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# THE EVOLUTION DARCY-BOUSSINESQ SYSTEM (A WEAK MAXIMUM PRINCIPLE AND THE UNIQUENESS)

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Abstract: An initial-boundary value problem for a Darcy-Boussinesq system is studied. A weak maximum principle and the uniqueness are proved.

Key words: Darcy-Boussinesq system, maximum principle, uniqueness.

Classification: 35 B 50, 76 R 99.

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Find u,p,T satisfying:

(1) $\operatorname{div} u = 0$ in $Q = \Omega \times (0,\theta)$, $\Omega \subseteq R^n (n = 2$ or $3)$, $\theta > 0$,

(2) $Bu + \nabla p = [1 - \alpha(T - T_m)]\, g$ in $Q$, $g \in H^2(\Omega)$,

(3) $\gamma \frac{\partial T}{\partial t} + u\nabla T = \operatorname{div}(A \nabla T)$ in $Q$,

(4) $u \cdot \nu = 0$ on $\partial\Omega \times (0,\theta)$, $\nu$ - outward normal,

(5) $T = \tau$ on $\partial\Omega \times (0,\theta)$, $\tau \in C(0,\theta; H^{3/2}(\partial\Omega))$,

(6) $T(0) = T_0 \in H^2(\Omega)$, $T_0 = \tau(0)$ on $\partial\Omega$,

where $\alpha > 0$, $T_m > 0$, $\gamma > 0$ and A, B are positive symetric tensors.

We pass to homogeneous boundary conditions introducing $S = T - (w_h + T_m)$, where for any $h > 0$ $w_h \in C(0,\theta; H^2(\Omega))$

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with:

(7) $w_h = \tau - T_m$ on $\partial\Omega \times (0, \theta)$,

(8) $|s \nabla w_h|_{L^2(\Omega)} \leq h \, |\nabla s|_{L^2(\Omega)}$, $(\forall)\; s \in H^1_0(\Omega)$, a.e. on $(0, \theta)$.

Denoting by $P_H$ the projection of $L^2(\Omega)$ on H, where $H = \{ v \in L^2(\Omega) \mid \operatorname{div} v = 0,\; v \cdot \nu = 0 \text{ on } \partial\Omega \}$ we are lead to:

Find $(u,S) \in L^2(0, \theta;\; H \times H^1_0(\Omega))$ satisfying

(9) $P_H(Bu - [1 - \alpha(S + w_h)]\, g) = 0$ a.e. on $(0, \theta)$,

(10) $\gamma(S' + w_h', T)_{L^2(\Omega)} + (u, T\nabla(S + w_h))_{L^2(\Omega)} +$

$+ (A\nabla(S + w_h), \nabla T)_{L^2(\Omega)} = 0$, $(\forall)\; T \in \mathcal{D}(\Omega)$, a.e. on $(0, \theta)$,

(11) $S(0) = S_0$ in $\Omega$, $S_0 = T_0 - (w_h + T_m)$.

<u>Remark</u>. $H^{\perp} = \{ v \in L^2(\Omega) \mid (\exists) g \in H^1(\Omega) \text{ such that } v = \nabla q \}$.

<u>Theorem 1</u>. If $(u,S)$ is a solution of (9)-(11), then

(12) $|S + w_h|_{L^\infty(\Omega)} \leq C_0 =$

$$= \max \left\{ \sup_{t \in [0,\theta]} |\tau - T_m|_{H^{3/2}(\partial\Omega)} ,\; \sup_{x \in \bar{\Omega}} |T_0 - T_m| \right\},$$

a.e. on $(0, \theta)$.

Proof. With the techniques of Lemma 3.1 [ D. Poliševski, Steady Convection in Porous Media - I, Int. J. Engng. Sci., to appear 1984 ] it can be proved that the corresponding $p \in H^{\perp}$ satisfy $|p|_{H^2(\Omega)} \leq C_1 \, |\nabla S|_{L^2(\Omega)} + C_2$; it follows $u \in L^2(0, \theta;\; H^1(\Omega))$ and thus we can choose in (10) :

$T = \operatorname{sgn}(S + w_h) \max \{ |S + w_h - C_0, 0 \} \in H^1_0(\Omega)$. It results

(13) $\frac{\gamma}{2} \frac{d}{dt} |T|^2_{L^2(\Omega)} + a_1 \, |\nabla T|^2_{L^2(\Omega)} \leq 0$ a.e. on $(0, \theta)$,

where $a_1 > 0$ is the first eigenvalue of A. Hence,

$|T(t)|_{L^2(\Omega)} \leqq |T(0)|_{L^2(\Omega)} = 0$ for a.a. $t \in (0, \theta)$, and $|\nabla T|_{L^2(\Omega)} = 0$ a.e. on $(0, \theta)$.

<u>Theorem 2</u>. The problem (9)-(11) has a unique solution.

Proof. $(u_i, s_i)$ $i = 1,2$ , solutions of (9)-(11) ; $u = u_1 - u_2$ , $S = S_1 - S_2$:

(14) $P_H(Bu + \alpha Sg) = 0$ a.e. on $(0, \theta)$ ,

(15) $\frac{\gamma}{2} \frac{d}{dt} |S|^2_{L^2(\Omega)} + (u, S\nabla(S_1 + w_h))_{L^2(\Omega)} + (A\nabla S, \nabla S)_{L^2(\Omega)} = 0$ a.e. on $(0, \theta)$,

(16) $|u|_{L^2(\Omega)} \leqq C_1 |S|_{L^2(\Omega)}$ a.e. on $(0, \theta)$ ,

(17) $\frac{\gamma}{2} \frac{d}{dt} |S|^2_{L^2(\Omega)} + a_1 |\nabla S|^2_{L^2(\Omega)} \leqq C_2 |u|_{L^2(\Omega)} |\nabla S|_{L^2(\Omega)}$ a.e. on $(0, \theta)$ ,

(18) $\frac{d}{dt} |S|^2_{L^2(\Omega)} \leqq C_3 |S|^2_{L^2(\Omega)}$ a.e. on $(0, \theta)$ .

Hence $|S(t)|^2_{L^2(\Omega)} \leqq |S(0)|^2_{L^2(\Omega)} \exp(C_3 t)$ for a.a. $t \in (0, \theta)$ and recalling (16) the proof is completed.

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