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**THE EVOLUTION DARCY-BOUSSINESQ SYSTEM
(A WEAK MAXIMUM PRINCIPLE AND THE UNIQUENESS)
Dan POLIŠEVSKI**

Abstract: An initial-boundary value problem for a Darcy-Boussinesq system is studied. A weak maximum principle and the uniqueness are proved.

Key words: Darcy-Boussinesq system, maximum principle, uniqueness.

Classification: 35 B 50, 76 R 99.

Find u, p, T satisfying:

- (1) $\operatorname{div} u = 0$ in $Q = \Omega \times (0, \theta)$, $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3),
 $\theta > 0$,
- (2) $Bu + \nabla p = [1 - \alpha(T - T_m)] g$ in Q , $g \in H^2(\Omega)$,
- (3) $\gamma \frac{\partial T}{\partial t} + u \nabla T = \operatorname{div} (A \nabla T)$ in Q ,
- (4) $u \cdot \nu = 0$ on $\partial \Omega \times (0, \theta)$, ν - outward normal,
- (5) $T = \tau$ on $\partial \Omega \times (0, \theta)$, $\tau \in C(0, \theta; H^{3/2}(\partial \Omega))$,
- (6) $T(0) = T_0 \in H^2(\Omega)$, $T_0 = \tau(0)$ on $\partial \Omega$,

where $\alpha > 0$, $T_m > 0$, $\gamma > 0$ and A, B are positive symmetric tensors.

We pass to homogeneous boundary conditions introducing $S = T - (w_h + T_m)$, where for any $h > 0$ $w_h \in C(0, \theta; H^2(\Omega))$

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with:

$$(7) \quad w_h = \tau - T_m \text{ on } \partial\Omega \times (0, \theta),$$

$$(8) \quad |s \nabla w_h|_{L^2(\Omega)} \leq h |\nabla s|_{L^2(\Omega)}, \quad (\forall) s \in H_0^1(\Omega), \text{ a.e. on } (0, \theta).$$

Denoting by P_H the projection of $L^2(\Omega)$ on H , where $H = \{v \in L^2(\Omega) \mid \operatorname{div} v = 0, v \cdot \nu = 0 \text{ on } \partial\Omega\}$ we are lead to:

Find $(u, S) \in L^2(0, \theta; H \times H_0^1(\Omega))$ satisfying

$$(9) \quad P_H(Bu - [1 - \alpha(S + w_h)]g) = 0 \text{ a.e. on } (0, \theta),$$

$$(10) \quad \gamma(S' + w_h', T)_{L^2(\Omega)} + (u, T \nabla(S + w_h))_{L^2(\Omega)} + (A \nabla(S + w_h), \nabla T)_{L^2(\Omega)} = 0, \quad (\forall) T \in \mathcal{D}(\Omega), \text{ a.e. on } (0, \theta),$$

$$(11) \quad S(0) = S_0 \text{ in } \Omega, \quad S_0 = T_0 - (w_h + T_m).$$

Remark. $H^\perp = \{v \in L^2(\Omega) \mid (\exists) g \in H^1(\Omega) \text{ such that } v = \nabla g\}$.

Theorem 1. If (u, S) is a solution of (9)-(11), then

$$(12) \quad |S + w_h|_{L^\infty(\Omega)} \leq C_0 = \max \left\{ \sup_{t \in [0, \theta]} |\tau - T_m|_{H^{3/2}(\partial\Omega)}, \sup_{x \in \bar{\Omega}} |T_0 - T_m| \right\},$$

a.e. on $(0, \theta)$.

Proof. With the techniques of Lemma 3.1 [D. Poliševski, Steady Convection in Porous Media - I, Int. J. Engng. Sci., to appear 1984] it can be proved that the corresponding $p \in H^1$ satisfy $|p|_{H^2(\Omega)} \leq C_1 |\nabla S|_{L^2(\Omega)} + C_2$; it follows $u \in L^2(0, \theta; H^1(\Omega))$ and thus we can choose in (10):

$$(13) \quad T = \operatorname{sgn}(S + w_h) \max \{ |S + w_h - C_0|, 0 \} \in H_0^1(\Omega). \text{ It results } \frac{\gamma}{2} \frac{d}{dt} |T|_{L^2(\Omega)}^2 + a_1 |\nabla T|_{L^2(\Omega)}^2 \leq 0 \text{ a.e. on } (0, \theta),$$

where $a_1 > 0$ is the first eigenvalue of A . Hence,

$$|T(t)|_{L^2(\Omega)} \leq |T(0)|_{L^2(\Omega)} = 0 \text{ for a.a. } t \in (0, \theta), \text{ and}$$

$$|\nabla T|_{L^2(\Omega)} = 0 \text{ a.e. on } (0, \theta).$$

Theorem 2. The problem (9)-(11) has a unique solution.

Proof. (u_i, s_i) $i = 1, 2$, solutions of (9)-(11);

$$u = u_1 - u_2, \quad S = S_1 - S_2:$$

$$(14) \quad P_H(Bu + \alpha Sg) = 0 \text{ a.e. on } (0, \theta),$$

$$(15) \quad \frac{\gamma}{2} \frac{d}{dt} |S|_{L^2(\Omega)}^2 + (u, S \nabla (S_1 + w_h))_{L^2(\Omega)} + (\Delta \nabla S, \nabla S)_{L^2(\Omega)} = 0$$

a.e. on $(0, \theta)$,

$$(16) \quad |u|_{L^2(\Omega)} \leq c_1 |S|_{L^2(\Omega)} \text{ a.e. on } (0, \theta),$$

$$(17) \quad \frac{\gamma}{2} \frac{d}{dt} |S|_{L^2(\Omega)}^2 + a_1 |\nabla S|_{L^2(\Omega)}^2 \leq c_2 |u|_{L^2(\Omega)} |\nabla S|_{L^2(\Omega)}$$

a.e. on $(0, \theta)$,

$$(18) \quad \frac{d}{dt} |S|_{L^2(\Omega)}^2 \leq c_3 |S|_{L^2(\Omega)}^2 \text{ a.e. on } (0, \theta).$$

Hence $|S(t)|_{L^2(\Omega)}^2 \leq |S(0)|_{L^2(\Omega)}^2 \exp(c_3 t)$ for a.a. $t \in (0, \theta)$
and recalling (16) the proof is completed.

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