Commentationes Mathematicae Universitatis Carolinae

Karel Čuda

Translation of nonstandard definitions to standard ones

Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 4, 615--634

Persistent URL: http://dml.cz/dmlcz/106329

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

25,4 (1984)

TRANSLANTION OF NONSTANDARD DEFINITIONS TO STANDARD ONES

Karel ČUDA

Abstract: An algorithm translating nonstandard definitions of notions to standard version is given. Counterexamples proving that our algorithm is in a certain sense the best one are described. It appears that in general this translation is much more complicated than in the case when the notion of a limit (and other similar notions) is translated by ε-δ method.

Key words: Enlargement, standard, internal, external, monad, figure.

Classification: Primary 03H05
Secondary 26E35,54J05

Introduction. The paper is the last of the series of three papers ([Č1], [Č2]) using the same idea but applied in different branches of nonstandard methods. We find a standard description of nonstandardly defined notions. Cauchy's ε - δ criterion for limits and Weierstrass' ε-δ method for exclusion of infinitesimals may serve as the first results in this area. In the paper we shall describe an algorithm which finds for any nonstandard definition of a notion its standard counterpart. The algorithm can be used also for the natural generalization of the notion "to be infinitely small", namely for the notion "to be an element of the monad of a filter". In this case we demand, however, the enlargement to be compact and we give some counterexamples proving the importance of this assumption. The algorithm uses auxiliary variables for subsets of the set we work at. Hence,in general, $\varepsilon - \delta$ method does not suffice. An example proving this fact will be also given here. Notions used in the counterexample are only by one step more complicated (from the syntactical point of view) than those ones of a limit, a derivation, etc. The counterexample proves that the given algorithm is from a certain point of view the best one - it uses namely auxiliary variables just from the power set of the "basic" set. This choice of variables is urgent and sufficient. The translation given in [N] uses auxiliary variables from increasing powersets of the basic set in dependence on the complexity of the nonstandard definition.

Both translations are complicated enough and one cannot expect that they contribute to the better understanding of the nonstandardly defined notions. The author believes that the complexity of the translation may point out the places that could be specific for nonstandard methods. Let us give here a notion of this kind. Let f_n be a sequence of real functions defined in a neighbourhood of a point x. We call a real number a to be a limit point of the sequence f_n in touch to x iff $(\exists y, y \doteq x)(\forall \alpha, IL(\alpha))(f_\alpha(y) \doteq a)$, where $y \doteq x$ means that y and x are infinitely close and $IL(\alpha)$ denotes that α is an infinitely large natural number (for a correct definition see §0). Let us note that $\lim_{n\to\infty} a_n = a$ is equivalent with $(\forall \alpha, IL(\alpha))(a_\alpha \doteq a)$ but the given notion is not equivalent to $(\forall \varepsilon)(\exists y \in o_\varepsilon(x))(\lim_{n\to\infty} f_n(y) = a)$. Some examples of nonstandard notions (syntactically) similar to the given one can be found in [H].

In the middle part of the paper we shall "word by word" modificate (using the technique of compact enlargements) the middle part of [Č2]. Also the numbering of theorems and definitions is consistent with this paper. The author believes that the replacement of this part by a citation and instructions of modification of [Č2] would spare the author's effort and some paper but could discourage possible readers.

§0 Preliminaries.

Definition 0.1: 1) A structure $^*\mathcal{U}$ is called an enlargement of $\mathcal{U}$ for c iff c, $\mathcal{P}_{fin}(c)$ are from the language of $\mathcal{U}$, $^*\mathcal{U}$ is an elementary extension of $\mathcal{U}$ and there is $a \in {}^*(\mathcal{P}_{fin}(c))$ ($^*(\mathcal{P}_{fin}(c))$ being the interpretation of $\mathcal{P}_{fin}(c)$ in $^*\mathcal{U}$) such that $(\forall x)$ $(\mathcal{U} \models x \in c \Rightarrow {}^*\mathcal{U} \models x \in a)$.

2) We call the interpretation $^*x$ of the element x of $\mathcal{U}$ in $^*\mathcal{U}$ the enlargement of x. The elements of $^*\mathcal{U}$ being such interpretations we call standard ones ($St(x) \equiv (\exists z)(x = {}^*z)$). Elements of $^*\mathcal{U}$ we call internal and subsets of $^*\mathcal{U}$ we call external. (Thus e.g. N (more exactly $\{^*n; n \in N\}$ where N denotes the set of natural numbers) is an external set, $\alpha \in {}^*N - N$ is an internal set (every natural number n is the set of all natural numbers less than n) and $^*N$ is a standard set (but having nonstandard elements).

Conventions 0.2: Sometimes we shall omit * if there is no danger of confusion (which is in use in the literature). We shall omit * mostly in the case of habitual relations and functions. Thus we write $\alpha \in {}^*N$ (instead of $\alpha \,{}^*\!\!\in {}^*N$), $x+y$ (instead of $x \,{}^*\!\!+ y$) etc. On the other hand we extend the meaning of $^*\mathcal{P}_{fin}(X)$ also for external sets X. We define $^*\mathcal{P}_{fin}(X) = \{x; (\forall t \in x)(t \in X) \& {}^*Fin(x)\}$. Thus e.g. $^*\mathcal{P}_{fin}(N) = \mathcal{P}_{fin}(N) \subset {}^*(\mathcal{P}_{fin}(N)) = {}^*\mathcal{P}_{fin}({}^*N)$. Similarly we extend the meaning of $\subset, \cap, -$ etc. for external sets.

Notations 0.3: We use n,m,k,l,... for elements of N (here we also identify n with $^*n$). We use $\alpha, \beta, \gamma, \dots$ for elements of $^*N$. We use x,y,t,u,... for internal sets, $X, Y, \rho, \sigma, \dots$ for external sets. There are also some exceptions. We use sometimes c instead of C for elements of $\mathcal{U}$, but in this case symbols $^*c$ or $^*(\mathcal{P}_{fin}(c))$ occur in its nearness.

Definition 0.4: 1) A natural number $\alpha \in {}^*N$ is called infinitely large ($IL(\alpha)$) iff $\alpha \in {}^*N-N$.

2) A real number $x \in {}^*R$ is called infinitely small ($IS(x)$) iff $(\exists \alpha, IL(\alpha))(|x| < 1/\alpha)$.

3) For c from $\mathcal{U}$ we say that a set $a \in {}^*(\mathcal{P}_{fin}(c))$ such that $c \subset a$ is c-infinitely large and use the notation $IL_c(a)$.

4) Let $\mathcal{F}$ from $\mathcal{U}$ be a filter on c. We put $\mu_{\mathcal{F}} = \{x \in {}^*c; (\forall X \in \mathcal{F})(x \in {}^*X)\}$. $\mu_{\mathcal{F}}$ is called the monad of $\mathcal{F}$. It would be also reasonable to use the notation $IS_{\mathcal{F}}(x)$ instead of $x \in \mu_{\mathcal{F}}$ as $IS(x) \equiv x \in \mu_{\mathcal{F}}$ if $\mathcal{F}$ denotes the neighbourhood filter of 0.

Note that if $^*\mathcal{U}$ is an enlargement for $\mathcal{F}$ then $\mu_{\mathcal{F}} \neq 0$ also in the case that $\bigcap \mathcal{F} = 0$.

Lemma 0.5: 1) $St(\beta) \equiv (\forall \alpha, IL(\alpha))(\beta < \alpha)$

2) Let $x \in {}^*c$ & $^*\mathcal{U}$ be an enlargement for c. $St(x) \& x \in {}^*c \equiv$ $\equiv (\forall a, IL_c(a))(x \in a)$.

Proof: 1) $\Leftarrow$ is obvious as $\neg n < n$. We prove $\Rightarrow$. Let $\alpha \leq \beta$. Put $M = \{k; k \leq \alpha\}$. $M \subseteq N$ and M is bounded from above (by β), hence M has the maximal element. Let $m = \max(M)$. If $m < \alpha$ then $m+1 \leq \alpha$ and $m+1 \in M$ which is in contradiction with $m = \max(M)$. Hence $m = \alpha$ and $St(\alpha)$.

2) $\Rightarrow$ obviously holds. We prove $\Leftarrow$. If $IL_c(a) \& x \notin c$ then $IL_c(a - \{x\})$ (see the definition of IL_c).

Definition 0.6: An enlargement $^*\mathcal{U}$ is called compact for c ($c \in \mathcal{U}$) iff for every internal set y the following property holds: $(\forall a, IL_c(a))(a \in y) \Rightarrow (\exists a_o \in \mathcal{P}_{fin}(c))(\forall a \in {}^*(\mathcal{P}_{fin}(c)), a \geq a_o)$ $(a \in y)$.

Saying that $^*\mathcal{U}$ is an (compact) enlargement we mean that $^*\mathcal{U}$ is an (compact) enlargement for every element of $\mathcal{U}$.

Fact: Every enlargement for N is compact for N.
The hint of the proof: $\mathcal{P}_{fin}(N)$ can be coded in N (e.g. using the dyadic expansion (see e.g. [Č2])). Moreover, a subset is standard iff its code is standard. Further for every IL(α) there is $\beta < \alpha$ such that β codes an IL_N set. For the proof of compactness use the co-overspreed lemma (consider the least of codes for sets in y from the definition of compactness).

Now we suppose that $\mathcal{U}$ has a set structure, too. By that we mean that $\in$ belongs to the language of $\mathcal{U}$ and extensionality holds (sets are equal iff they have the same elements). Hence $\subset, \cap, \cup, -$ have usual boolean properties. Furthermore we suppose that for every $c \in \mathcal{U}$ we have $\mathcal{P}_{fin}(c) \in \mathcal{U}$.

Theorem 0.7: If $^*\mathcal{U}$ is an enlargement compact for c and φ is an internal formula (only internal sets can be used as parameters and quantified - φ is a formula in the sense of $^*\mathcal{U}$) then the following equivalence holds $(\forall a, IL_c(a))(\forall x)$ $(\exists b, IL_c(b))\, \varphi(t,a,b,x,\vec{z}) \equiv (\forall b_o \in \mathcal{P}_{fin}(c))(\exists a_o \in \mathcal{P}_{fin}(c))$ $(\forall a \geq a_o)(\forall x)(\exists b \geq b_o)\, \varphi(t,a,b,x,\vec{z})$. Furthermore, if $\vec{z}$ are standard then the righthand side of the equivalence can be understood in the standard sense (as a usual formula of the structure $\mathcal{U}$).
Proof: see [Č].

Note that if we use an analogous equivalence for IL(α), we obtain $\varepsilon-\delta$ translations of notions (uniform) continuity, (uniform) limit, derivation etc. (see [Č]).

Theorem 0.7 may be considered as a first form of a translation algorithm. The form is suitable for the translation of notions similar to the notion of limit.

Let us consider (on the other hand) the formula $(\forall\alpha, \mathrm{IL}(\alpha))$ $(\exists a, \mathrm{IL}_c(a))(\mathrm{card}(a)<\alpha)$. This formula holds in any compact enlargement. But an example proving that there is an enlargement in which the given formula does not hold can be found in [L] (it follows the lemma 2.7.8). Hence to the given formula there is no standard counterpart which is independent on the enlargement.

Similarly: Let M be a set of ultrafilters on N described by the following nonstandard definition: $\mathcal{F}\in M \equiv (\forall\alpha, \mathrm{IL}(\alpha))$ $(\exists\beta, \beta\le\alpha)(\exists f, f: {}^*N \to {}^*N \,\&\, \mathrm{St}(f))(\forall X\in {}^*\mathcal{F} \,\&\, \mathrm{St}(X))(f(\beta)\in X)$. (Note that $\mathcal{F} = \mathrm{Fil}(f(\beta))$ in the notation from [CH H]). In the case of compact enlargement M contains all the ultrafilters. In the general case M depends on the enlargement. That is the reason why we restrict ourselves, in finding the translation, to the case of compact enlargements. It is not known to the author if there is a natural parameter and a translation algorithm for the general case. In the case of nonstandard models of arithmetic we use, as such a parameter, the standard system of the model (see [Č2]).

§1 Set considerations.

Lemma 1.1 (Quantifiers changing lemma): Let ${}^*\mathrm{Fin}(a)$, let $\rho\subseteq a$ be an external set and let $\varphi(t,\vec{z})$ be a set formula. There is an external set $\sigma\subseteq {}^*\mathcal{P}_{fin}(a)$ and a set formula $\psi(t,\vec{z})$ such that $(\forall t\in\rho)\varphi(t,\vec{z}) \equiv (\exists \bar{t}\in\sigma)\psi(\bar{t},\vec{z})$.

Moreover, σ is definable from ρ, a by the operations ${}^*\mathcal{P}_{fin}$, −.

Proof: Put $\sigma = {}^*\mathcal{P}_{fin}(a) - {}^*\mathcal{P}_{fin}(a-\rho)$, $\psi(t,\vec{z}) \equiv (\forall t\in\bar{t})\varphi(t,\vec{z})$ and consider the internal set $\{t\in a;\ \varphi(t,\vec{z})\}$.

Corollary 1.2: Let $\varphi(t,St,\vec{z})$ be a normal formula (only the quantification of internal sets is allowed) using the predicate St for elements of ${}^*c$. A set formula $\psi(t,x,\vec{z},\vec{u},u)$ can be found (by an algorithm) such that for any suitably defined external set $\sigma\subset u$ and suitably defined parameters $\vec{u}$,u we have $\varphi(t,St,\vec{z}) \equiv (\exists x\in\sigma)\psi(t,x,\vec{z},\vec{u},u)$. σ is defined from c (the external set of all the standard elements of the standard set ${}^*c$), and from an arbitrary a such that $IL_c(a)$ using the operations ${}^*\mathcal{P}_{fin}$, $\times$, −. Parameters $\vec{u}$,u are defined from a using the operations ${}^*\mathcal{P}_{fin}$, $\times$.

Proof: By the induction based on the complexity of the formula φ. For induction steps let us note the following hints: $St(t) \equiv (\exists x\in c)(t=x)$. For conjunction use, in an obvious manner, cartesian product. For negation use the dual formula and Q.ch.l. (L.1.1). For a quantifier use the commutative law for the same type of quantifiers and Q.ch.l. if necessary.

Remarks: 1) The lemma and its corollary can be generalized for several "small" external sets (instead of c) and corresponding "large" sets as parameters (see [Č2]). As an example let us mention the iterated ultrapower where "small" means enlargements with small indices and "large" enlargements with large indices.

2) If St occurs only in the prefix of φ then it is sufficient to modify only the prefix. In this case the modification and the definition of σ and $\vec{u}$ are dependent only on the syntactical form of the prefix of φ.

§2 Topological considerations.

Definition 2.1: Let $\sim$ be an equivalence relation.

1) $\mathfrak{Fig}_{\sim}(X) \equiv (\forall x,y)(x \in X \& y \sim x \Rightarrow y \in X)$ (we say that X is a figure in $\sim$).

2) $\mathrm{Fig}_{\sim}(X) = \{y;(\exists x \in X)(y \sim x)$ (the figure of X).

3) $\mu_{\sim}(x) = \mathrm{Fig}_{\sim}(\{x\})$ (the monad of x).

Fact: $\mathfrak{Fig}(\mathrm{Fig}(X))$.

Definition 2.2: 1) We use w for words defined by the following inductive definition: (i) The empty word Λ is a word.

(ii) If $w_{1/2}$ are words, then $(w_1 \times w_2)$ is a word.

(iii) If w is a word then $\mathcal{P}w$ is a word.

(iv) Each word is obtained by finitely many applications of (ii) and (iii) on empty words.

2) For $a \in {}^*(\mathcal{P}_{fin}(c))$ (standard or nonstandard) and for a word w we define the set u_a^w and the (external) equivalence $\overset{w}{\underset{a}{\equiv}}$ on u_a^w by the recursion based on the complexity of w.

(i) $u_a^\Lambda = {}^*\mathcal{P}_{fin}(a)$, $x \overset{\Lambda}{\underset{a}{\equiv}} y \equiv x \cap c = y \cap c$ (x,y have the same standard elements).

(ii) $u_a^{(w_1 \times w_2)} = u_a^{w_1} \times u_a^{w_2}$, $\langle x_1, x_2\rangle \overset{(w_1 \times w_2)}{\underset{a}{\equiv}} \langle y_1, y_2\rangle \equiv$
$\equiv x_1 \overset{w_1}{\underset{a}{\equiv}} y_1 \& x_2 \overset{w_2}{\underset{a}{\equiv}} y_2$.

(iii) $u_a^{\mathcal{P}w} = {}^*\mathcal{P}_{fin}(u_a)$, $x \overset{\mathcal{P}w}{\underset{a}{\equiv}} y \equiv \mathrm{Fig}_{\overset{w}{\underset{a}{\equiv}}}(x) = \mathrm{Fig}_{\overset{w}{\underset{a}{\equiv}}}(y)$.

Remark: For $a \in \mathcal{P}_{fin}(c)$ (i.e. standard and finite) all the equivalences are identical with the equality.

Theorem 2.3: 1) $(\forall a, \mathrm{IL}_c(a))\ \mathfrak{Fig}_{\overset{\Lambda}{\underset{a}{\equiv}}}(c)$.

2) $\mathfrak{Fig}_{\overset{w}{\underset{a}{\equiv}}}(\sigma) \Rightarrow \mathfrak{Fig}_{\overset{w}{\underset{a}{\equiv}}}(u_a^w - \sigma)$.

3) $\mathfrak{Fig}_{\overset{w_1}{\underset{a}{\equiv}}}(\sigma_1) \& \mathfrak{Fig}_{\overset{w_2}{\underset{a}{\equiv}}}(\sigma_2) \equiv \mathfrak{Fig}_{\overset{(w_1 \times w_2)}{\underset{a}{\equiv}}}(\sigma_1 \times \sigma_2)$.

4) $\mathfrak{Fig}_{\overset{w}{\underset{a}{\equiv}}}(\sigma) \Rightarrow \mathfrak{Fig}_{\overset{\mathcal{P}w}{\underset{a}{\equiv}}}({}^*\mathcal{P}_{fin}(\sigma))$.

Proof: Only 4) is not obvious. Let us prove 4). We have to show that $x \subseteq \sigma \& y \overset{\mathcal{O}\mu}{\underset{a}{=}} x \Rightarrow y \subseteq \sigma$ (*finiteness of y follows from the definition of $u_a^{\mathcal{O}\mu}$). But $y \subseteq \mathrm{Fig}_{\overset{\mu}{\underset{a}{=}}}(y) = \mathrm{Fig}_{\overset{\mu}{\underset{a}{=}}}(x) \subseteq \sigma$, as $\mathcal{F}ig_{\overset{\mu}{\underset{a}{=}}}(\sigma)$.

Corollary 2.4: The set u from Cl.2 may be chosen as u_a^{μ} for a suitable μ, a; the external set σ as a figure in $\overset{\mu}{\underset{a}{=}}$ and u_i from $\vec{u}$ as $u_a^{\mu_i}$ for suitable subwords μ_i of μ.

Remark: The given step can be done also for several "input" classes, if we suppose that they are figures in suitable equivalences.

§3 A construction of standard equivalent formulas.

Theorem 3.1: If $b \subseteq a \in^*(\mathcal{P}_{fin}(c))$ then $u_b^{\mu} \subseteq u_a^{\mu}$ and $(\forall x, y \in u_b^{\mu})(x \overset{\mu}{\underset{b}{=}} y \equiv x \overset{\mu}{\underset{a}{=}} y)$.

Proof: By the induction based on the complexity of μ. Only the step for $\mathcal{O}\mu$ is not obvious. Let us prove this step. Let $x, y \in u_b^{\mathcal{O}\mu}$, $x \overset{\mathcal{O}\mu}{\underset{a}{=}} y$ and $t \in x$. There is $s \in y$ such that $s \overset{\mu}{\underset{a}{=}} t$. As $x, y \in u_b^{\mathcal{O}\mu}$ we have $s, t \in u_b^{\mu}$. Using the induction assumption we obtain $s \overset{\mu}{\underset{b}{=}} t$ and hence $\mathrm{Fig}_{\overset{\mu}{\underset{b}{=}}}(x) \subseteq \mathrm{Fig}_{\overset{\mu}{\underset{b}{=}}}(y)$. The proof of the assertion where x, y are changed and the proof of $\Rightarrow$ are analogous.

Definition 3.2: For $a, b \in^*(\mathcal{P}_{fin}(c))$ such that $b \subseteq a$ and a word μ we define the function ${}_a f_b^{\mu} : u_a^{\mu} \xrightarrow{on} u_b^{\mu}$. We proceed by the recursion based on the complexity of μ.

(i) ${}_a f_b^{\Lambda}(x) = x \cap b$

(ii) ${}_a f_b^{(\mu_1 \times \mu_2)}(\langle x_1, x_2 \rangle) = \langle {}_a f_b^{\mu_1}(x_1), {}_a f_b^{\mu_2}(x_2) \rangle$

(iii) ${}_a f_b^{\mathcal{O}\mu}(x) = ({}_a f_b^{\mu})''x$.

Lemma 3.3: 1) ${}_a f_b^{\mu}$ is described by a set formula with parameters a, b, μ.

2) For $x \in u_b^{\mu}$ we have ${}_a f_b^{\mu}(x) = x$.

3) If $a \subseteq b \subseteq d$ then ${}_b f_a^{\mu} \circ {}_d f_b^{\mu} = {}_d f_a^{\mu}$.

Proof: By the induction based on the complexity of μ.

Theorem 3.4: 1) For $a,b\in{}^*(\mathcal{P}_{fin}(c))$ such that $b\leq a$, and for $x,y\in u_a^{\mu}$ the following implication holds: $x\overset{\mu}{\underset{a}{=}}y \Rightarrow {}_af_b^{\mu}(x)\overset{\mu}{\underset{ba}{=}}{}_af_b^{\mu}(y)$.

2) If $IL_c(b)$ then the opposite implication holds, too.

3) If $IL_c(a)$ and $x,y\in u_a^{\mu}$ then $x\overset{\mu}{\underset{a}{=}}y \equiv (\forall b\in\mathcal{P}_{fin}(c))({}_af_b^{\mu}(x)= {}_af_b^{\mu}(y)) \equiv (\exists b, IL_c(b))(b\leq a \,\&\, {}_af_b^{\mu}(x)={}_af_b^{\mu}(y))$.

Proof: 1) By the induction based on the complexity of μ. Only the induction step for $\mathcal{P}\mu$ is not obwious. Let us prove this step. Let $t\in{}_af_b^{\mathcal{P}\mu}(x)$ and let $\bar{t}\in x\cap(({}_af_b^{\mu})^{-1}{}''\{t\})$. There is $\bar{v}\in y$ such that $\bar{v}\overset{\mu}{\underset{a}{=}}\bar{t}$. By the induction assumption we have $t\overset{\mu}{\underset{ba}{=}}f_b(\bar{v})$ and hence $\mathrm{Fig}_{\overset{\mu}{\underset{b}{=}}}({}_af_b^{\mathcal{P}\mu}(x))\leq \mathrm{Fig}_{\overset{\mu}{\underset{b}{=}}}({}_af_b^{\mathcal{P}\mu}(y))$. If we change x,y then we proceed analogously.

2) We use again the induction and only the step for $\mathcal{P}\mu$ is not obvious. Let $t\in x$. It is sufficient to find $\bar{s}\in y$ such that $t\overset{\mu}{\underset{a}{=}}\bar{s}$. Let $s\in{}_af_b^{\mathcal{P}\mu}(y)$ be such that $s\overset{\mu}{\underset{ba}{=}}f_b^{\mu}(t)$ (the existence follows from the assumption of the implication). Let $\bar{s}\in y$ be such that $s={}_af_b^{\mu}(\bar{s})$. By the induction assumption we have $\bar{s}\overset{\mu}{\underset{a}{=}}t$.

3) The fact that the second assertion is implied by the first one can be proved by 1) and by the fact that for $b\in\mathcal{P}_{fin}(c)$ $\overset{\mu}{\underset{b}{=}}$ is the identity. The fact that the third assertion is implied by the second one follows from the compact enlargement property. Using 2) we prove that the first assertion is implied by the third one.

Corollary 3.5: If $c\subset b\subseteq a\in{}^*(\mathcal{P}_{fin}(c))\,\&\,x\in u_a^{\mu}$ then ${}_af_b^{\mu}(x)\overset{\mu}{\underset{a}{=}}x$.

Proof: Put $y={}_af_b^{\mu}(x)$. $y\in u_b^{\mu}$ hence ${}_af_b^{\mu}(y)=y={}_af_b^{\mu}(x)$ (see L3.3.2)). Thus $y\overset{\mu}{\underset{a}{=}}x$ (see T3.4.3)).

Theorem 3.6: Let $b\subseteq a$ and $IL_c(b), IL_c(a)$. If $\sigma_{b/a}\subseteq u_{b/a}^{\mu}$ are figures in $\overset{\mu}{\underset{b/a}{=}}$ then $({}_af_b^{\mu})''\sigma_a=\sigma_a\cap u_b$ and $({}_af_b)^{-1}{}''\sigma_b= \mathrm{Fig}_{\overset{\mu}{\underset{a}{=}}}(\sigma_b)$. Hence $\sigma_a = ({}_af_b^{\mu})^{-1}{}''(({}_af_b^{\mu})''\sigma_a)$ and $\sigma_b = ({}_af_b^{\mu})''(({}_af_b^{\mu})^{-1}{}''\sigma_b)$.

Proof: For $x \in \sigma_a$ we have $x \overset{\mathcal{M}}{\underset{a}{=}} {}_a f_b(x) \in \sigma_a \cap u_a^{\mathcal{M}}$. The assertions are now easy consequences of this fact.

Theorem 3.7: The operations $-, \times, \mathcal{P}$ commute with f in the following sense: Let $b \subseteq a \subseteq d$ and let $IL_c(a), IL_c(b), IL_c(d)$.

1) If $\sigma_{1/2} \subseteq u_a^{\mathcal{M}}$ are figures in $\overset{\mathcal{M}}{\underset{a}{=}}$ then ${}_a f_b^{\mathcal{M}}{}'' \sigma_1 - {}_a f_b^{\mathcal{M}}{}'' \sigma_2 =$ $= {}_a f_b^{\mathcal{M}}{}''(\sigma_1 - \sigma_2)$.

2) If $\sigma_{1/2} \subseteq u_a^{\mathcal{M}_1 \mathcal{M}_2}$ are figures then $(({}_a f_b^{\mathcal{M}_1})'' \sigma_1) \times$ $\times (({}_a f_b^{\mathcal{M}_2})'' \sigma_2) = ({}_a f_b^{(\mathcal{M}_1 \times \mathcal{M}_2)})''(\sigma_1 \times \sigma_2)$.

3) If $\sigma \subseteq u_a^{\mathcal{M}}$ is a figure then ${}^*\mathcal{P}_{fin}(({}_a f_b^{\mathcal{M}})'' \sigma) =$ $= ({}_a f_b^{\mathcal{P}\mathcal{M}})'' \; {}^*\mathcal{P}_{fin}(\sigma)$.

For $({}_d f_a^{\mathcal{M}})^{-1}$ assertions analogous to 1),2),3) hold.

Proof: We use T3.6. We prove only the most complicated case, namely 3). Let $x \in {}^*\mathcal{P}_{fin}(({}_a f_b^{\mathcal{M}})'' \sigma) = {}^*\mathcal{P}_{fin}(\sigma \cap u_b^{\mathcal{M}})$. Thus $x \subseteq \sigma$ & $\& x \in u_b^{\mathcal{P}\mathcal{M}} \Rightarrow x = {}_a f_b^{\mathcal{P}\mathcal{M}}(x)$ (see L3.3.2)) $\Rightarrow x \in ({}_a f_b^{\mathcal{P}\mathcal{M}})'' \; {}^*\mathcal{P}_{fin}(\sigma)$. Let on the other hand $x = {}_a f_b^{\mathcal{P}\mathcal{M}}(y)$ & $y \subseteq \sigma$. We have to prove that $(\forall t \in x)(t \in ({}_a f_b^{\mathcal{M}})'' \sigma \; (= \sigma \cap u_b^{\mathcal{M}}))$. Let for an arbitrary $t \in x$ an element $s \in y$ be such that $t = {}_a f_b^{\mathcal{M}}(s)$ (see the definition of $f^{\cdot}$). We have $t \overset{\mathcal{M}}{\underset{a}{=}} s$ (see C3.5), $t \in u_b^{\mathcal{M}}$ hence $t \in \sigma \cap u_b^{\mathcal{M}}$ as σ is a figure. We now give the proof for $({}_d f_a^{\mathcal{M}})^{-1}$. Let $x \in {}^*\mathcal{P}_{fin}(({}_d f_a^{\mathcal{M}})^{-1}{}'' \sigma) =$ $= {}^*\mathcal{P}_{fin}(\mathrm{Fig}_{\frac{a}{d}}(\sigma))$ (i.e. $x \subseteq \mathrm{Fig}_{\frac{a}{d}}(\sigma) = \overline{\sigma}$). We have to prove that ${}_d f_a^{\mathcal{P}\mathcal{M}}(x) = ({}_d f_a^{\mathcal{M}})'' x \subseteq \sigma$. If t is an arbitrary element of x then ${}_d f_a^{\mathcal{M}}(t) \in \overline{\sigma} \cap u_a^{\mathcal{M}} = \sigma$ (see T3.6). Let on the other hand $x \in$ $\in (({}_d f_a^{\mathcal{P}\mathcal{M}})^{-1}{}''({}^*\mathcal{P}_{fin}(\sigma)))$. Hence $({}_d f_a^{\mathcal{M}})'' x \subseteq \sigma$. If t is an arbitrary element of x then ${}_d f_a^{\mathcal{M}}(t) \in \sigma$. Thus $x \in {}^*\mathcal{P}_{fin}(({}_d f_a^{\mathcal{M}})^{-1}{}'' \sigma)$.

Definition 3.8: Let $IL_c(a)$ and let $\sigma_a \subseteq u_a^{\mathcal{M}}$ be a figure in $\overset{\mathcal{M}}{\underset{a}{=}}$. We define an external set $\mathcal{K}_{\sigma_a}$ of standard functions F such that $\mathrm{dom}(F) = c$ in the following manner: $F \in \mathcal{K}_{\sigma_a} \equiv (\exists x \in \sigma_a)$ $(\forall b \in \mathcal{P}_{fin}(c))(F(b) = {}_a f_b^{\mathcal{M}}(x))$.

Remark: The system $\mathcal{K}_{\sigma_a}$ will play an essential role in the elimination method. $F \in \mathcal{K}_{\sigma_a}$ are functions from $\mathcal{P}_{fin}(c)$ into sets obtained by operations x and $\mathcal{P}_{fin}$ applied successively on finite subsets of c and hence may be usually identified (e.g. if choice is at disposal) with subsets of c. On this fact lays the strengthening of our method in comparison with the method in [N].

Theorem 3.9: Let $IL_c(a)$ and let $\sigma_a \subseteq u_a^{\mathcal{M}}$ be a figure in $\overset{\mathcal{M}}{\underset{a}{=}}$.

1) $t \in \sigma_a \equiv (\exists F \in \mathcal{K}_{\sigma_a})(\forall d \in \mathcal{P}_{fin}(c))(F(d) = {}_a f_d^{\mathcal{M}}(t) \& t \in u_a^{\mathcal{M}})$.

2) For $b \geq a \& IL_c(b)$ let us put $\sigma_b = Fig_{\overset{a}{\underset{b}{=}}}(\sigma_a)$. We have $\mathcal{K}_{\sigma_a} = \mathcal{K}_{\sigma_b}$. ($\mathcal{K}$ does not depend on the choice of a - it has a standard sense.)

Proof: 1) $\Rightarrow$ see the definition of $\mathcal{K}_{\sigma_a}$. $\Leftarrow$ For t satisfying the righthand side let $\bar{t} \in \sigma_a$ be such that $(\forall d \in \mathcal{P}_{fin}(c))$ $({}_a f_d^{\mathcal{M}}(\bar{t}) = {}_a f_d^{\mathcal{M}}(t))$ (for the existence of $\bar{t}$ see the definition of $\mathcal{K}_{\sigma_a}$). We have $t \overset{\mathcal{M}}{\underset{a}{=}} \bar{t}$ (see T3.4) and hence $t \in \sigma_a$ (as $Fig_{\overset{\mathcal{M}}{=}}(\sigma_a)$).

2) For $x \in \sigma_b$ and $d \in \mathcal{P}_{fin}(c)$ we have ${}_b f_d^{\mathcal{M}}(x) = {}_a f_d^{\mathcal{M}}({}_b f_a^{\mathcal{M}}(x))$ (see L3.3.3)) and ${}_b f_a^{\mathcal{M}}(x) \in \sigma_a$ (see T3.6).

Corollary 3.10: For each normal formula $\varphi(x, St, \vec{z})$ (using the predicate St - "to be a standard element of c") there are a set formula $\psi(x, y, \vec{z})$ and a set $\mathcal{K}$ ($\in \mathcal{U}$) of functions $F: \mathcal{P}_{fin}(c) \longrightarrow \mathcal{R}(c)$ (where $\mathcal{R}(c)$ denotes $\mathcal{P}_{fin}(\mathcal{P}_{fin} \ldots (\mathcal{P}_{fin}(c)) \ldots)$ for a suitable number of iterations) such that for every $\vec{z}$, t (internal sets) $\varphi(t, St, \vec{z}) \equiv (\exists F \in \mathcal{K})(\forall a \in \mathcal{P}_{fin}(c))$ $\psi(t, F(a), \vec{z})$.

Moreover, if $\vec{z} = {}^*\vec{y}$ ($\vec{z}$ are standard) then we have $\varphi({}^*t, St, {}^*\vec{y}) \equiv (\exists F \in \mathcal{K})(\forall a \in \mathcal{P}_{fin}(c)) \psi^{St}({}^*t, F(a), {}^*\vec{y})$ (where ψ^{St} means ψ in the standard sense - all the quantifiers are restricted to standard elements) as the enlargement is an elementary extension. The formula on the righthand side is standard - it is a standard definition of the predicate defined nonstandardly on the lefthand side of the equivalence.

Proof: Let us denote (1),(2) the lefthand side and the right-hand side of the equivalence, respectively. Using C1.2 and C2.4 we find an equivalent formula to (1) of the form $(\exists \bar{t} \in \sigma_a)\bar{\psi}(t,\bar{t},\vec{z})$ for an arbitrary chosen a such that $IL_c(a)$. We know that $\sigma_a \subseteq u_a^{\mu}$ is a figure in $\underset{a}{\overset{\mu}{=}}$ for a suitable word μ. Using T3.9 we obtain an equivalent formula of the form

(3) $(\exists F \in \mathcal{K}_{\sigma_a})(\forall d \in \mathcal{P}_{fin}(c))\bar{\bar{\psi}}(F(d),d,a,\vec{z})$.

We know that $\mathcal{K}_{\sigma_a}$ is not dependent on the choice of a and that (by T3.9.2),T1.3,T3.6)

(i) $a_1 \subset a_2 \Rightarrow (\bar{\bar{\psi}}(F(d),d,a_1,t,\vec{z}) \Rightarrow \bar{\bar{\psi}}(F(d),d,a_2,t,\vec{z}))$.

a does not occur in the formula φ. Using the logical law $\varphi \equiv \psi(a) \vdash \varphi \equiv (\exists a)\psi(a)$ we obtain the equivalent formula

(4) $(\exists F \in \mathcal{K}_{\sigma_a})(\exists a, IL_c(a))(\forall d \in \mathcal{P}_{fin}(c))\bar{\bar{\psi}}(F(d),d,a,t,\vec{z})$.

We prove that (4) is equivalent to (5).

(5) $(\exists F \in \mathcal{K}_{\sigma_a})(\forall d \in \mathcal{P}_{fin}(c))(\exists a \in {}^*\mathcal{P}_{fin}(c), a \supset d)\bar{\bar{\psi}}(F(d),d,a,t,\vec{z})$.

(4)$\Rightarrow$(5) is obvious. Let us prove (5)$\Rightarrow$(4). Let us fix F. Using (i) we obtain from (5) the formula $(\forall m \in \mathcal{P}_{fin}(\mathcal{P}_{fin}(c)))$ $(\exists a \in {}^*(\mathcal{P}_{fin}(c)))((\forall d \in m)(a \supset d) \& \bar{\bar{\psi}})$. Using the compact enlargement argument we obtain that there is $m, IL_{\mathcal{P}_{fin}(c)}(m)$ such that $(\exists a \in {}^*\mathcal{P}_{fin}(c))((\forall d \in m)(a \supset d) \& \bar{\bar{\psi}}({}^*F(d),d,a,t,\vec{z}))$. Now as $d \subset a$ for every $d \in \mathcal{P}_{fin}(c)$, we have $a \supset c$ and therefore (4) holds. For completeness of the proof it is sufficient now to put $\psi(x,y,\vec{z}) \equiv (\exists y_1,y_2)(y = \langle y_1,y_2 \rangle \& (\exists a \in {}^*(\mathcal{P}_{fin}(c)), a \supset y_2) \& \bar{\bar{\psi}}(y_1,y_2,a,x,\vec{z}))$ and $\mathcal{K} = \{F; dom(F) = \mathcal{P}_{fin}(c) \& (\exists \bar{F} \in \mathcal{K}_{\sigma_a})(\forall b \in \mathcal{P}_{fin}(c))(F(b) = \langle \bar{F}(b), b \rangle)\}$.

To finish the whole procedure it suffices only to give a description of $\mathcal{K}_{\sigma_a}$ in the usual set theoretical standard language. This is done in the section 4.

§4 Standard description of $\mathcal{K}$.

In this section we have to solve a problem typical for the beginning of the ε-δ method in the calculus. Namely : How to find new definitions of notions defined with the help of infinitesimals. The new definitions may be more complicated, may be less objective, but must not use infinitesimals. In our case we consider the operations $-$, $\times$, ${}^{*}\mathcal{P}$ for parts of formally finite sets.

Theorem 4.3: Let $\mathrm{IL}_c(a)$.

1) $\mathcal{K}_{u_a^{(\mu_1 \times \mu_2)}} = \{F; \mathrm{dom}(F) = \mathcal{P}_{fin}(c) \& (\exists F_1 \in \mathcal{K}_{u_a^{\mu_1}})$ $(\exists F_2 \in \mathcal{K}_{u_a^{\mu_2}})(\forall d \in \mathcal{P}_{fin}(c))(F(d) = \langle F_1(d), F_2(d)\rangle)\}$.

2) $\mathcal{K}_{u_a^{\mathcal{P}\mu}} = \{F; \mathrm{dom}(F) = \mathcal{P}_{fin}(c) \& (\forall d \in \mathcal{P}_{fin}(c))(F(d) \subseteq u_d^{\mu}) \&$ $\& (\forall b, d \in \mathcal{P}_{fin}(c))(b \subset d \Rightarrow F(b) = {}_d f_b^{\mathcal{P}\mu}(F(d)))$

Proof: It is an easy application of the definition of $\mathcal{K}_{\sigma_a}$ and the assertions from §3.

Definition 4.4: 1) $\mathcal{K}_1 \otimes \mathcal{K}_2 = \{F; \mathrm{dom}(F) = \mathcal{P}_{fin}(c) \& (\exists F_1 \in \mathcal{K}_1)$ $(\exists F_2 \in \mathcal{K}_2)(\forall d \in \mathcal{P}_{fin}(c))(F(d) = \langle F_1(d), F_2(d)\rangle)\}$.

2) For $F \in \mathcal{K}_{u_a^{\mu}}$ and $H \in \mathcal{K}_u^{\mathcal{P}\mu}$ let us define $F \circledcirc H \equiv (\forall d \in \mathcal{P}_{fin}(c))$ $(F(d) \in H(d))$.

3) For $\mathcal{K} \subseteq \mathcal{K}_{u_a^{\mu}}$ let us define $\mathcal{K}^{\textcircled{\mathcal{P}\mu}} = \{H \in \mathcal{K}_{u_a^{\mathcal{P}\mu}}; (\forall F \circledcirc H)$ $(F \in \mathcal{K})\}$.

Theorem 4.5: Let $a \in {}^{*}(\mathcal{P}_{fin}(c))$ & $c \subseteq a$.

1) If $\sigma_{1/2} \subseteq u_a^{\mu}$ are figures in $\frac{\mu}{a}$ then $\mathcal{K}_{\sigma_1 - \sigma_2} = \mathcal{K}_{\sigma_1} - \mathcal{K}_{\sigma_2}$.

2) If $\sigma_{1/2} \subseteq u_a^{\mu_{1/2}}$ are figures in $\frac{\mu_{1/2}}{a}$ then $\mathcal{K}_{\sigma_1 \times \sigma_2} =$ $= \mathcal{K}_{\sigma_1} \otimes \mathcal{K}_{\sigma_2}$.

3) If $\sigma \subseteq u_a^{\mu}$ is a figure in $\frac{\mu}{a}$ then $\mathcal{K}_{{}^{*}\mathcal{P}_{fin}(\sigma)} = \mathcal{K}_{\sigma}^{\textcircled{\mathcal{P}\mu}}$.

Proof: Only the proof of 3) is not obvious and hence we prove it. $\subseteq$ – let $H \in \mathcal{K}_{{}^{*}\mathcal{P}_{fin}(\sigma)}$, let $y \in u_a^{\mathcal{P}\mu}$ be an element corresponding to H $((\forall d \in \mathcal{P}_{fin}(c))(H(d) = {}_a f_d^{\mathcal{P}\mu}(y)))$, hence $y \subseteq \sigma$. Let

$F \circledS H$. We know that for every $d \in \mathcal{P}_{fin}(c)$, $d \leq a$ & ${}^*F(d) \in_a f_d^{\mathcal{P}\mu}(y)$ & & $(\forall b \subset d)({}^*F(b) =_d f_b({}^*F(d)))$. Using the compact enlargement property we obtain that the last formula is satisfied also for d such that $IL_c(d)$&$d \leq a$. Hence ${}^*F(d) \in_a f_d^{\mathcal{P}\mu}(y) \subset \sigma$ and $F \in \mathcal{K}_\sigma$. Thus $H \in \mathcal{K}_\sigma^{(\mathcal{P}\mu)}$. 2 - let $H \in \mathcal{K}_\sigma^{(\mathcal{P}\mu)}$ and let $y \in u_a^{\mathcal{P}\mu}$ be an element corresponding to H. We have to prove $y \subseteq \sigma$. Let x be an arbitrary element of y. Let $F \in \mathcal{K}_{u_a^\mu}$ be a function corresponding to x. For any $d \in \mathcal{P}_{fin}(c)$ we have $F(d) \in H(d)$ as $F(d) =_a f_d^\mu(x) \in_a f_d^{\mathcal{P}\mu}(y) = H(d)$. Hence $F \circledS H$ and $F \in \mathcal{K}_\sigma$. Hence $x \in \sigma$ (see T3.9).

In the theorem we have given the inductive steps for a standard definition of the set $\mathcal{K}$ used in C3.10, which completes our procedure.

§5 Counterexamples.

In the last section we give two examples. The first one proves that the usage of auxiliary variables from the powerset of the basic set is necessary. The second one gives reasons for our restriction on compact enlargements.

Example 1. Let Sat^n denote the satisfactory relation on N for formulas of the arithmetic of the order n+1. E.g. Sat^0 is the relation such that $(Sat^0)''(\ulcorner\varphi\urcorner) = \{\vec{n}; \varphi(\vec{n})$, where $\ulcorner\varphi\urcorner$ denotes the Gödel's number of the formula φ of the first order arithmetic and $\vec{n}$ are (evaluation of) free variables of φ. Sat^1 is defined analogously for formulas φ where the quantification of subsets of N is allowed. An easy diagonal consideration proves that Sat^n cannot be defined by a formula of the n+1 order arithmetic. We prove that Sat^n is defined by a "two changes of quantification" formula using the generalized IL predicate in any compact enlargement of $\mathcal{P}^n(N)$.

For the description of Sat^n we use the set theoretical formulas replacing N by HF (hereditarily finite sets) as this is technically much easier. From technical reasons we also identify c with $\mathcal{P}_{fin}(c)$ for iterated powersets of HF. Further we use $x \in \mathcal{P}_{fin}(\mathcal{P}^n(HF))$ just to stress that we have in our mind the set structure of x (as a finite set). This identification can be described by a standard formula and hence it preserves the predicate "to be standard". Remember that:

1) $x \in {}^*\mathcal{P}^n(HF) \& St(x) \equiv \{x\} \in {}^*\mathcal{P}^{n+1}(HF) \& St(\{x\})$

2) $St(\langle x,y\rangle) \equiv St(x) \& St(y)$

3) If we put $IL^n(x) = IL_{\mathcal{P}^n(HF)}(x)$ then we obtain (using the compact enlargement argument) $(\forall x, IL^n(x))\, \varphi(x,\vec{z}) \equiv$
$\equiv (\exists x_o \in \mathcal{P}_{fin}(\mathcal{P}^n(HF)))(\forall x \in {}^*(\mathcal{P}_{fin}(\mathcal{P}^n(HF))), x \supseteq x_o)\, \varphi(x,\vec{z})$
for every internal formula φ and every internal parameters $\vec{z}$.

Let (1) $GS(\ulcorner\varphi\urcorner)$ denote the generating sequence of $\ulcorner\varphi\urcorner$ (the sequence of Gödel's numbers of elements of the generating sequence of φ).

(2) $\vec{t} \in FV(\ulcorner\varphi\urcorner)$ means that $\vec{t}$ are (evaluation of) free variables of φ. Let $t/FV(\ulcorner\varphi\urcorner)$ be the restriction (of the evaluation mapping) on free variables of φ.

(3) $\vec{t} \smile t_k$ be the prolongation of the sequence $\vec{t}$. (If we use e.g. functions for the representation then $\vec{t} \smile t_k = \vec{t} \cup \langle t_k, k\rangle$.)

Let us now give the definition for x to code the satisfactory relation.

Definition 5.1: $Cdsat^n(x) \equiv x \in {}^*(\mathcal{P}_{fin}(\mathcal{P}^n(HF)))\&$
$\& (\forall \ulcorner\varphi\urcorner \in {}^*N \& St(\ulcorner\varphi\urcorner))(\ulcorner\varphi\urcorner \in dom(x))\ \&$
$\& (\forall \ulcorner\varphi\urcorner \in dom(x))(GS(\ulcorner\varphi\urcorner) \subseteq dom(x))\ \&$
$\& (\forall \ulcorner a_k \in a_1\urcorner \in dom(x), St(\ulcorner a_k \in a_1\urcorner))(\forall \langle t_k, t_1\rangle, St(\langle t_k, t_1\rangle))$
$(\langle t_k, t_1\rangle \in x''\{\ulcorner a_k \in a_1\urcorner\} \equiv t_k \in t_1)\ \&$
$\& (\forall \ulcorner\varphi_1 \& \varphi_2\urcorner \in dom(x), St(\ulcorner\varphi_1 \& \varphi_2\urcorner))(\forall \vec{t}, St(\vec{t}) \& \vec{t} \in FV(\ulcorner\varphi_1 \& \varphi_2\urcorner))$

$$(\vec{t} \in x''\{\ulcorner\varphi_1 \& \varphi_2\urcorner\} \equiv (\vec{t} \wedge \mathrm{FV}(\varphi_1) \in x''\{\ulcorner\varphi_1\urcorner\} \& \vec{t} \wedge \mathrm{FV}(\varphi_2) \in x''\{\ulcorner\varphi_2\urcorner\}) \&$$

$$\& (\forall \ulcorner\neg\varphi\urcorner \in \mathrm{dom}(x), \mathrm{St}(\ulcorner\neg\varphi\urcorner))(\forall \vec{t}, \mathrm{St}(\vec{t}) \& \vec{t} \in \mathrm{FV}(\ulcorner\neg\varphi\urcorner))$$

$$(\vec{t} \in x''\{\ulcorner\varphi\urcorner\} \equiv t \notin x''\{\ulcorner\neg\varphi\urcorner\}) \&$$

$$\& (\forall \ulcorner(\exists a_k)\varphi\urcorner \in \mathrm{dom}(x), \mathrm{St}(\ulcorner(\exists a_k)\varphi\urcorner))(\forall \vec{t}, \mathrm{St}(\vec{t}) \& \vec{t} \in \mathrm{FV}(\ulcorner(\exists a_k)\varphi\urcorner))$$

$$(t \in x''\{\ulcorner(\exists a_k)\varphi\urcorner\} \equiv (\exists t_k, \mathrm{St}(t_k))(\vec{t} \cup t_k \in x''\{\ulcorner\varphi\urcorner\})).$$

Let us now consider the syntactical form of the given formula. The formula is a conjunction of formulas of the form $(\forall z, \mathrm{St}(z))\psi(t,z)$ and $(\forall z, \mathrm{St}(z))(\psi_1(t,z) \equiv (\exists y, \mathrm{St}(y))\psi_2(t,z,y))$ where ψ are set formulas. Using prenex operations and the mentioned facts, we can find an equivalent formula of the form $(\forall z, \mathrm{St}(z))(\exists y, \mathrm{St}(y))\psi(t,y,z)$. If $\mathrm{IL}_{\mathrm{Sat}^n}(x)$ then $\mathrm{Cdsat}^n(x)$. Thus we have for any standard $\langle t, \ulcorner\varphi\urcorner\rangle : \langle t, \ulcorner\varphi\urcorner\rangle \in \mathrm{Sat}^n \equiv (\exists x)(\mathrm{Cdsat}^n(x) \& \langle t, \ulcorner\varphi\urcorner\rangle \in x)$. Another use of the prenex operations gives an equivalent formula of the form $(\exists x)(\forall z, \mathrm{St}(z))(\exists y, \mathrm{St}(y))\psi(t,x,y,z) \equiv (\exists x)(\forall z, \mathrm{St}(z))(\exists y)(\forall \bar{y}, \mathrm{IL}^n(\bar{y}))(y \in \bar{y} \& \psi) \equiv$ (using comp. enl. arg.) $(\exists x)(\forall z, \mathrm{St}(z))(\forall \bar{y}, \mathrm{IL}^n(\bar{y}))(\exists y)(y \in \bar{y} \& \psi) \equiv (\exists x)(\forall \bar{y}, \mathrm{IL}^n(\bar{y}))(\forall z, \mathrm{St}(z))\psi_1 \equiv (\exists x)(\forall \bar{y}, \mathrm{IL}^n(\bar{y}))(\exists \bar{z}, \mathrm{IL}^n(\bar{z}))\psi_2$.

Note that the formula on the righthand side is only by one step more complicated than formulas having the "easy" translation mentioned in §0.

The following counterexample uses ideas of such great mathematicians as Sochor, Keisler and Luxemburg.

<u>Example 2.</u> We prove that the set M of ultrafilters on N described by the nonstandard definition $M = \{\mathcal{F}; (\forall \alpha, \mathrm{IL}(\alpha))(\exists \beta < \alpha)(\exists f : N \to N)(\forall X \in \mathcal{F})({}^*f(\beta) \in {}^*X)\}$ is dependent on the enlargement which is used and hence there is no standard formula equivalent to the definition of M.

We prove firstly that, for any compact enlargement, M consists of all ultrafilters. For this it suffices, at first, to prove that for every ultrafilter $\mathcal{F}$ on N the formula $(\forall z \in \mathcal{P}_{fin}(\mathcal{F}))({}^*(\bigcap z) \cap \alpha \neq 0)$ holds. Then we shall use the compact enlargement argument and we shall put f to be the identity mapping. To prove the above mentioned formula let us note that $N \subset \alpha$ for every $IL(\alpha)$.

Now we construct an example of an enlargement in which M consists only of principal ultrafilters and ultrafilters equivalent, in the Rudin-Keisler ordering of ultrafilters, to a minimal ultrafilter. Let us note at first that the ultrapower of an enlargement is an enlargement (see [L] the place mentioned in §0). To see this fact it suffices to realize that the constant of an IL_c element of the enlargement (being the basic structure for the ultrapower) is an IL_c element in the sense of ultrapover. Remember some notation and facts from [CH H]. For $x \in {}^*c$ we put $Fil(x) = \{X \in \mathcal{P}(c); x \in {}^*X\}$. If $\mathcal{F}$ is an ultrafilter on c then $\mathcal{F} = Fil(x)$ for every $x \in \mu(\mathcal{F})$. Let $\mathcal{F}_1, \mathcal{F}_2$ be two ultrafilters on c. $\mathcal{F}_1 \underset{RK}{\leq} \mathcal{F}_2$ (the Rudin-Keisler ordering on ultrafilters) iff there is $f: c \to c$ such that for one (and also for all) $x \in \mu(\mathcal{F}_2)$ we have ${}^*f(x) \in \mu(\mathcal{F}_1)$. All the mentioned facts are immediate consequences of properties of enlargements. Now we construct the promised enlargement. Let $\mathcal{F}$ be a minimal (in $\underset{RK}{\leq}$) ultrafilter on N (thus $\mathcal{F}$ is a selective ultrafilter). Let ${}^{\#}\mathcal{U}$ be an enlargement for ultrafilters on N. We put ${}^*\mathcal{U} = Ul_{\mathcal{F}}({}^{\#}\mathcal{U})$ which is the ultrapower of ${}^{\#}\mathcal{U}$. Let $\delta \in {}^*N$ be the equivalence class containing $Id \upharpoonright N$. It is obvious that $Fil(\delta) = \mathcal{F}$. Let $\alpha \in {}^*N$ and $\alpha < \delta$. Let $f: N \to {}^{\#}N$ be a function contained in the equivalence class of α. Thus we have $X = \{n; f(n) < n\} \in \mathcal{F}$. Without loss of the generality we may suppose that $X = N$. Hence $(\forall n)(f(n) \in N)$ and $f: N \to N$. Considering the construction of ultrapower we may see that ${}^*f(\delta)=\alpha$. From the selectivity of $\mathcal{F}$ we obtain that on a certain set from $\mathcal{F}$, f is either constant or one-one. Hence $\alpha\in N$ or $\delta={}^*g(\alpha)$ for a suitable $g:N\rightarrow N$. Thus, in our example, the set M consists from principal ultrafilters and ultrafilters equivalent (in $\leq_{RK}$) with $\mathcal{F}$.

References

[Č] K. ČUDA: The relation between $\varepsilon-\delta$ procedures and the infinitely small in nonstandard methods, Set Theory and Hierarchy Theory V, Lecture Notes in Mathematics 619. s.143-152.

[Č1] K. ČUDA: An elimination of infinitely small quantities and infinitely large numbers (within the framework of AST), Comment. Math. Univ. Carolinae 21(1980). s.433-445.

[Č2] K. ČUDA: An elimination of the predicate "To be a standard number" in nonstandard models of arithmetic, Comment. Math. Univ. Carolinae 23(1982).s.785-803.

[CH H] G. CHERLIN, J. HIRSCHFELD: Ultrafilters and ultraproducts in non-standard analysis, Contributions to Non-Standard Analysis, Studies in Logic 69, s.261-280.

[L] W.A.J. LUXEMBURG: A general theory of monads, Applications of Model Theory to Algebra, Analysis, and Probability, Holt, Rinehart and Winston, Inc. s.18-86.

[H] A. E. HURD: Nonstandard analysis of dynamical systems. I: Limit motions, stability, Transactions of the Am. Math. Soc. vol 160,Oct. 1971. s.1-25.

[N] E. NELSON: Internal set theory: A new approach to nonstandard analysis, BAMS vol 83 (1977).s.1165--1198.

[S] A. SOCHOR: Differential calculus in the Alternative set theory, Set Theory and Hierarchy Theory V, Lecture notes in Mathematics 619. s.273-284.

Matematický ústav, Karlova universita, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(Oblatum 25.6. 1984)