tau$, where D^τ is the Cantor cube of weight τ. Let $p: X \to Z$ be a natural projection. Since D^τ is AE(0), it is easy to check (see, for example, [5]) that p is 0-soft.

Consequently, a composition $f = hp$ is also 0-soft. For each point y from Y its inverse image $f^{-1}(y)$ coincides with the product $h^{-1}(y) \times D^{\tau}$. Since $h^{-1}(y)$ is a zero-dimensional AE(0) of weight $\leq \tau$ we can conclude, using a characterization of D^{τ} given by Ščepin (see [11], theorem 1), that $f^{-1}(y)$ is homeomorphic to D^{τ}. To complete the proof, we only have to remark that $\dim X = 1$ and $wX = \tau$.

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