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BIEQUIVALENCES AND TOPOLOGY IN THE ALTERNATIVE SET THEORY

Jaroslav GURIČAN and Pavol ZLATOŠ

Abstract: The topological problematics in the AST is enriched by simultaneous study of indescernibility phenomena represented by a π-equivalence together with accessibility phenomena represented by a σ-equivalence. A pair $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$ where $\doteq$ is a π-equivalence and $\overset{\cdot}{\leftrightarrow}$ is a σ-equivalence is called a biequivalence if both $\doteq$ and $\overset{\cdot}{\leftrightarrow}$ have the same set-theoretically definable domain and $\doteq$ is a subclass of $\overset{\cdot}{\leftrightarrow}$. Basic properties of biequivalences, compatible biequivalences (each infinite set of pairwise accessible elements contains two indiscernible elements) and compact classes are listed. Some questions concerning continuous functions and relations are studied. In particular, some compactness results concerning spaces of continuous functions and relations are established.

Key words: π-, σ-equivalence, biequivalence, monad, galaxy, figure, compact, revealed, continuous, function, relation.

Classification: Primary 54J05
Secondary 54D45, 54C05, 54C60

This paper goes on investigating of topological problematics in the Alternative Set Theory (AST) in the spirit of Vopěnka's book [V], i.e. on the base of some "indiscernibility" equivalence enabling to formalize such notions as "nearness" and "continuity" in a different and - at least in our opinion - more natural way than in the classical topology. The relation of "indiscernibility" or "infinitesimal nearness" serves as a mathematization of the horizon of discerning ability either of a man or of a measuring device. The majority of observations, however, meets with one more horizon yet - the horizon of "accessibility" or "reachability within sight".

This is in fact the most common appearance of the phenomenon of horizon in everyday life, nevertheless, as far as we know, it was not studied by the classical topology up to this time. Our article is the first attempt to fill this gap within the framework of the AST.

Needless to emphasize, neither the indiscernibility nor the accessibility relations as occurring e.g. by optical observations are transitive. Hence, though they both are naturally reflexive and symetric relations, they need not be equivalences. If all 1
the same restrict our study to equivalences of indiscernibility and accessibility, it will be a useful idealization enabling to treat the problematics by means similar in some sense to the classical ones. Last but not least, the understanding of the finite and infinite within the AST throws quite a different light upon this question, as far as for a general equivalence relation R and an arbitrary sequence (set function) $\langle x_0, x_1, \ldots, x_\nu \rangle$ such that $(\forall \alpha < \nu)\ \langle x_\alpha, x_{\alpha+1} \rangle \in R$, the conclusion $\langle x_0, x_\nu \rangle \in R$ follows only for "small" i.e. finite natural numbers ν. For a "large" i.e. infinite natural number ν $\langle x_0, x_\nu \rangle \notin R$ may well happen.

Each observation produces a sequence of "sharp" discernibility criteria leading to the horizon of discernibility and a sequence of "sharp" accessibility criteria leading to the horizon of accessibility. Two objects are indiscernible under such an observation if all criteria fail to distinguish between them, they are accessible if they are accessible at least according to one such criterion. The phenomenon of indiscernibility was formalized by the notion of a π-equivalence (i.e. an equivalence which is a π-class) in [V]. We will formalize the phenomenon of accessibility by the notion of a σ-equivalence (i.e. an equivalence which is a σ-class). A simultaneous investigation of both these notions requires the satisfaction of a single natural condition: any two indiscernible points are accessible.

A pair of classes $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ where $\doteq$ is a π-equivalence and $\overset{\bullet}{\leftrightarrow}$ is a σ-equivalence is called a biequivalence if $\doteq$ is a subclass of $\overset{\bullet}{\leftrightarrow}$, i.e. iff for all x,y $x \doteq y$ implies $x \overset{\bullet}{\leftrightarrow} y$. $x \doteq y$ is read "x is indiscernible from y" or "x and y are infinitesimally near" and $x \overset{\bullet}{\leftrightarrow} y$ is read as "x is accessible from y" or "x and y are finitely distant" etc. $\neg x \doteq y$ is abbreviated to $x \not\doteq y$ ("x is discernible from y") and so is $\neg x \overset{\bullet}{\leftrightarrow} y$ to $x \overset{\bullet}{\not\leftrightarrow} y$ ("x is not accessible from y").

The aim of the first part of this paper is to list only some very basical results concerning biequivalences. A more detailed study of several naturally arising questions is postponed into the nearest future. The traditional education countenanced by the modern physics contributes to the general extension of the opinion that the macrostructure (e.g. that of the Universe) is determined by the microstructure. Our investigation remains still tributary to this viewpoint, as well. The main attention will be paid to the study of indiscernibility phenomena (π-equivalences). The accessibility phenomena (σ-equivalences) will play rather an auxiliary role: they enable a natural restriction of the domain of our investigation (e.g. to a single galaxy - the class of objects accessible from a given object). Such restrictions often bring substantial simplifications. This restriction principle also motivates the study of π-equivalences identifying sets whose shapes in a given π-equivalence have the same trace on a given σ-class.

The second part of our paper deals with the notions of continuous function and relation. The connections between several possible concepts of continuity are investigated. For a large family of "well behaved" relations all these notions coincide. Various "natural" $\mathfrak{X}$-equivalences which can be introduced on classes of functions or relations are shown to give the same result for continuous relations from a given $\mathfrak{X}$-equivalence to another one. Using this fact, some compactness properties for continuous relations between compacta can be proved.

The authors are indebted to the members of the Prague seminar on the AST especially to Petr Vopěnka for valuable discussions.

Preliminaries

The reader is assumed to be familiar with [V]. FZ denotes the class of finite integers. Variables k,m,n are used sometimes also for finite integers not just for natural numbers.

The composition of classes X and Y is defined by

$$X \circ Y = \{\langle x,y\rangle ;\ (\exists z)(\langle x,z\rangle \in X \ \&\ \langle z,y\rangle \in Y)\},$$

and the ν-th iterated composition of the class X is

$$X^0 = \{\langle x,x\rangle ;\ x \in V\} = \mathrm{Id},$$

$$X^\nu = \{\langle x,y\rangle ;\ (\exists f)(\mathrm{dom}(f) = \nu + 1 \ \&\ x = f(0) \ \&\ f(\nu) = y \ \&\ (\forall \alpha < \nu)\langle f(\alpha), f(\alpha+1)\rangle \in X)\}$$

if $\nu \in N - \{0\}$ and $X^\nu = (X^{-\nu})^{-1}$ if $-\nu \in N - \{0\}$. The notation X^ν will be used in this sense solely for X being a relation. According to the context it will be always clear whether X^ν denotes the iterated composition or the ν-th cartesian power

$$X^\nu = \{f;\ \mathrm{dom}(f) = \nu \ \&\ \mathrm{rng}(f) \subseteq X\}$$

(here the function f with domain ν is identified with the "ν-tuple" $\langle f(0), f(1), \ldots, f(\nu-1)\rangle$).

Recall that a class X is called revealed if for each its countable subclass Y there is a set u such that $Y \subseteq u \subseteq X$; X is a fully revealed class if for each normal formula $\varphi(x_0, X_0)$ of the language FL (or FL_V - it is the same - see [S-V 1]) the class $\{x; \varphi(x,X)\}$ is revealed. A relation R will be called conditionally revealed if for each revealed class $X \subseteq dom(R)$ the restricted relation $R \restriction X$ is revealed.

Theorem 1. The following conditions are equivalent for every relation R:

(1) R is conditionally revealed;

(2) for every set $u \subseteq dom(R)$ $R \restriction u$ is revealed;

(3) for every at most countable relation $S \subseteq R$ such that $dom(S) \subseteq u \subseteq dom(R)$ for some set u there is a set relation r such that $S \subseteq r \subseteq R$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Let $S \subseteq R$, $S \preccurlyeq FN$, $dom(S) \subseteq u \subseteq dom(R)$. Then $S \subseteq R \restriction u$. Since $R \restriction u$ is revealed there is an r such that $S \subseteq r \subseteq R \restriction u \subseteq R$.

(3) $\Rightarrow$ (1): Let $X \subseteq dom(R)$ be a revealed class, $S \subseteq R \restriction X$, $S \approx FN$. Then dom(S) is an at most countable subclass of X. Hence, $dom(S) \subseteq u \subseteq X \subseteq dom(R)$ for some u. Thus there is an r such that $S \subseteq r \subseteq R$. Then also $S \subseteq r \restriction X$. Since X and r are revealed, $r \restriction X$ is also revealed and $S \subseteq s \subseteq r \restriction X \subseteq R \restriction X$ holds for some s.

A relation S is called a prolongation of the relation R if $R = S \restriction dom(R)$. Thus a function G is a prolongation of a function F iff $F \subseteq G$.

Corollary. Every revealed relation is conditionally revealed. More generally, every relation with revealed prolongation is conditionally revealed. If R is a relation and dom(R) is revealed then R is conditionally revealed iff R is revealed.

We record without proof one more result:

Theorem 2. If R and S are revealed (conditionally revealed) relations then the relation $R \circ S$ is revealed (conditionally revealed).

1. Biequivalences, compatibility and compact classes

For well known reasons (see [V]) it suffices to assume that all the equivalences considered have the same domain - the whole universal class V. All the results obtained for them apply to equivalences with arbitrary set-theoretically definable domains, particularly, to equivalences on sets. In such a case both the π- and the σ-equivalences taking part in a biequivalence are assumed to have the same domain, called the domain of the biequivalence. Note that a common domain of a π- and a σ-equivalence is necessarily a set-theoretically definable class.

A codable system $\{R_n; n \in FZ\}$ of set-theoretically definable, reflexive and symetric relations is called a bigenerating sequence provided for each n holds $R_n \circ R_n \subseteq R_{n+1}$. Similarly as for π-equivalences in [V], the following theorem can be proved.

Theorem 3. A pair of classes $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$ is a biequivalence iff there is a bigenerating sequence $\{R_n; n \in FZ\}$ such that $\doteq$ is the intersection and $\overset{\cdot}{\leftrightarrow}$ is the union of all the relations R_n.

In this case $\{R_n; n \in FZ\}$ is called the bigenerating sequence of the biequivalence $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$. Then for each m one can obtain a generating sequence $\{P_n; n \in FN\}$ of the π-equivalence $\doteq$ putting $P_0 = V^2$ and $P_n = R_{m-n}$ for $n > 0$. Similarly, for each m $\{S_n;\ n \in FN\}$ where $S_0 = Id$ and $S_n = R_{m+n}$ for $n > 0$, can be called the generating sequence of the σ-equivalence $\overset{\cdot}{\leftrightarrow}$. The precise definition is left to the reader.

Given a biequivalence $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$ the notions of the monad and the galaxy of a point x and those of the figure and the expansion of a class X can be introduced as follows:

$$Mon(x) = \{ y;\ y \doteq x \} \quad , \quad Fig(X) = \{ y;\ (\exists x \in X)\ y \doteq x \} ,$$

$$Gal(x) = \{ y;\ y \overset{\cdot}{\leftrightarrow} x \} \quad , \quad Exp(X) = \{ y;\ (\exists x \in X)\ y \overset{\cdot}{\leftrightarrow} x \} .$$

When the distinction between several biequivalences e.g. $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$, $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$, etc. will be necessary, we will write $Mon^{\cdot}(x)$, $Gal^{\cdot}(x)$, $Fig^{+}(X)$, etc.

Many results from [V] concerning compact σ-equivalences (equivalences of indiscernibility in the terminology of [V]) remain valid for arbitrary σ-equivalences or can be generalized easily to them. We state here only those generalizations which are necessary for our aims. The reference to [V] enables to shorten some proofs to mere sketches or completely to drop them.

We shall formulate a further condition imposed on biequivalences resulting from the following observation: No infinite set can be grasped perfectly at once in its totality within discernation of each of its individual elements. Thus any infinite set of pairwise accessible elements has to contain at least two indiscernible elements; or which is the same, any infinite set of pairwise discernible elements has to reach beyond the horizon, i.e. it must contain at least two inaccessible elements.

A biequivalence $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$ is called compatible if for each infinite set u holds

$$(\forall x,y \in u)\ x \overset{\cdot}{\leftrightarrow} y \Rightarrow (\exists x,y \in u)(x \neq y \ \&\ x \doteq y),$$

or equivalently

$$(\forall x,y \in u)(x \doteq y \Rightarrow x = y) \Rightarrow (\exists x,y \in u)\ x \not\leftrightarrow y .$$

Let us point out two extremal cases:
A π-equivalence $\doteq$ is called compact if $\langle \doteq, V^2 \rangle$ is a compatible biequivalence. Dually, a σ-equivalence $\overset{\cdot}{\leftrightarrow}$ is called discrete if $\langle \mathrm{Id}, \overset{\cdot}{\leftrightarrow} \rangle$ is a compatible biequivalence. Thus $\overset{\cdot}{\leftrightarrow}$ is discrete iff each of its galaxies is at most countable.

Example 1. The π-equivalence "$x \overset{\circ}{=} y$ iff for each set-theoretical formula $\varphi(x_0) \in FL$ holds $\varphi(x) \equiv \varphi(y)$" is compact (see [V]). It is the equivalence of the orbital partition of the group of all automorphisms of the universe V acting on V.

Example 2. The σ-equivalence "$x \overset{\circ}{\leftrightarrow} y$ iff there is a set-theoretical formula $\varphi(x_0,x_1) \in FL$ such that

$$(\forall x_0)(\exists ! x_1)\varphi(x_0,x_1)\ \&\ (\forall x_1)(\exists ! x_0)\ \varphi(x_0,x_1)\ \&\ \varphi(x,y)"$$

is discrete. It is the equivalence of the orbital partition of the group of all set-theoretically definable without parameters one-to-one maps $F:V \longrightarrow V$. Note that for each such an F and each automorphism A holds $F \circ A = A \circ F$ and both the groups have only the identity map in common. Thus the least group of one-to-one maps $V \longrightarrow V$ containing both the mentioned groups is isomorphic to their direct product. Though $\overset{\circ}{=}$ and $\overset{\circ}{\leftrightarrow}$ are dual in some sense, $\langle \overset{\circ}{=}, \overset{\circ}{\leftrightarrow} \rangle$ is not a biequivalence.

Example 3. The biequivalence $\langle \doteq, \overset{\cdot}{\leftrightarrow} \rangle$ on the class of all rational numbers RN defined by

$$x \doteq y \equiv (\forall n \in FN)\ |x - y| < 1/(n+1)$$

$$x \overset{\cdot}{\leftrightarrow} y \equiv (\exists n \in FN)\ |x - y| < n$$

is compatible.

Example 4. Using the biequivalence $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ from the previous Example one can define for each $a \in RN$ a compatible biequivalence $\langle \overset{a}{=}, \overset{a}{\leftrightarrow} \rangle$ on RN as follows:

$$x \overset{a}{=} y \equiv x = a = y \vee (x \neq a \neq y \ \& \ (x - a)/(y - a) \doteq 1)$$

$$x \overset{a}{\leftrightarrow} y \equiv x = a = y \vee (x \neq a \neq y \ \& \ 0 \not\doteq (x - a)/(y - a) \overset{\bullet}{\leftrightarrow} 1).$$

These biequivalences seem to be promising for the study of functions on rationals and complex rationals near their singularities.

Example 5. For each set u let us introduce a biequivalence $\langle \doteq^u, \overset{\bullet}{\leftrightarrow}{}^u \rangle$ on the class $RN^u = \{ f; \text{dom}(f) = u \ \& \ \text{rng}(f) \subseteq RN \}$ by

$$f \doteq^u g \equiv (\forall x \in u) \ f(x) \doteq g(x)$$

$$f \overset{\bullet}{\leftrightarrow}{}^u g \equiv (\forall x \in u) \ f(x) \overset{\bullet}{\leftrightarrow} g(x)$$

where $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ is taken from Example 3. Then $\langle \doteq^u, \overset{\bullet}{\leftrightarrow}{}^u \rangle$ is compatible iff u is finite. Moreover, every set-theoretically definable class $X \subseteq RN^u$ containing the monad of at least one $f \in RN^u$ contains a set $v \mathrel{\hat{\approx}} u$ of pairwise discernible elements.

The reader is kindly asked to complete the proofs of the assertions from Examples 1 - 5.

Let us recall that given a symetric relation R a class X is called an R-net if for all $x,y \in X$ $\langle x,y \rangle \in R$ implies $x = y$. X is a maximal R-net on Z if X is an R-net and $Z \subseteq R''X$. A relation R is called an upper (lower) bound of the $\mathfrak{X}$-equivalence $\doteq$ ($\mathfrak{C}$-equivalence $\overset{\bullet}{\leftrightarrow}$) if R is set-theoretically definable, reflexive, symetric and $\doteq$ is a subclass of R (R is a subclass of $\overset{\bullet}{\leftrightarrow}$). R is called a mean bound of the biequivalence $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ if it is simultaneously an upper bound of $\doteq$ and a lower bound of $\overset{\bullet}{\leftrightarrow}$. Clearly, each relation from the bigenerating sequence is a mean bound of $\langle \doteq, \dot{\leftrightarrow} \rangle$.

Let $\doteq$ be a $\mathfrak{T}$-equivalence. A class X is called pseudocompact in $\doteq$ if every infinite subset of X contains at least two different indiscernible elements. According to this definition every subclass of a pseudocompact class is pseudocompact. Pseudocompactness is a rather weak property since there are e.g. uncountable classes without any infinite subsets which are automatically pseudocompact.

Notice that for an upper bound R of $\doteq$ and any class X holds $X \subseteq R''X$, and even more

$$(\forall x \in X)(\exists y \in X)\ \mathrm{Mon}(x) \subseteq R''\{y\} ,$$

hence the codable class $\{R''\{y\};\ y \in X\}$ forms an "open cover" of X.

<u>Theorem 4</u>. Let $\doteq$ be a $\mathfrak{T}$-equivalence and X be a revealed class. The following conditions are equivalent:

(1) X is pseudocompact in $\doteq$;

(2) for each upper bound R of $\doteq$ there is a finite maximal R-net $u \subseteq X$ on X;

(3) for each upper bound R of $\doteq$ there is a finite set $u \subseteq X$ such that $X \subseteq R''u$.

<u>Proof</u>. (1) $\Rightarrow$ (2): If for each n there were an R-net $u \subseteq X$ with exactly n elements then by the prolongation axiom an infinite R-net $u \subseteq X$ could be obtained, contradicting the pseudocompactness of X. Hence, there is an $n \in FN$ such that each R-net $u \subseteq X$ has at most n-elements. Then every R-net $u \subseteq X$ with maximal possible number of elements is maximal on X.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1): Let $\{R_n;\ n \in FN\}$ be a generating sequence of $\doteq$ and for each n $u_n \subseteq X$ be a finite set such that $X \subseteq R_n''u_n$. Let

$v \subseteq X$ be an infinite set. For each $n \in FN$ there is an $x_n \in u_n$ such that the set $v \cap R_n``\{x_n\}$ is infinite. By the axiom of prolongation there is an $x \in X$ such that the class $v \cap Mon(x)$ contains at least two elements.

A class X is called compact in the $\mathcal{X}$-equivalence $\doteq$ if it is pseudocompact and revealed. We let to the reader the proof of the following

<u>Theorem 5</u>. Let X be a compact class in the $\mathcal{X}$-equivalence $\doteq$. Then for each $\nu \in N-FN$ there is a set $u \gtrsim \nu$ such that $u \subseteq X \subseteq Fig(u)$. Hence $Fig(X) = Fig(u)$ is a compact $\mathcal{X}$-class.

<u>Corollary</u>. For a revealed class X the following conditions are equivalent:
(1) X is pseudocompact; (2) X is compact; (3) Fig(X) is pseudocompact; (4) Fig(X) is compact; (5) Fig(X) is a compact $\mathcal{X}$-class; (6) $(\forall \nu \in N-FN)(\exists u)(u \gtrsim \nu \ \& \ u \subseteq X \ \& \ Fig(X) = Fig(u))$.

<u>Theorem 6</u>. Let $\{R_n;\ n \in FN\}$ be a generating sequence of the $\mathcal{X}$-equivalence $\doteq$, $\{X_n,\ n \in FN\}$ be a sequence of classes and $\{u_n;\ n \in FN\}$ be a sequence of sets such for each n $X_n \subseteq X_{n+1}$ and $X_n \subseteq R_n``u_n$. Then for every set u such that $\bigcup\{u_n;\ n \in FN\} \subseteq u$ holds $\bigcup\{X_n;\ n \in FN\} \subseteq Fig(u)$.

<u>Proof</u>. If $x \in X_n$ then there is a sequence $\{y_k;\ k \in FN\}$ such that $y_k \in u_{n+k} \cap R''_{n+k}\{x\}$. By the axiom of prolongation, there is a $y \in u$ such that $x \doteq y$.

Using Theorem 6 one can prove similarly as in [V]

<u>Theorem 7</u>. Let $\doteq$ be a $\mathcal{X}$-equivalence, $\{X_n;\ n \in FN\}$ be a sequence of revealed classes and $X = \bigcup\{X_n;\ n \in FN\}$. The following conditions are equivalent:

(1) X is pseudocompact;

(2) for each $n \in FN$ X_n is pseudocompact (hence compact);

(3) for each infinite natural number ν there is a set $u \hat{\lesssim} \nu$ such that $X \subseteq \mathrm{Fig}(u)$;

(4) for each infinite set $u \subseteq X$ there is an infinite set $v \subseteq u$ such that $(\forall x,y \in v)\ x \doteq y$.

A class which is the union of countably many compact classes will be called σ-compact (in $\doteq$).

Corollary. If X is a σ-compact class then $\mathrm{Fig}(X)$ is a σ-compact $\sigma\pi$-class.

Corollary. A biequivalence $\langle \doteq, \dot{\leftrightarrow} \rangle$ is compatible iff for each x $\mathrm{Gal}(x)$ is σ-compact in $\doteq$.

Theorem 8. Let $\doteq$ be a π-equivalence and A be a σ-class which is a figure in $\doteq$. If A is σ-compact then there is a compact π-equivalence $\pm$ such that $(\doteq \restriction A) = (\pm \restriction A)$.

Proof. Put $x \pm y \equiv \mathrm{Mon}'(x) \cap A = \mathrm{Mon}'(y) \cap A$.

Theorem 9. Let $\doteq$ and $\pm$ be two π-equivalences, K be a compact class in $\pm$ and C be a conditionally revealed relation such that $K \subseteq \mathrm{dom}(C)$, for each $x \in K$ the class $C''\{x\}$ is compact in $\doteq$ and

$$(\forall x,y \in \mathrm{dom}(C))(x \pm y \Rightarrow C''\{x\} = C''\{y\}).$$

Then the class $C''K$ is compact in $\doteq$.

Proof. It suffices to show that for every revealed relation C such that $\mathrm{dom}(C)$ is compact in $\pm$, $C''\{x\}$ is compact in $\doteq$ for each $x \in \mathrm{dom}(C)$ and for all $x,y \in \mathrm{dom}(C)$ $x \pm y$ implies $C''\{x\} = C''\{y\}$ also $\mathrm{rng}(C)$ is compact in $\doteq$. Obviously, $\mathrm{rng}(C)$ is revealed. Let $u \subseteq \mathrm{rng}(C)$ be an infinite set. As for each $x \in \mathrm{dom}(C)$ the class $u \cap C''\{x\}$ is revealed, it suffices to show that for some $x \in \mathrm{dom}(C)$ the class $u \cap C''\{x\}$ is infinite. Then it will contain an infinite subset and the compactness of $C''\{x\}$ will complete the proof. Contrarywise, assume that for each finite set $w \subseteq \mathrm{dom}(C)$

$u \cap C"w$ is also finite. Then one can construct two sequences $\{a_n; n \in FN\}$ and $\{b_n; n \in FN\}$ such that for each n holds $b_n \in dom(C)$ and $a_n \in (u - C"\{b_i; i < n\}) \cap C"\{b_n\}$. Then $F = \{\langle b_n, a_n\rangle ; n \in FN\} \subseteq C$ is a one-one countable function and $rng(F) \subseteq u$. Then there is a one-one set function f such that $F \subseteq f \subseteq C$ and $rng(f) \subseteq u$. Then $dom(f)$ is an infinite subset of $dom(C)$. There has to be an infinite set $v \subseteq dom(f)$ such that $(\forall x,y \in v)\ x \stackrel{\pm}{=} y$. According to the last property of C, for all $x,y \in v$ holds $f(y) \in C"\{y\} = C"\{x\}$, thus $(\forall x \in v)\ f"v \subseteq u \cap C"\{x\}$. As f is one-one, $f"v$ is infinite. This contradiction proves the Theorem.

In particular, putting $(\doteq) = (\stackrel{\pm}{=}) = C$ in the last Theorem, one obtains a new proof of the fact that the figure of a compact class is compact.

<u>Theorem 10</u>. Let $\doteq$ be a π-equivalence. The following conditions are equivalent:

(1) there is a σ-equivalence $\stackrel{\bullet}{\leftrightarrow}$ such that $\langle \doteq, \stackrel{\bullet}{\leftrightarrow}\rangle$ is a compatible biequivalence;

(2) there is an upper bound R of $\doteq$ such that the class $R"\{x\}$ is compact for each x.

<u>Proof</u>. (1) $\Rightarrow$ (2): Any mean bound R of the compatible biequivalence $\langle \doteq, \leftrightarrow\rangle$ has the required property.

(2) $\rightarrow$ (1): Let S be an upper bound of $\doteq$ such that $S \circ S \subseteq R$. Then the relation $C = S \circ (\doteq)$ is revealed, satisfies $x \doteq y \rightarrow C"\{x\} = C"\{y\}$ and the class $C"\{x\} \subseteq R"\{x\}$ is compact for each x. Applying Theorem 9, the compactness of all the classes $(C^n)"\{x\}$ $(n \in FN - \{0\})$ follows by an induction argument. Hence all the classes $(S^n)"\{x\} \subseteq (C^n)"\{x\}$ are compact, as well. Then $(\stackrel{\bullet}{\leftrightarrow}) = \bigcup\{S^n; n \in FN\}$ is a σ-equivalence, $\doteq$ is a subclass of

$\stackrel{\bullet}{\leftrightarrow}$, and the biequivalence $\langle \doteq, \stackrel{\bullet}{\leftrightarrow} \rangle$ has σ-compact galaxies.

Remark. Theorem 10 reminds of the following result from the classical topology:

"For a topological space X the following conditions are equivalent:

(1) X is the direct sum (disjoint disconnected union) of σ-compact spaces;

(2) the topology of X is induced by a uniformly locally compact uniformity;

(3) X is locally compact and paracompact."

(See e.g. [K].)

As any galaxy of a biequivalence is a clopen class (see [V]), the conditions (1) of Theorem 10 and of our Remark seem to be very similar. (However in the AST the domain of a biequivalence can be well connected even if it consists of more than one galaxy.) The relationship of conditions (2) is even more transparent. This could suggest the idea to combine the Stone result on paracompactness of metrizable spaces (see [K]) from the classical topology and the Mlček's metrization theorem for π-equivalences (see [M2]) from the AST. Thus one could expect that the following condition is equivalent to conditions (1) and (2) of Theorem 10:

(3) For each x there is a set-theoretically definable class X such that $\mathrm{Mon}(x) \subseteq X$ and X is compact in $\doteq$.

Though (2) $\Rightarrow$ (3) is trivial, the following example shows that this implication cannot be reversed.

Example 6. Let ν be an infinite natural number and $\doteq^{\nu}$ be the π-equivalence on RN^{ν} introduced in Example 5. We put

$$T = \{\langle f,g,\alpha\rangle \in RN^{\nu} \times RN^{\nu} \times N;\ \{\lambda < \nu;\ f(\lambda) \neq g(\lambda)\} \lesssim 1 \ \&\ (\forall \lambda < \nu)\ |f(\lambda) - g(\lambda)| \in \{0, 1/2^{\alpha+1}\}\},$$

$$a = \{ \langle g, \alpha \rangle \in RN^{\nu} \times \nu ;\ g(0) = \alpha \ \& \ (\forall \lambda < \nu)(\lambda \neq 0 \rightarrow g(\lambda) = 0\ ) \} .$$

Then the set $T''a \subseteq RN^{\nu}$ is the union of sets $T'' \{ \langle g, \alpha \rangle \}$ $(\alpha < \nu)$ each of them consisting of the corresponding function g and 2ν functions differing from g in exactly one argument by the value $\pm 1/2^{\alpha+1}$. Then the π-equivalence $\doteq^{\nu} \upharpoonright T''a$ satisfies (3) but not (2) of the last Theorem.

If $\{ \overset{\cdot}{\leftrightarrow}_n ;\ n \in FN \}$ is a sequence of σ-equivalences then the least equivalence

$$\mathring{\bigcup} \{ \overset{\cdot}{\leftrightarrow}_n ;\ n \in FN \} = \bigcup \{ ((\overset{\cdot}{\leftrightarrow}_0) \circ \ldots \circ (\overset{\cdot}{\leftrightarrow}_n))^n ;\ n \in FN \}$$

containing all the equivalences $\overset{\cdot}{\leftrightarrow}_n$ is a σ-equivalence, again.

<u>Theorem 11</u>. Let $\{ \doteq_n ;\ n \in FN \}$ be a sequence of π-equivalences and $\{ \overset{\cdot}{\leftrightarrow}_n ;\ n \in FN \}$ be a sequence of σ-equivalences such that for all m,n $\langle \doteq_m , \overset{\cdot}{\leftrightarrow}_n \rangle$ is a compatible biequivalence. Then $\langle \bigcap \{ \doteq_n ;\ n \in FN \} ,\ \mathring{\bigcup} \{ \overset{\cdot}{\leftrightarrow}_n ;\ n \in FN \} \rangle$ is also a compatible biequivalence.

<u>Proof</u>. We put $(\doteq) = \bigcap \{ \doteq_n ;\ n \in FN \}$ and $(\overset{\cdot}{\leftrightarrow}) = \mathring{\bigcup} \{ \overset{\cdot}{\leftrightarrow}_n ;\ n \in FN \}$. Obviously, $\langle \doteq , \overset{\cdot}{\leftrightarrow} \rangle$ is a biequivalence. By a slight modification of the proof of compactness of the intersection of countably many compact π-equivalences (see [V]) it can be shown that for each n the biequivalence $\langle \doteq , \overset{\cdot}{\leftrightarrow}_n \rangle$ is compatible. Let $\{ S_{nk} ;\ k \in FN \}$ be a generating sequence of $\overset{\cdot}{\leftrightarrow}_n$. Let

$$T_k = (S_{0k} \circ \ldots \circ S_{kk})^k .$$

Then $(\overset{\cdot}{\leftrightarrow}) = \bigcup \{ T_k ;\ k \in FN \}$. Similarly as in the proof of Theorem 10 one can show that for each x and each k the class $T_k'' \{ x \}$ is compact in $\doteq$.

A biequivalence $\langle \doteq , \overset{\cdot}{\leftrightarrow} \rangle$ is called tighter than the biequivalence $\langle \pm , \overset{\pm}{\leftrightarrow} \rangle$ (and $\langle \pm , \overset{\pm}{\leftrightarrow} \rangle$ is looser than $\langle \doteq , \overset{\cdot}{\leftrightarrow} \rangle$) if

$(\overset{+}{=}) \subseteq (\dot{=})$ and $(\overset{\bullet}{\leftrightarrow}) \subseteq (\overset{+}{=})$.

Theorem 12. If the biequivalence $\langle \dot{=}, \overset{\bullet}{\leftrightarrow} \rangle$ is tighter than the biequivalence $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ and $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is compatible then $\langle \dot{=}, \overset{\bullet}{\leftrightarrow} \rangle$ is also compatible.

For every σ-class A $\{\langle u,v \rangle ; u \cap A = v \cap A\}$ is a π-equivalence which is compact iff A is at most countable. More generally, let $\dot{=}$ be a π-equivalence and A be a σ-class which is a figure in $\dot{=}$. The power equivalence of $\dot{=}$ restricted to A is defined as follows:

$$u \doteq_A v \equiv \mathrm{Fig}(u) \cap A = \mathrm{Fig}(v) \cap A.$$

Since A is a figure in $\dot{=}$, we have

$$u \doteq_A v \equiv \mathrm{Fig}(u \cap A) = \mathrm{Fig}(v \cap A).$$

Theorem 13. Let $\dot{=}$ be a π-equivalence and A be σ-class and a figure in $\dot{=}$. For arbitrary u,v holds

$$u \doteq_A v \equiv (\exists w)(u \cap A = w \cap A \ \& \ \mathrm{Fig}(w) = \mathrm{Fig}(v)).$$

Proof. Let $\{A_n ; n \in FN\}$ be an increasing sequence of set--theoretically definable classes whose union is A and $\{R_n ; n \in FN\}$ be a generating sequence of $\dot{=}$. The reader can easily verify that $(\forall n)(\exists k)\ R_k{}''A_n \subseteq A$ since A is a figure. Without loss of generality we can assume that the sequences were chosen in such a way that $R_n{}''A_n \subseteq A$ holds for each n. Let us define a sequence of sets by

$$w_n = (u \cap R_n{}''A_n) \cup (v - A_n).$$

Then for each n holds $u \cap A_n = w_n \cap A_n$, and $\mathrm{Fig}(w_n) = \mathrm{Fig}(v)$.

To prove the last claim assume $x \in w_n$. If $x \notin A_n$ then $x \in v$ or $x \in u \cap (R_n{}''A_n - A_n) \subseteq u \cap A$. If $x \in A_n$ then $x \in u \cap A$. In any case $x \in Fig(v)$. Similarly, if $x \in v$ then either $x \notin A_n$ and $x \in w_n$, or $x \in A_n$ $x \in v \cap A_n \subseteq Fig(v) \cap A_n = Fig(u) \cap A_n \subseteq Fig(u \cap R_n"A_n) \subseteq Fig(w_n)$. By the axiom of prolongation there is a w such that $u \cap A = w \cap A$ and $Fig(w) = Fig(v)$.

Now, let w satisfy both the conditions. Then

$$Fig(u) \cap A = Fig(u \cap A) = Fig(w \cap A) = Fig(w) \cap A = Fig(v) \cap A.$$

Corollary. For every $\mathfrak{X}$-equivalence $\doteq$ and every σ-class A which is a figure in $\doteq$ the restricted power equivalence $\doteq_A$ is a $\mathfrak{X}$-equivalence. It is the least equivalence E such that for all u,v $u \cap A = v \cap A \Rightarrow \langle u,v\rangle \in E$ and $Fig(u) = Fig(v) \Rightarrow \langle u,v\rangle \in E$.

The reader will easily find examples that neither the Theorem nor the Corollary have to be true without the assumption that A is a figure.

Theorem 14. Let $\doteq$ be a $\mathfrak{X}$-equivalence and A be a σ-class which is a σ-compact figure in $\doteq$. Then the $\mathfrak{X}$-equivalence $\doteq_A$ is compact.

Proof. Let $\{A_n;\ n \in FN\}$ be an increasing sequence of set-theoretically definable (hence compact) classes whose union is A. For each n we put $u \doteq_n v$ iff $Fig(u \cap A_n) \cap A_n = Fig(v \cap A_n) \cap A_n$. Obviously, each $\doteq_n$ is a $\mathfrak{X}$-equivalence. We claim that it is compact. Given any infinite set s, there are either $u \neq v$ in s such that $u \cap A_n = v \cap A_n$, or $f(u) = u \cap A_n$ is a one-to-one map of s onto a subset of $P(A_n)$. Since $\doteq\restriction A_n$ is a compact $\mathfrak{X}$-equivalence, also its (unrestricted) power equivalence is compact (see [V]). Thus there are $u \neq v$ in s such that $u \doteq_n v$. Now, it suffices to show that the compact $\mathfrak{X}$-equivalence $\bigcap\{\doteq_n;\ n \in FN\}$ is finer than $\doteq_A$. If $u \doteq_n v$ for each n then

$$\bigcup\{Fig(u \cap A_n) \cap A_n;\ n \in FN\} = \bigcup\{Fig(v \cap A_n) \cap A_n;\ n \in FN\}.$$

The equality $\bigcup\{\mathrm{Fig}(u \cap A_n) \cap A_n;\ n \in FN\} = \mathrm{Fig}(u) \cap A$ concludes the proof. In fact, one inclusion is trivial. Let $x \in \mathrm{Fig}(u) \cap A$. Then $x \doteq y$ for some $y \in u$. Since A is a figure there is an n such that $x, y \in A_n$. Then $x \in \mathrm{Fig}(u \cap A_n) \cap A_n$.

Corollary. Let $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ be a compatible biequivalence. Then for each point a $\doteq_{\mathrm{Gal}(a)}$ is a compact $\mathfrak{T}$-equivalence.

2. Continuous relations

Throughout this section $\doteq$ and $\overset{+}{=}$ denote two fixed $\mathfrak{T}$-equivalences with generating sequences $\{R_n;\ n \in FN\}$ and $\{S_n;\ n \in FN\}$, respectively. The variables R and S always denote upper bounds of $\doteq$ and $\overset{+}{=}$, respectively. Sometimes $\doteq$ and $\overset{+}{=}$ will be considered as parts of biequivalences $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ and $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$.

The product of the biequivalences $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ and $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ is the biequivalence $\langle \overset{\times}{=}, \overset{\times}{\leftrightarrow} \rangle$ with domain V^2 defined in the following natural way:

$$\langle a,x \rangle \overset{\times}{=} \langle b,y \rangle \equiv a \doteq b \;\&\; x \overset{+}{=} y,$$

$$\langle a,x \rangle \overset{\times}{\leftrightarrow} \langle b,y \rangle \equiv a \overset{\bullet}{\leftrightarrow} b \;\&\; x \overset{+}{\leftrightarrow} y.$$

On the base of Theorem 4 it is routine to check

Theorem 15. The biequivalence $\langle \overset{\times}{=}, \overset{\times}{\leftrightarrow} \rangle$ is compatible iff both $\langle \doteq, \overset{\bullet}{\leftrightarrow} \rangle$ and $\langle \overset{+}{=}, \overset{+}{\leftrightarrow} \rangle$ are compatible. In particular, $\overset{\times}{=}$ is compact iff both $\doteq$ and $\overset{+}{=}$ are compact; $\overset{\times}{\leftrightarrow}$ is discrete iff both $\overset{\bullet}{\leftrightarrow}$ and $\overset{+}{\leftrightarrow}$ are discrete.

A relation C is called pseudocontinuous from $\overset{+}{=}$ to $\doteq$ in the point $x \in \mathrm{dom}(C)$ if for each $y \in \mathrm{dom}(C)$ $y \overset{+}{=} x$ implies $\mathrm{Fig}^{\bullet}(C''\{y\}) = \mathrm{Fig}^{\bullet}(C''\{x\})$. C is called pseudocontinuous from $\overset{+}{=}$ to $\doteq$ on the class $X \subseteq \mathrm{dom}(C)$ if it is pseudocontinuous in each point $x \in X$; C is pseudocontinuous from $\overset{+}{=}$ to $\doteq$ if it is pseudocontinuous on $\mathrm{dom}(C)$.

Thus a function F is pseudocontinuous from $\doteq$ to $\dot{=}$ on the class $X \subseteq \mathrm{dom}(F)$ iff

$$(\forall x \in X)(\forall y \in \mathrm{dom}(F))(x \doteq y \Rightarrow F(x) \dot{=} F(y)).$$

Some further notions can be easily reduced to the notions already introduced. A relation C is pseudocontinuous from $\doteq$ to $\dot{=}$ with respect to the class M (in the point $x \in M \subseteq \mathrm{dom}(C)$, on the class $X \subseteq M \subseteq \mathrm{dom}(C)$) if $C \upharpoonright M$ is pseudocontinuous from $\doteq$ to $\dot{=}$ (in x, on X). Notice that if $M = \mathrm{Fig}^{+}(M) \subseteq \mathrm{dom}(C)$ then C is pseudocontinuous from $\doteq$ to $\dot{=}$ with respect to M iff it is pseudocontinuous on M.

In the sequel any continuity notion always means continuity from $\doteq$ to $\dot{=}$.

Theorem 16. Let C be a relation. Then C is pseudocontinuous iff $(\dot{=}) \circ C \circ (\doteq \upharpoonright \mathrm{dom}(C)) = (\dot{=}) \circ C$.

Proof. Let C be pseudocontinuous. If $a \dot{=} b$, $\langle b,y \rangle \in C$ and $y \doteq x \in \mathrm{dom}(C)$ then $a \in \mathrm{Fig}'(C''\{y\}) = \mathrm{Fig}'(C''\{x\})$ and $\langle a,x \rangle \in (\dot{=}) \circ C$. The other inclusion is trivial. Now, assume that the above equality holds. Let $x,y \in \mathrm{dom}(C)$, $x \doteq y$. Then

$$\begin{aligned}\mathrm{Fig}'(C''\{x\}) &= ((\dot{=}) \circ C)''\{x\} = ((\dot{=}) \circ C \circ (\doteq \upharpoonright \mathrm{dom}(C)))''\{x\} \\ &= ((\dot{=}) \circ C \circ (\doteq \upharpoonright \mathrm{dom}(C)))''\{y\} = ((\dot{=}) \circ C)''\{y\} \\ &= \mathrm{Fig}'(C''\{y\}).\end{aligned}$$

Note that every relation C satisfying the last presumption of Theorem 9 is pseudocontinuous. If C is a conditionally revealed pseudocontinuous relation then the relation $D = (\dot{=}) \circ C \circ (\doteq \mathrm{dom}(C)) = (\dot{=}) \circ C$ is also conditionally revealed and satisfies the last presumption of Theorem 9. If $C''\{x\}$ is compact in $\dot{=}$ then $D''\{x\} = \mathrm{Fig}'(C''\{x\})$ is also compact by the virtue of Theorem 5. Now, given any revealed class $X \subseteq \mathrm{dom}(C)$ the class $C''X$ is compact in $\dot{=}$ iff $D''X = \mathrm{Fig}'(C''X)$ is compact in $\dot{=}$. We have proved the following generalization of Theorem 9:

Theorem 17. Let C be a conditionally revealed pseudocontinuous relation. Let $K \subseteq \mathrm{dom}(C)$ be a compact class in $\stackrel{+}{=}$ such that for each $x \in K$ $C''\{x\}$ is compact in $\doteq$. Then the class $C''K$ is compact in $\doteq$.

As each one point set is compact, Theorem 17 has the following

Corollary. Let F be a conditionally revealed pseudocontinuous function. If $K \subseteq \mathrm{dom}(F)$ is a compact class in $\stackrel{+}{=}$ then $F''K$ is compact in $\doteq$.

Even for functions with compact domains the notion of pseudocontinuity is too weak to formalize the continuity phenomena.

Example 7. Let $I = \{x \in RN;\ 0 \leq x \leq 1\}$ be the unit interval of rational numbers, $\doteq$ be the common indiscernibility on rationals introduced in Example 1 and FRN be the class of all finite rational numbers (see [V]). Then the "Dirichlet function" on I

$$F(x) = \begin{cases} 1 & \text{if} \quad x \in I \cap \mathrm{Fig}(FRN) \\ 0 & \text{if} \quad x \in I - \mathrm{Fig}(FRN) \end{cases}$$

is pseudocontinuous from $\doteq\restriction I$ to $\doteq\restriction I$.

Extending the classical definition of continuity from functions to relations, a relation C will be called continuous (from $\stackrel{+}{=}$ to $\doteq$) in a point $x \in \mathrm{dom}(C)$ if for each upper bound R of $\doteq$ there is an upper bound S of $\stackrel{+}{=}$ such that $(C \circ S)''\{x\} \subseteq (R \circ C)''\{x\}$. C is called continuous on the class $X \subseteq \mathrm{dom}(C)$ if it is continuous in each $x \in X$; C is continuous if it is continuous on $\mathrm{dom}(C)$. Finally, C is called uniformly continuous if for each upper bound R of $\doteq$ there is an upper bound S of $\stackrel{+}{=}$ such that $C \circ S \restriction \mathrm{dom}(C) \subseteq R \circ C$.

The reader can asily verify the following facts:

(1) every uniformly continuous relation is continuous;

(2) if C is continuous in x and $C''\{x\}$ is revealed then C is pseudocontinuous in x;

(3) if C is continuous and $C''\{x\}$ is revealed for each $x \in \mathrm{dom}(C)$ then C is pseudocontinuous.

Mainly for the simplicity and transparentness of the notion of pseudocontinuity we examine some fairly weak conditions under which pseudocontinuity implies continuity or uniform continuity.

A relation D will be called an approximate prolongation of the relation C with respect to the π-equivalence $\doteq$ if for each $x \in \mathrm{dom}(C)$ holds $\mathrm{Fig}^{\cdot}(C''\{x\}) = \mathrm{Fig}^{\cdot}(D''\{x\})$. In the sequel an "approximate prolongation" always means an approximate prolongation with respect to the π-equivalence $\doteq$ fixed at the beginning of the section. Note that if D is an approximate prolongation of C then $\mathrm{dom}(C) \subseteq \mathrm{dom}(D)$ and $(\doteq)\circ D$ is a prolongation of $(\doteq)\circ C$. Obviously every prolongation of C is an approximate prolongation of C.

<u>Theorem 18</u>. Let C be a relation and $x \in \mathrm{dom}(C)$. Assume that there is a set-theoretically definable class X such that $\mathrm{Mon}^{+}(x) \subseteq X$ and a revealed approximate prolongation D of $C \upharpoonright X$ such that $D''\{x\}$ is fully revealed and D is pseudocontinuous in x. Then C is continuous in x.

<u>Proof</u>. Let $m \in FN$ be such that $S_m''\{x\} \subseteq X$. Let R be such an upper bound of $\doteq$ that for each $n \geqslant m$ there is a pair $\langle b_n, y_n\rangle \in C$ such that $\langle y_n, x\rangle \in S_n$ and $b_n \notin (R\circ C)''\{x\}$. Let R_1 be an upper bound of $\doteq$ satisfying $R_1\circ R_1 \subseteq R$. Then for each $n \geqslant m$ $\langle b_n, y_n\rangle \in (\doteq)\circ D$ and $b_n \notin (R_1\circ D)''\{x\}$. By the axiom of prolongation there is a pair $\langle b, y\rangle \in (\doteq)\circ D$ such that $y \doteq x$ and $b \notin (R_1\circ D)''\{x\}$. This contradiction proves the Theorem.

Our next result is a direct consequence of the last Theorem.

__Theorem 19__. Let C be a relation. Assume that for each $x \in \mathrm{dom}(C)$ there is an $X \in Sd_V$ such that $\mathrm{Mon}^+(x) \subseteq X$ and a revealed prolongation D of $C \restriction X$ such that $D''\{x\}$ is fully revealed and D is pseudocontinuous in x. Then C is continuous.

Theorems 18 and 19 have the following

__Corollary__. Let C be a revealed relation.

(1) If $x \in \mathrm{dom}(C)$ and the class $C''\{x\}$ is either fully revealed or pseudocompact (hence compact) in $\doteq$ then C is continuous in x iff C is pseudocontinuous in x.

(2) If for each $x \in \mathrm{dom}(C)$ the class $C''\{x\}$ is either fully revealed or compact in $\doteq$ then C is continuous iff C is pseudocontinuous.

__Proof__. It is enough to prove (1). The case when $C''\{x\}$ is fully revealed easily follows from Theorem 18. So let $C''\{x\}$ be compact and u be a set such that $\mathrm{Fig}^{\dot{}}(C''\{x\}) = \mathrm{Fig}^{\dot{}}(u)$. Then the class $X = V$ and the relation

$$D = (C - (V \times \{x\})) \cup (u \times \{x\})$$

satisfy the presumptions of Theorem 18.

Note that for a function F all the classes $F''\{x\} = \{F(x)\}$ where $x \in \mathrm{dom}(F)$ are both fully revealed and compact.

Thus pseudocontinuity implies continuity under some assumptions on local approximate prolongability to a revealed relation. To ensure uniform continuity the existence of certain global revealed approximate prolongations is needed. The next Theorem corresponds rather to the last Corollary than to Theorems 18, 19.

__Theorem 20__. Let C be a relation and D be a revealed approximate pseudocontinuous prolongation of C. If D is either fully revealed or for each $x \in dom(D)$ the class $D''\{x\}$ is compact in $\doteq$ then C is uniformly continuous.

Proof. Assume that D is fully revealed and R is such an upper bound of $\doteq$ that for each n there is a pair $\langle x_n, y_n \rangle \in S_n \cap dom(C)^2$ and an a_n such that $\langle a_n, x_n \rangle \in C$ and $\langle a_n, y_n \rangle \notin R \circ C$. Let R_1 be an upper bound of $\doteq$ satisfying $R_1 \circ R_1 \subseteq R$. Then for each n also holds $\langle x_n, y_n \rangle \in S_n \cap dom(D)^2$, $\langle a_n, x_n \rangle \in (\doteq) \circ D$ and $\langle a_n, y_n \rangle \notin R_1 \circ D$. By the axiom of prolongation there is a pair $\langle x, y \rangle \in (\doteq) \cap dom(D)^2$ and an a such that $\langle a, x \rangle \in (\doteq) \circ D$ and $\langle a, y \rangle \notin R_1 \circ D$ - a contradiction.

To prove the second case we record the following obvious

Lemma. A relation C is uniformly continuous iff for each countable class $X \subseteq dom(C)$ the restricted relation $C \upharpoonright X$ is uniformly continuous.

Now, let all the classes $D''\{x\}$ $(x \in dom(D))$ be compact and $X = \{x_k ; k \in FN\} \subseteq dom(C)$ be a countable class.

As all the classes $Fig(C''\{x_k\})$ are compact, for each n there is a sequence $\{u_{nk} ; k \in FN\}$ of finite sets such that $(\forall n,k)\ u_{nk} \subseteq D''\{x_k\} \subseteq R_n''u_{nk}$. Then there is a set relation d such that

$$\bigcup \{u_{nk} \times \{x_k\} \ ; \ \langle k,n \rangle \in FN^2\} \subseteq d \subseteq D \ ,$$

and consequently $(\forall k)\ Fig'(d''\{x_k\}) = Fig'(D''\{x_k\})$.
Hence there is a set $w \subseteq dom(d)$ containing X such that $(\forall x \in w)\ Fig'(d''\{x\}) = Fig'(D''\{x\})$ (all the classes $Fig'(D''\{x_k\})$ are π-classes). Then the set relation $d \upharpoonright w$ is a pseudocontinuous approximate prolongation of $C \upharpoonright X$. By the first part of Theorem 20 which was already proved $C \upharpoonright X$ is uniformly continuous. The Lemma completes the proof.

Theorems 18, 19, 20 are much more general than we really need. In most cases the functions and relations studied will be at least fully revealed (or even set-theoretically definable or sets). Our theorems can be then used to obtain results like the following:

Theorem 21. (1) Let C be a fully revealed relation. Then C is continuous (in the point $x \in \mathrm{dom}(C)$, on the class $X \subseteq \mathrm{dom}(C)$) iff C is pseudocontinuous (in x, on X).

(2) Let F be a revealed function. Then F is uniformly continuous iff it is pseudocontinuous.

(3) Let C be a fully revealed relation. Then C is uniformly continuous iff it is pseudocontinuous.

When studying relations on the universe V endowed with different π-equivalences $\dot{=}$ and $\overset{+}{=}$, the power of the π-equivalence $\overset{\times}{=}$ defined on the domain $P(V^2)$ by

$$r \overset{\times}{=}_p s \equiv \mathrm{Fig}^{\times}(r) = \mathrm{Fig}^{\times}(s)$$

seems to be the most promising framework for classifying their shapes. When studying functions then the π-equivalence

$$\mathrm{dom}(f) = \mathrm{dom}(g) \ \& \ (\forall x \in \mathrm{dom}(f))\ f(x) \dot{=} g(x))$$

seems to be more interesting and natural. It can be generalized to arbitrary relations as follows

$$r \overset{\cdot\cdot}{=} s \equiv (\forall x)\ \mathrm{Fig}^{\cdot}(r''\{x\}) = \mathrm{Fig}^{\cdot}(s''\{x\}).$$

The problem of finding the "best" π-equivalence on $P(V^2)$ classifying the behaviour of relations with respect to the original π-equivalences $\dot{=}$ and $\overset{+}{=}$ has a common solution for continuous relations.

Lemma. Let C be a relation and D be a pseudocontinuous relation such that $\mathrm{dom}(D)$ is a figure in $\overset{+}{=}$. The following conditions are equivalent:

(1) $(\forall x,y)(x \overset{+}{=} y \Rightarrow \mathrm{Fig}^{\cdot}(C''\{x\}) \subseteq \mathrm{Fig}^{\cdot}(D''\{y\}))$;

(2) $(\forall x)\,(\mathrm{Fig}^{\bullet}(C''\{x\}) \subseteq \mathrm{Fig}^{\bullet}(D''\{x\}))$;

(3) $\mathrm{Fig}^{\times}(C) \subseteq \mathrm{Fig}^{\times}(D)$.

<u>Proof</u>. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): If $\langle a,x\rangle \in C$ then $a \doteq b$ for some $b \in D''\{x\}$ and $\langle a,x\rangle \overset{\times}{=} \langle b,x\rangle \in D$.

(3) $\Rightarrow$ (1): Since D is pseudocontinuous and $\mathrm{dom}(D)$ is a figure in $\overset{+}{=}$, Theorem 16 yields

$$\mathrm{Fig}^{\times}(D) = (\doteq)\circ D\circ(\overset{+}{=}) = (\doteq)\circ D\circ(\overset{+}{=}\restriction \mathrm{dom}(D)) = (\doteq)\circ D.$$

If $x \overset{+}{=} y$ and $\langle a,x\rangle \in C$ then $\langle a,y\rangle \in \mathrm{Fig}^{\times}(D)$ and $a \doteq b$, $\langle b,y\rangle \in D$ for some b. Thus $a \in \mathrm{Fig}^{\bullet}(D''\{y\})$.

<u>Theorem 22</u>. Let C and D be pseudocontinuous relations such that $\mathrm{dom}(C)$ and $\mathrm{dom}(D)$ are figures in $\overset{+}{=}$. The following conditions are equivalent:

(1) $(\forall x,y)\,(x \overset{+}{=} y \Rightarrow \mathrm{Fig}^{\bullet}(C''\{x\}) = \mathrm{Fig}^{\bullet}(D''\{y\}))$;

(2) $(\forall x)\,(\mathrm{Fig}^{\bullet}(C''\{x\}) = \mathrm{Fig}^{\bullet}(D''\{x\}))$;

(3) $\mathrm{Fig}^{\times}(C) = \mathrm{Fig}^{\times}(D)$.

<u>Proof</u> is trivial in view of the Lemma.

Note that the Lemma and Theorem 22 apply to arbitrary equivalences $\doteq$, $\overset{+}{=}$ (without the assumption that they are π-classes) under the obvious extension of the definition of pseudocontinuity. Theorem 16 remains true, as well.

Let ν be an infinite natural number. Put

$$\alpha \overset{+}{=} \beta \equiv \alpha/\nu \doteq \beta/\nu \quad \text{for } \alpha,\beta \leq \nu$$

where $\doteq$ is the common π-equivalence on RN. Then $\overset{+}{=}$ is a compact π-equivalence on $\nu + 1$. In fact $\overset{+}{=}$ "coincides" with $\doteq$ on the set $\{\frac{\alpha}{\nu};\ 0 \leq \alpha \leq \nu\}$. Let us consider the linear space $RN^{\nu+1}$ of all $\nu + 1$ - tuples of rationals with the operations defined componentwise in the obvious way and with the norm

$$\|f\| = \sum_{\alpha=0}^{\nu} |f(\alpha)|/\nu .$$

<u>Theorem 23</u>. Let $f,g \in RN^{\nu+1}$ be continuous functions from $\stackrel{+}{=}$ to $\doteq$. Then each of the conditions (1) - (3) of Theorem 22 is equivalent to

(4) $\|f - g\| \doteq 0$.

<u>Proof</u>. It is enough to consider the case $g = \{0\} \times (\nu + 1)$. If for each $\alpha \leq \nu$ holds $f(\alpha) \doteq 0$ then obviously $\|f\| \doteq 0$. Let $|f(\alpha)| > 1/n$ for some $\alpha \leq \nu$, $n \in FN - \{0\}$. Then there are $\gamma, \delta \leq \nu$ such that $\gamma \leq \alpha \leq \delta$, $\gamma \not\doteq \delta$ and $|f(\beta)| > 1/2n$ for each β, $\gamma \leq \beta \leq \delta$. Then

$$\|f\| \geq \sum_{\beta=\gamma}^{\delta} |f(\beta)|/\nu > (\delta - \gamma + 1)/2n\nu \not\doteq 0.$$

The result extends with some effort also to other norms e.g. to

$$\|f\|^2 = \sum_{\alpha=0}^{\nu} f(\alpha)^2/\nu .$$

Condition (2) of Theorem 22 itself defines a $\mathfrak{X}$-equivalence on $RN^{\nu+1}$ induced by the norm

$$\|f\| = \max \{ |f(\alpha)| ; \alpha \in \nu + 1 \} .$$

The reader will easily find examples of motions in the time ν with respect to the $\mathfrak{X}$-equivalence $\doteq$ on RN omitting any of the implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (2), (4) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) between the conditions of Theorems 22 and 23. Thus the pseudocontinuity assumption cannot be removed.

Applying the results on (restricted) power equivalences and Theorem 15 to Theorems 22 and 23 a series of compactness results concerning continuous relations can be obtained. Let us quote the following two examples (not the most general ones):

<u>Theorem 24</u>. Let $\doteq$ and $\stackrel{+}{=}$ be two compact $\mathfrak{X}$-equivalences and

$v = \mathrm{dom}(\stackrel{+}{=})$ be a set. Then for every infinite set u of continuous relations from $\stackrel{+}{=}$ to $\doteq$ with common domain v there are two different relations $r, s \in u$ such that $r \mathrel{\ddot{=}} s$.

Theorem 25. Let $\langle \doteq, \stackrel{\cdot}{\leftrightarrow} \rangle$ and $\langle \stackrel{+}{=}, \stackrel{+}{\leftrightarrow} \rangle$ be two compatible bi-equivalences and $\mathrm{Gal}^+(z)$ be a semiset. Let u be an infinite set of functions such that

$$(\forall f \in u)\ \mathrm{Gal}^+(z) \subseteq \mathrm{dom}(f),$$

$$(\forall f, g \in u)(\forall x, y \in \mathrm{Gal}^+(z))\ f(x) \stackrel{\cdot}{\leftrightarrow} g(y)$$

and each $f \in u$ is continuous on $\mathrm{Gal}^+(z)$. Then there are two different functions $f, g \in u$ such that

$$(\forall x \in \mathrm{Gal}^+(z))\ f(x) \doteq g(x).$$

From any of Theorems 24 and 25 one can derive a corollary on "filling the screen by continuous curves". This result was communicated to the authors by P. Vopěnka.

Corollary. Let $\doteq$ and $\stackrel{+}{=}$ be two compact π-equivalences and $\mathrm{dom}(\stackrel{+}{=}) = v$ be a set. Let R be a relation such that the codable class $R''\{x\}/\doteq$ is uncountable for at least one $x \in v$. Then in each set u of continuous functions with domain v such that $R \subseteq \mathrm{Fig}^*(\bigcup u)$ there are at least two functions $f \neq g$ such that $(\forall x \in v)\ f(x) \doteq g(x)$.

Proof. Obviously, u cannot be finite.

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