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SHORT BRANCHES IN RUDIN-FROLÍK ORDER

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Rudin-Frolík order of types of ultrafilters in βN has the following properties:

(1) each type of ultrafilters has at most $2^{\aleph_0}$ predecessors, [2],

(2) the cardinality of each branch is at least $2^{\aleph_0}$.

Thus, in Rudin-Frolík order the cardinality of branches can be only $2^{\aleph_0}$ or $(2^{\aleph_0})^+$. It was shown in [1] that there exists a chain order - isomorphic to $(2^{\aleph_0})^+$. Hence, the existence of a branch of cardinality $(2^{\aleph_0})^+$ is proved.

The following result solves the problem of the existence of a branch having smaller cardinality.

Theorem. In Rudin-Frolík order there exists an unbounded chain order-isomorphic to ω_1.

By the properties (1) and (2) the branch containing this chain has cardinality $2^{\aleph_0}$.

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RESULTS ON DISJOINT COVERING SYSTEMS ON THE RING OF INTEGERS

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A system of congruence classes

(1) $a_1(\mathrm{mod}\ n_1),\ a_2(\mathrm{mod}\ n_2),\ \ldots,\ a_k(\mathrm{mod}\ n_k)$

will be called a disjoint covering system (DCS) if for every integer x there is exactly one $i \in \{1, 2, \ldots, k\}$ such that $x \equiv a_i(\mathrm{mod}\ n_i)$. The integers $n_1, n_2, \ldots, n_k$ will be called moduli of (1) and their least common multiple will be called the common modulus of (1).

If $k > 1$ then no two moduli of (1) are relatively prime. This condition can be expressed in the form

(2) $$\bigwedge_{i=1}^{k} \bigwedge_{j=1}^{k} \varphi(n_i, n_j)$$

where $\varphi(x, y)$ is the formula

$$\exists z\, \exists u\, \exists v\, (z \neq 1 \wedge z.u = x \wedge z.v = y)$$

Consider more generally the formulae of the form

(3) $$\bigwedge_{i_1=1}^{k} \bigwedge_{i_2=1}^{k} \cdots \bigwedge_{i_r=1}^{k} \psi(n_{i_1}, n_{i_2}, \ldots, n_{i_r})$$

which are true for all DCS (1) with $k > 1$, where $\psi(x_1, \ldots, x_r)$ is a first-order formula with the only non-logical symbol "." for multiplying. The main result of [1] is that every such formula (3) is a consequence of (2). Hence the condition (2) is the strongest among all conditions of the form (3) which hold for all non-trivial DCS (i. e., DCS different from $\{Z\}$). The proof uses product-invariant relations, i. e. the relations which are invariant with respect to all automorphism of the semigroup $(N, .)$.

For every prime p the DCS

(4) $0 \pmod p,\ 1 \pmod p,\ \ldots,\ p-1 \pmod p$

has the following property:

The union of any subset X of (4), $1 < \mathrm{card}(X) < k$ is not a congruence class (by any modulus).

All DCS (except $\{Z\}$) with this property will be called irreducible DCS, abbreviation IDCS. There are IDCS which are not of the form (4). For example, the congruence classes

0, 4 (mod 6), 1, 3, 5, 9 (mod 10), 2 (mod 15), 7, 8, 14, 20, 26, 27 (mod 30) form an IDCS with the common modulus 30 (it is Porubský's example of a nonnatural DCS in essential). In [2] many IDCS are constructed and it is proved that an IDCS with the common modulus n exists if and only if n is a prime (then only (4) can be obtained) or n is divisible by at least three different primes. Further, an operation of splitting is defined which allows to obtain all DCS from the degenerated DCS $\{Z\} = \{0 \pmod 1\}$ and the IDCS. If only IDCS of the form (4) are used then so called natural DCS are exactly obtained.

For every prime p denote $\mathcal{F}(p) = p - 1$, and extend the function $\mathcal{F}$ to the set N by the formula $\mathcal{F}(x.y) = \mathcal{F}(x) + \mathcal{F}(y)$.

The Mycielski's conjecture stated $k \geqq 1 + \mathcal{F}(n_i)$ for every DCS (1) and every $i \in \{1, 2, \ldots, k\}$. The main result of 3 is that for all DCS which are not natural (hence e. g. for all IDCS which are not of the form (4)) it holds

(5) $$k \geq 6 + \mathcal{F}(n_i) .$$

The proof is rather complicated but elementary. The constant 6 in (5) is the best possible. We stated the hypothesis that the modulus n_i in (5) can be replaced by the common modulus of (1).

The IDCS with the common modul pqr (where p, q, r are distinct primes) are completely described, and the number of them is determined, in [4].

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The aim of this, and the subsequent note, is to announce a selection of results presented at the Colloquium on Topology held in Eger in August 1983, and at the Semester of Topology in Banach Center in April 1984. I feel that it is time to prove deeper results about Suslin sets derived from Borel sets in compact spaces.

1. By a space we mean a completely regular T_2 topological space. We denote by $\mathcal{S}(\mathfrak{M})$ the collection of Suslin sets derived from the collection of sets $\mathfrak{M}$. Recall that $\mathcal{S}(\mathcal{S}(\mathfrak{M})) = \mathcal{S}(\mathfrak{M}) \supset \mathfrak{M}_\sigma \cup \mathfrak{M}_\delta$. We denote by $\mathcal{S}_d(\mathfrak{M})$ the sets in $\mathcal{S}(\mathfrak{M})$ with disjoint Suslin representation. Denote by Σ the space ω^ω with product topology where ω has the discrete topology.

Lemma 1. Let Y be a subset of a space X. Then

(a) $Y \in \mathcal{S}$ (closed(X)) iff some closed set in $X \times \Sigma$ projects onto Y.

(b) $Y \in \mathcal{S}$ (open(X)) iff some open set in $X \times \Sigma$ projects onto Y.

(c) $Y \in \mathcal{S}$ (open(X) $\cup$ closed(X)) (= $\mathcal{S}$ (Borel(X)) iff the intersection of a closed set and a G_δ set in $X \times \Sigma$ projects onto Y.

Note that (a) is classical, and (c) is essentially due to Fremlin [Fre].

2. Theorem 1. The following conditions on a space X are equivalent:

(1a) Some Čech complete subspace of $X \times \Sigma$ projects onto X.

(1b) If X is a subspace of Z then $X \in \mathcal{S}$(Borel(Z)).

(1c) X is obtained by Suslin operation from locally compact sets in some $Z \supset X$.

(1d) There exists a complete sequence of σ-relatively open covers of X.

A space X satisfying the equivalent conditions in Theorem 1 will be called Čech-analytic (following [Fre]). To be sure note that a cover $\mathcal{U}$ of X is called σ-relatively open if $\mathcal{U} = \bigcup\{\mathcal{U}_n \mid n \in \omega\}$ such that each $\mathcal{U}_n$ is an open cover of $\bigcup \mathcal{U}_n$. It was proved in [Ž] that if $X \in \mathcal{S}$(Borel(K)) for some compactification of X, then it holds for any compactification of X. Fremlin [Fre] introduced implicitly (1a) and showed the equivalence with Žolkov's definition. If the space X is hereditarily Lindelöf then (1d) implies that X has a complete sequence of countable covers, and hence it is ω-analytic (= K-analytic in Choquet and Sneider terminology) by [F]. The following result is a solution of a problem of Fremlin.

Theorem 2. A space X is ω-analytic iff it is Čech analytic and there exists an usco-compact correspondence from a separable metric space onto X.

The proof is based on the following

Lemma 2. Let f be a perfect mapping ot X onto a metrizable space Y, and let $\{\mathcal{U}_n\}$ be a sequence of families of open sets in X. There exists a factorization $f = h \circ g$ such that $g: X \longrightarrow S$, $h: S \longrightarrow Y$ are perfect, S is metrizable, and for each n

$\{y \mid g^{-1}y \subset \cup \mathcal{U}_n\} = \cup\{\{y \mid g^{-1}y \subset U\} \mid U \in \mathcal{U}_n\}$.

3. Theorem 3. The following conditions on a space X are equivalent:

(2a) Some Čech complete subspace of $X \times \Sigma$ injectively projects onto X.

(2b) If X is a subspace of Z then $X \in \mathcal{S}_d(\mathrm{Borel}(Z))$.

(2c) X is obtained by the disjoint Suslin operation from locally compact subsets in some $Z \supset X$.

(2d) There exists a complete sequence $\{\cup\{\mathcal{M}_s \mid s \in \omega^n\} \mid n \in \omega\}$ of covers such that each $\mathcal{M}_s$ is an open cover of $M_s = \cup \mathcal{M}_s$, $M_s = \cup\{M_{si} \mid i \in \omega\}$ for each s, and if $\sigma \in \Sigma$, $M_n \in \mathcal{M}_{\sigma|n}$ then $\cap\{\overline{\cap\{M_i \mid i \leq n\}} \mid n \in \omega\} \in \cap\{M_{\sigma|n} \mid n \in \omega\}$.

A space satisfying the equivalent condition in Theorem 3 will be called Čech-Luzin. Any Čech-Luzin space X is absolutely bi-Suslin (Borel), and I do not know whether or not the converse holds.

The basic stability results follow easily from (ia) and the fact that any countable ($\neq 0$) power of Σ is homeomorphic to Σ.

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DISTINGUISHED SUBCLASSES OF ČECH-ANALYTIC SPACES

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This is a free continuation of [F_3]. Recall that if $\mathcal{F}$ is a set of families of subsets of X then a family $\{X_a \mid a \in A\}$ in X is called $\mathcal{F}$ σ-decomposable if there exist families $\{X_{an} \mid a \in A\}$ in $\mathcal{F}$, $n \in \omega$, such that $X_a = \cup\{X_{an} \mid n \in \omega\}$ for each a. So it is clear what is meant by discretely σ-decomposable. We shall call a family $\{X_a\}$ in a topological space uniformly discrete if it is discrete in the finest uniformity inducing the topology. A family $\{X_a\}$ is called isolated if it is discrete in $\cup\{X_a\}$.

Following [F-H_1], if $\varkappa$ is an infinite cardinal then a space X is called $\varkappa$-analytic (or topologically $\varkappa$-analytic, abb. T $\varkappa$-analytic) if there exists an usco-compact correspondence from the metric space $\varkappa^{\omega}$ onto X such that the image of each discrete family (equivalently, discretely decomposable family) is uniformly discretely (or discretely, resp.) σ-decomposable. If the values are disjoint, then the space is called $\varkappa$-Luzin (or topologically $\varkappa$-Luzin, resp.), and if the values are singletons or empty then we speak about point-$\varkappa$-analytic etc. spaces. Analytic means $\varkappa$-analytic for some $\varkappa$, and similarly Luzin etc. The theory of analytic and Luzin spaces was developed in [F-$H_{1,2,3}$]. A discussion of topologically analytic spaces appeared in [H-J-R].