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# DISCRETE SPECTRUM OF OPERATOR VALUED FRIEDRICHS MODELS

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Abatract: The operator valued Friedrichs model is studied. It is proved that there is only a finite number of eigenvalues outside the continuous spectrum.

Key words: Friedrichs model, Fredholm theory, Puiseux series.

Classification: 45B05, 81C10

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Several problems of mathematical physics lead to the study of a spectrum of a self-adjoint operator (operator valued Friedrichs models) acting on the Hilbert space $L_2(S^\nu,\mathcal{H})$ according to the following formula

(1) $(Hf)(x) = u(x)f(x)+\int_{S^\nu} K(x,y)f(y)dy, \quad f\in L_2(S^\nu,\mathcal{H})$

Here $S^\nu$ is a $\nu$-dimensional torus, $\mathcal{H}$ is an n-dimensional complex Hilbert space, and the matrices

$$u(x) = \begin{pmatrix} u_{11}(x)\ldots u_{1n}(x) \\ \ldots\ldots\ldots\ldots \\ u_{n1}(x)\ldots u_{nn}(x)\end{pmatrix}$$

and

$$K(x,y) = \begin{pmatrix} K_{11}(x,y)\ldots K_{1n}(x,y) \\ \ldots\ldots\ldots\ldots\ldots \\ K_{n1}(x,y)\ldots K_{nn}(x,y)\end{pmatrix}$$

are self-adjoint. We shall suppose that $u_{ij}(x) = u_{ji}(x)$ and $K_{ij}(x,y) =$

$= K_{ji}(x,y) = K_{ij}(x,y)$, $i,j = 1,2,\dots,n$ are real-analytic functions on $S^{\nu}$ and $S^{\nu}\times S^{\nu}$, respectively.

A spectrum of operator of the form (1) was first investigated by Friedrichs [1] for $u(x) = x$, and in [2] for an arbitrary real-analytic function $u(x)$.

Here we shall give a more detailed description of the spectrum of operator (1), namely, we shall prove that there is only a finite number of eigenvalues outside the continuous spectrum.

Let us denote by $\Sigma_{cont}(H)$ the continuous spectrum of the operator H, and by $\Gamma_x$ the set

$$\Gamma_x = \{z \in \mathbb{C}^1 : \sigma(x,z) = 0\},$$

where $\mathbb{C}^1$ is the complex plane, $\sigma(x,z)$ is a determinant of $u(x)-zE$.

It is well known that the self-adjointness of $u(x)$, $x \in S^{\nu}$ implies that $\Gamma_x \subset R^1$, where $R^1$ is the real line.

<u>Proposition 1</u>. It is

$$\Sigma_{cont}(H) = \bigcup_{x\in S^{\nu}} \Gamma_x \tag{2}$$

<u>Proof</u>. Let $z \in \bigcup_{x\in S^{\nu}} \Gamma_x$, i.e. $\sigma(x,z) = 0$ for some $x \in S^{\nu}$. Then the operator $u(x)-zE$, where E is the identity operator in $\mathcal{H}$, is not invertible . Therefore, the operator

$$[(H_0-zI)f](x) = (u(x)-zE)f(x), \quad f \in L_2(S^{\nu},\mathcal{H})$$

where I is the identity operator in $L_2(S^{\nu},\mathcal{H})$, is not invertible in the space of bounded operators on $L_2(S^{\nu},\mathcal{H})$ i.e. $z \in \Sigma_{cont}(H_0)$. Since

$$\int_{S^{\nu}\times S^{\nu}} \|K(x,y)\|^2 dx\, dy < \infty \tag{3}$$

we infer that the operator

$$[(H-H_0)f](x) = \int_{S^{\nu}} K(x,y)f(y)dy, \quad f \in L_2(S^{\nu},\mathcal{H})$$

belongs to the class of Hilbert-Schmidt operators. Using the well known theorem of H. Weyl (see [3]) we conclude that the continuous spectra of both H and $H_0$ coincide. Thus $z \in \Sigma_{cont}(H)$.

Now let $z \in \Sigma_{cont}(H)$. Using again the mentioned theorem of H. Weyl we have also $z \in \Sigma_{cont}(H_0)$, and thus $z \in \Gamma_x$ for some $x \in S^\nu$, i.e. $z \in \bigcup_{x \in S^\nu} \Gamma_x$.

Theorem 1. The resolvent $R_z(H)$ of H exists. It can be expressed by the formula

$$(R_z f(x) = [u(x)-zE]^{-1} f(x) + [u(z)-zE]^{-1} \int_{S^\nu} \frac{\mathfrak{D}(x,y;z)}{\Delta(z)} f(y)dy$$

for all $z \in \mathbb{C}^1$, Im $z \neq 0$ where $\Delta(z)$, and $\mathfrak{D}(x,y;z)$ are defined below (in (11),(13)).

Proof. We shall find an explicit formula for $R_z(H) \equiv$ the inverse of H-zI. Let for some $g \in L_2(S^\nu, \mathcal{H})$,

(4) $[(H-zI)f](x) = (u(x)-zE)f(x) + \int_{S^\nu} K(x,y)f(y)dy = g(x), \quad f \in L_2(S^\nu, \mathcal{H})$

Since u(x) is self-adjoint in $\mathcal{H}$, the determinant $\delta(x,z)$ of the matrix u(x)-zE is nonvanishing for all $z \in \mathbb{C}^1$, Im $z \neq 0$, and hence the inverse operator

$$(u(x)-zE)^{-1} = \frac{1}{\delta(x,z)} \begin{pmatrix} \vartheta_{11}(x,z) \dots \vartheta_{n1}(x,z) \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \\ \vartheta_{1n}(x,z) \dots \vartheta_{nn}(x,z) \end{pmatrix}$$

exists. Here $\vartheta_{ji}(x,z)$ denotes the signed minor of the element $u_{ij}(x,z)$ of the matrix u(x)-zE. Introducing the notation

(5) $\hat{f}(x) = [u(x)-zE]f(x), \quad f \in L_2(S^\nu, \mathcal{H})$

we can write (4) as

$$\hat{f}(x) + \int_{S^\nu} K(x,y)[u(y)-zE]^{-1}\hat{f}(y)dy = g(x), \quad \hat{f} \in L_2(S^\nu, \mathcal{H})$$

which can be formulated as a system

$$(6)\quad \begin{cases} \hat{f}_1(x)+ \int_{S^\nu} K_{11}(x,y;z)\hat{f}_1(y)dy+\ldots+\int_{S^\nu} K_{1n}(x,y;z)\hat{f}_n(y)dy=g_1(x) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{f}_n(x)+ \int_{S^\nu} K_{n1}(x,y;z)\hat{f}_1(y)dy+\ldots+\int_{S^\nu} K_{nn}(x,y;z)\hat{f}_n(y)dy=g_n(x) \end{cases}$$

of integral equations. Here

$$\hat{f}(x)=(\hat{f}_1(x),\ldots,\hat{f}_n(x)),\ g(x)=(g_1(x),\ldots,g_n(x)),\ \hat{f}_i, g_i \in L_2(S^\nu,\mathbb{C}^1)$$
$$i = 1,2,\ldots,n,$$

and $L_2(S^\nu, \mathbb{C}^1)$ is the Hilbert space of all square integrable complex functions defined on the $\nu$-dimensional torus $S^\nu$ , and .

$$(7)\quad K_{ij}(x,y;z)= \frac{1}{\sigma(y,z)} \sum_{s=1}^{m} K_{is}(x,y)\ \vartheta_{is}(y,z).$$

We shall now rewrite (6) as an integral equation equivalent to the system (6). To this end we denote by M the union of disjoint copies of $S^\nu$ , i.e.

$$M = \bigcup_{j=1}^{m} (S^\nu)_j,\ (S^\nu)_j = S^\nu\ ,\ j = 1,2,\ldots,n.$$

Define now a measure on M such that its restriction to each $(S^\nu)_j = S^\nu$ , $j = 1,2,\ldots,n$ coincides with the Lebesgue measure. For each $z \in \mathbb{C}^1$, $\operatorname{Im} z \neq 0$ we define the function (kernel) $K(\lambda,\mu;z)$ on $M\times M$ as

$$K(\lambda,\mu;z) = K_{ij}(x,y;z),\ \lambda= x \in (S^\nu)_i,\ \mu= y \in (S^\nu)_j.$$

Finally we define the following functions on M:

$$f(\lambda) = f_i(x),\ g(\lambda) = g_i(x),\ \lambda = x \in (S^\nu)_i,\ i = 1,2,\ldots,n.$$

Then the system of integral equations (6) is equivalent with

$$f(\lambda)+ \int_M K(\lambda,\mu;z)f(\mu)d\mu = g(\lambda),\ f,g \in L_2(M, \mathbb{C}^1),$$

where $L_2(M, \mathbb{C}^1)$ is the Hilbert space of all square integrable complex valued functions on M.

<u>Proposition 2</u>. Any $z \in \mathbb{C}^1\setminus \Sigma_{cont}(H)$ is an eigenvalue of H if and only if the homogeneous equation

$$(8)\qquad f(\lambda) + \int_M K(\lambda,\mu;z)f(\mu)d\mu = 0$$

has a nonzero solution $f \in L_2(M, \mathbb{C}^1)$.

Proof. Any $z \in \mathbb{C}^1 \setminus \Sigma_{cont}(H)$ is an eigenvalue of H iff for some $f \in L_2(S^\nu, \mathcal{H})$ the following relation holds true:

(9) $(u(x)-zE)f(x)+\int_M K(x,y)f(y)dy = 0.$

By the same argument as before it is possible to show that (9) is equivalent to the system of homogeneous integral equations

$$(10)\quad \begin{cases} f_1(x)+\int_{S^\nu} K_{11}(x,y;z)f_1(y)dy+\ldots+\int_{S^\nu} K_{1n}(x,y;z)f_n(y)dy=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x)+\int_{S^\nu} K_{n1}(x,y;z)f_1(y)dy+\ldots+\int_{S^\nu} K_{nn}(x,y;z)f_n(x)dy=0. \end{cases}$$

Further, from the definition of $L_2(M, \mathbb{C}^1)$ and the kernel $K(\lambda, \mu; z)$ it follows that for any $z \in \mathbb{C}^1 \setminus \Sigma_{cont}(H)$, the system (10) has a nonzero solution iff the homogeneous integral equation (8) has a nonzero solution from $L_2(M, \mathbb{C}^1)$.

To finish the proof of Theorem 1 we use the self-adjointness of H to infer from Proposition 2 that for each $z \in \mathbb{C}^1$, $\operatorname{Im} z \neq 0$ the homogeneous equation

$$f(\lambda) + \int_M K(\lambda, \mu; z) f(\mu) d\mu = 0,$$

has no nonzero solution. Besides, since

$$\int_{M\times M} |K(\lambda, \mu; z)|^2 d\lambda\, d\mu = \sum_{i,j=1}^{n} \int_{S^\nu \times S^\nu} |K_{ij}(x,y;z)|^2 dx\, dy < \infty$$

it follows that the operator

$$[K(z)f](\lambda) = \int_{S^\nu} K(\lambda, \mu; z) f(\mu) d\mu, \quad f \in L_2(M, \mathbb{C}^1)$$

is of Hilbert-Schmidt type. Therefore, it follows from Fredholm theorem (see [4]) that the equation (7) has a unique solution $f \in L_2(M, \mathbb{C}^1)$, for any $g \in L_2(M, \mathbb{C}^1)$. This solution can be expressed as

$$f(\lambda) = g(\lambda) - \frac{1}{\Delta(z)} \int_M \mathcal{D}(\lambda, \mu; z) g(\mu) d\mu,$$

where $\Delta(z)$, and $\mathfrak{D}(\lambda,\mu,z)$ denote the Fredholm determinant, and minor, respectively. Considering the restriction of f on $(S^\nu)_i$, $i = 1,2,\dots,n$, we obtain the solution of the system (6) in the following form:

$$\hat{f}_i(x) = g_i(x) + \int_M \frac{\mathfrak{D}_i(x,\mu;z)}{\Delta(z)} g(\mu)\,d\mu =$$

$$= g_i(x) + \sum_{j=1}^{n} \int_{S^\nu} \frac{\mathfrak{D}_{ij}(x,y;z)}{\Delta(z)} g_j(y)\,dy,\quad i = 1,2,\dots,n.$$

Here $\mathfrak{D}_{ij}(x,y;z)$ and $\Delta(z)$ are given by the following formulas:

$$\mathfrak{D}_{ij}(x,y;z) = K_{ij}(x,y;z) + \sum_{s=1}^{\infty} \frac{1}{s!} DS^{(ij)}(x,y;z),$$

$$d_s^{(ij)}(x,y;z) = \sum_{j_1,j_2,\dots,j_s=1}^{n}$$

$$\int_{S^\nu}\cdots\int_{S^\nu} \begin{vmatrix} K_{ij}(x,y;z)K_{ij_1}(x,t_1;z)\dots K_{ij_s}(x,t_s;z) \\ K_{j_1j}(t_1,y;z)K_{j_1j_1}(t_1,t_1;z)\dots K_{j_1j_s}(t_1,t_s;z) \\ \dots\dots\dots\dots\dots\dots \\ K_{j_sj}(t_s,y;z)K_{j_sj_1}(t_s,t_1;z)\dots K_{j_sj_s}(t_s,t_s;z) \end{vmatrix} \times$$

$$\times\, dt_1\, dt_2 \dots dt_n,$$

(11) $$\Delta(z) = 1 + \sum_{s=1}^{\infty} \frac{1}{s!} d_s(z),$$

(12) $$d_s(z) = \sum_{j_1,\dots,j_s=1}^{n}$$

$$\int_{S^\nu}\cdots\int_{S^\nu} \begin{vmatrix} K_{j_1j_1}(t_1,t_1;z)\dots K_{j_1j_s}(t_1,t_s;z) \\ \vdots\dots\dots\dots\dots \\ K_{j_sj_1}(t_s,t_1;z)\dots K_{j_sj_s}(t_s,t_s;z) \end{vmatrix} dt_1\dots dt_n .$$

Therefore it follows from the formula (11) and (5) that the resolvent of H acts on $L_2(S^\nu,\mathcal{H})$ according to the formula

$$[R_zf](x) = [u(x)-zE]^{-1}f(x) - \frac{[u(x)-zE]^{-1}}{\Delta(z)} \int_{S^\nu} \mathfrak{D}(x,y;z)f(y)\,dy$$

where

(13) $$\mathfrak{D}(x,y;z) = \begin{pmatrix} \mathfrak{D}_{11}(x,y;z)\dots\mathfrak{D}_{1n}(x,y;z) \\ \dots\dots\dots\dots \\ \mathfrak{D}_{n1}(x,y;z)\dots\mathfrak{D}_{nn}(x,y;z) \end{pmatrix}$$

The boundedness of $R_z$ follows from the explicit formula (13). Thus, the theorem 1 is proved.

Theorem 2. The operator (1) has only a finite number of eigenvalues not belonging to the continuous spectrum.

We shall restrict ourselves to the case $\nu = 1$ and $u_{ij}(x)=0$ for $i \neq j$ to avoid certain technical difficulties of the general case. In addition, without loss of generality we can assume that $u_j(x) \equiv u_{jj}(x)$ and $K_{j_1 j_2}(x,y)$, $j,j_1,j_2 = 1,2,\ldots,n$ are $2\pi$ periodical functions defined on $[0,2\pi]$ and $[0,2\pi]\times[0,2\pi]$, respectively. We notice that in this special case the continuous spectrum of H consists of

$$\Sigma_{cont}(H) = \bigcup_{j=1}^{m} [A_j,B_j],$$

where $A_j = \inf_x u_j(x)$, $B_j(x) = \sup_x u_j(x)$,
and the function $d_q(z)$ , $q = 1,2,\ldots$ from (12) can be written as

$$d_q(z) = \sum_{j_1,\ldots,j_q=1}^{m} d_{j_1 j_2 \ldots j_q}(z) = \sum_{j_1,\ldots,j_q=1}^{m}$$

$$\int_0^{2\pi}\cdots\int_0^{2\pi} \begin{vmatrix} K_{j_1 j_1}(t_1,t_1)\ldots K_{j_1 j_q}(t_1 t_q) \\ \cdots\cdots\cdots\cdots\cdots \\ K_{j_q j_1}(t_q t_1)\ldots K_{j_q j_q}(t_q,t_q) \end{vmatrix} \frac{dt_1\ldots dt_q}{(u_{j_1}(t_1)-z)\ldots(U_{j_q}(t_q)-z)}$$

The following lemma plays a crucial role in the proof of Theorem 2.

Lemma 1. Let $A' \in \Sigma_{cont}(H)$ and $u_j^{-1}(A') = \{x_{j1}, x_{j2},\ldots,x_{jm_j}\}$, $j = 1,2,\ldots,n$. Then there is an $\varepsilon$-neighborhood $V_\varepsilon'(A') = \{z \in \mathbb{C}^1 : 0 < |z-A'| < \varepsilon\}$ of $z = A'$ such that the restriction $\Delta(z)/\mathbb{C}_+^1$ of the $\Delta(z)$, where $\mathbb{C}_+^1 = \{z \in \mathbb{C}^1 : \operatorname{Im} z > 0\}$ is the half-plane, has an analytic continuation onto $V_\varepsilon'(A')$. This analytic continuation $\Delta^*(z)$ is a multivalued function with the branching point $z = A'$ and can be in $V'_\varepsilon(A')$ expanded into the series

$$\Delta^*(z) = \sum_{s=-\hat{q}} F_{A',s}(K)(z-A')^{s/p}, \quad z \in V'_\varepsilon(A').$$

Here

$$\hat{q} = P \sum_{j=1}^{n} \sum_{s=1}^{m_j} \frac{R_{js}-1}{R_{js}},$$

and $R_{js} - 1 = R(x_{js}) - 1$ denote the multiplicity of the root $x = x_{js}$ of the function $u'_j(x)$, $j = 1,2,\dots,m_j$, $P$ is the lowest common multiple of the numbers

$$\{R_{11},\dots,R_{1m_1},\dots,R_{n1},\dots,R_{nm_n}\}.$$

The proof of this lemma is based on Lemma 2 which we shall prove first.

<u>Lemma 2</u>. Let $A' \in \Sigma_{cont}(H)$. Then for any $q = 1,2,\dots$ there is a neighborhood $V'_\varepsilon(A')$ of $z = A'$, and a function $d^*_q(z)$ defined on it, such that

$$d^*_q(z)/_{V'_\varepsilon(A') \cap \mathbb{C}^1_+} = d_q(z)/_{V'_\varepsilon(A') \cap \mathbb{C}^1_+},$$

where $\mathbb{C}^1_+ = \{z \in \mathbb{C}^1 : \mathrm{Im}\, z > 0\}$. The function $d^*_q(z)$ is a multivalued function with the branching point $z = A'$.

<u>Proof</u> of Lemma 2. For any $A' \in \Sigma_{cont}(H) = \bigcup_{j=1}^{n} [A_j, B_j]$ we denote by $u_j^{-1}(A') \subset [0,2\pi]$, $j = 1,2,\dots,n$ its pre-image with respect to the mapping $u_j$. It is obviously finite, i.e. we can write $u_j^{-1}(A') = \{x_{j1}, x_{jm_j}\}$. Let us denote by $u_j(\xi)$ and $K_{j_1 j_2}(\xi_1,\xi_2)$ the analytic continuations of $u_j(x)$, and $K_{j_1 j_2}(x_1,x_2)$, into $Q \subset \mathbb{C}^1$, and $Q \times Q \subset \mathbb{C}^2$, respectively, where $Q \subset \mathbb{C}^1$ is some complex neighborhood of the segment $[0,2\pi]$.

Because $u_j(\xi)$, $j = 1,2,\dots,n$ is regular in $x = x_{j\nu}$, $\nu = 1,2,\dots,m_j$ there are some $\varepsilon > 0$ and $\delta > 0$ (in the following we shall assume that these numbers are sufficiently small) such that for each $z \in V_\varepsilon'(A')$ the equation

$$u_j(\varsigma) - z = 0$$

has exactly $R_{j\nu}$, $\nu = 1,2,\dots,m_j$ solutions in the disc $|x_{j\nu} - \varsigma| < \delta$. These solutions are branches of some $R_{j\nu}$-valued analytical functions, whose branching point $z = A'$ has an order $R_{j\nu}$ and can be in $V_\varepsilon'(A')$ expanded into the series

$$(14) \qquad \psi_j^\nu(z) = x_{j\nu} + c_{j1}^\nu (z-A')^{1/R_j} + c_{j2}^\nu (z-A')^{2/R_{j\nu}} + \dots ,$$

where

$$(15) \qquad c_{j1}^\nu = \left[ \frac{R_{j\nu}}{u_j^{(R_{j\nu})}(x_{j\nu})} \right]^{1/R_{j\nu}}, \quad \nu = 1,2,\dots,m_j .$$

This statement follows from the theorem about inverse function of an analytic function (see [5]).

We put

$$(z-A')_s^{1/R_{j\nu}} = |z-A'|^{1/R_{j\nu}} \exp\left\{ i \, \frac{\arg(z-A')}{R_{j\nu}} + \frac{2\pi i}{R_{j\nu}} \, . \, s \right\}$$

$s = 0,1,\dots,R_{j\nu} - 1$ and call this value the s-th value of the root $(z-A')^{1/R_{j\nu}}$. Correspondingly we call the

$$\psi_{js}^\nu(z) = x_{j\nu} + c_{j1}^\nu (z-A')_s^{1/R_{j\nu}} + c_{j2}^\nu [(z-A')^{1/R_{j\nu}}]^2 + \dots$$

the s-th value of the multivalued function $\psi_j^\nu(z)$.

<u>Proposition 3</u>. For any $\delta > 0$ there exists $\varepsilon > 0$ such that for each $z \in V_\varepsilon'(A')$, $\operatorname{Im} z > 0$ the number of values $\psi_{j0}^\nu(z), \psi_{j1}^\nu(z), \dots, \psi_{jp_{j\nu}-1}^\nu$ of $\psi_j^\nu(z)$ which belong to $\{\varsigma \in \mathbb{C}^1 : \operatorname{Im} \varsigma > 0,\ |\varsigma - x_{j\nu}| < \delta\}$, equals to $P_{j\nu}$. Here, $P_{j\nu}$ is the integer part of

$$\frac{1}{2}\{ R_{j\nu} + [\operatorname{sgn} u_j^{(R_{j\nu})}(x_{j\nu})]^{R_{j\nu}} \} .$$

<u>Proof</u>. Since $\operatorname{Im} z > 0$ we observe that $P_{j\nu}'$ values of $(z-A')^{1/R_{j\nu}}$,

where $P'_{j\nu}$ is the integer part of $\frac{1}{2}(R_{j\nu} + 1)$, belong to the upper half-plane. From this fact and from (15) it follows that $P_{j\nu}$ values

$$c^{\nu}_{j1}(z-A')_0^{1/R_{j\nu}},\ c^{\nu}_{j1}(z-A')_1^{1/R_{j\nu}}, \ldots, c^{\nu}_{j1}(z-A')_{P_{j\nu}-1}^{1/R_{j\nu}}$$

belong to the upper half-plane. The smallness of $\varepsilon > 0$ then implies that $P_{j\nu}$ values

$$\psi^{\nu}_{j0}(z),\ \psi^{\nu}_{j1}(z), \ldots, \psi^{\nu}_{jP_j-1} \text{ of } \psi^{\nu}_j(z)$$

belong to $\{\zeta \in \mathbf{C}^1 . \operatorname{Im}\zeta > 0,\ |\zeta - x_{j\nu}| < \sigma\}$.

<u>Proposition 4</u>. The function $[u'_j(\psi^{\nu}_j)]^{-1}$ can be in $V'_{\varepsilon}(A')$ expressed in the form

$$[u'_j(\psi^{\nu})]^{-1} = \frac{(R_{j\nu} - 1)!}{u_j^{(R_{j\nu})}(x_{j\nu})(c^{\nu}_{j1})^{\frac{R_j - 1}{R_j}}} \cdot [1 + \sum_{k=1}^{\infty} \hat{\varepsilon}_k (z-A')^{k/R_{j\nu}}]$$

and the function $K_{j_1 j_2}(\psi^{\nu_1}_{j_1}, \psi^{\nu_2}_{j_2})$ in the form

$$K_{j_1 j_2}(\psi^{\nu_1}_{j_1}, \psi^{\nu_2}_{j_2}) = K_{j_1 j_2}(x_{j_1\nu_1}, x_{j_2 \nu_2}) +$$

$$+ \sum_{s_1, s_2 = 1}^{\infty} \hat{c}_{s_1 s_2} (z-A')^{s_1/R_{j_1\nu_1}} (z-A')^{s_2/R_{j_2\nu_2}} .$$

<u>Proof</u>. Since $\zeta = x_{j\nu}$, $\nu = 1,2,\ldots,m_j$ is a zero point of the order $R_{j\nu} - 1$ of the function $u'_j(\zeta)$, we can expand this function into the following series:

$$(16)\quad u'_j(\zeta) = \frac{u_j^{(R_{j\nu})}(x_{j\nu})}{(R_{j\nu}-1)!}(\zeta - x_{j\nu})^{R_{j\nu}-1} + \frac{u_j^{(R_{j\nu}+1)}(x_{j\nu})}{R_\nu !}(\zeta - x_{j\nu})^{R_{j\nu}} + \ldots$$

Substituting (14) into (16) we obtain

$$(17)\quad u'_j(\psi^{\nu}_j) = \frac{u_j^{(R_{j\nu})}(x_{j\nu})}{(R_{j\nu}-1)!}(\zeta - x_{j\nu}) + \frac{u_j^{(R_{j\nu}+1)}(x_{j\nu})}{\nu !}[\psi^{\nu}_j(z) - A']^{R_{j\nu}-1} + \ldots =$$

$$= \frac{u_j^{(R_{j\nu})}(x_{j\nu})(c_{j1}^{\nu})^{R_{j\nu}-1}}{(R_{j\nu}-1)!}\left[(z-A')^{1/R_{j\nu}}\right]^{R_{j\nu}-1}\left[1+c_{j2}^{\nu}(z-A')^{1/R_{j\nu}}+\dots\right].$$

For a sufficiently small $\varepsilon > 0$ and $z \in V_\varepsilon'(A')$ we have an inequality

$$|c_{j2}^{\nu}(z-A')^{1/R_{j\nu}}+\dots| < 1.$$

This inequality, together with (17), implies the statement of Proposition 4 for the function $[u_j'(\psi^\nu)]^{-1}$. The second assertion of Proposition 4 is proved in an analogous way.

Coming back to the proof of Lemma 2 let $\varepsilon > 0$ and $\delta > 0$ be such that the segments

$$(x_{j1}-\delta, x_{j1}+\delta), \dots, (x_{jm_j}-\delta, x_{jm_j}+\delta), \quad j = 1,2,\dots,n$$

do not intersect and

$$u_j^{-1}(V_\varepsilon'(A')) \subset \bigcup_{\nu=1}^{m} \{\xi \in \mathbb{C}^1 : |\xi - x_{j\nu}| < \delta\}.$$

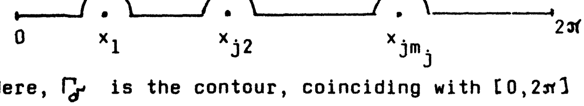
Investigate the function

$$d_j(z) = \int_0^{2\pi} \frac{K_{jj}(\xi,\xi)}{u_j(\xi)-z}\, d\xi, \quad j = 1,2,\dots,n$$

which is regular in $V_\varepsilon'(A') \cap \mathbb{C}_+^1$. We can write the function $d_j(z)$ in terms of its residua as follows:

$$(18)\quad d_j(z) = 2\pi i \sum_{\nu=1}^{m_j} \sum_{s=0}^{P_{j\nu}-1} \operatorname{res} \frac{K_{jj}(\xi,\xi)}{u_j(\xi)-z} + \int_{\Gamma_\delta} \frac{K_{jj}(\xi,\xi)}{u_j(\xi)-z}\, d\xi =$$

$$= 2\pi i \sum_{\nu=1}^{m_j} \sum_{s=0}^{P_{j\nu}-1} \frac{K_{jj}(\psi_{js}^{\nu},\psi_{js}^{\nu})}{u_j'(\psi_{js}^{\nu})} + \int_{\Gamma_\delta} \frac{K_{jj}(\xi,\xi)}{u_j(\xi)-z}\, d\xi, \quad z \in V_\varepsilon'(A') \cap \mathbb{C}_+^1$$




Here, $\Gamma_\delta$ is the contour, coinciding with $[0,2\pi]$ outside of all intervals

$$(x_{j1}-\delta, x_{j1}+\delta), \dots, (x_{jm_j}-\delta, x_{jm_j}+\delta)$$

and containing all the half-circles

$\{\xi \in \mathbb{C}^1 : |\xi - x_{j\nu}| = \delta,\ \operatorname{Im}\xi \geq 0\}$.

Since $\xi \in \Gamma_\delta$ we conclude that

$$u_j(\xi) \,\overline{\in}\, V_\varepsilon(A') = \{z \in \mathbb{C}^1 : |z - A'| < \varepsilon\}\ .$$

Therefore, the function $\int_{\Gamma_\delta} \frac{K_{jj}(\xi,\xi)}{u_j(\xi)-z}\, d\xi$ is regular in $V_\varepsilon(A')$. Using the representation (18) of $d_j(z)$, $j = 1,2,\dots,n$, the existence of an analytical continuation of $d_j(z)$ into the region $V_\varepsilon'(A')$ through the interval $(A'-\varepsilon, A')$ and also through $(A', A'+\varepsilon)$ follows. Both these analytical continuations coincide. We denote by $d_j^*(z)$ the analytical continuation of $d_j(z)$. From the proposition 4 it follows that $d_j^*(z)$ is a multivalued function with the branching point $z = A'$, expressed in the Puiseux series in the powers of $z - A'$.

Consider now the function

$$d_{j_1j_2}(z) = \int_0^{2\pi}\!\!\int_0^{2\pi} \begin{vmatrix} K_{j_1j_1}(\xi_1,\xi_1)K_{j_1j_2}(\xi_1,\xi_2) \\ K_{j_2j_1}(\xi_2,\xi_1)K_{j_2j_2}(\xi_2,\xi_2) \end{vmatrix} \frac{d\xi_1\, d\xi_2}{(u_{j_1}(\xi_1)-z)(u_{j_2}(\xi_2)-z)}.$$

Using the "theorem about residua" to the function $d_{j_1j_2}(z)$ several times, we obtain:

(19) $\quad d_{j_1j_2}(z) =$

$$= \int_0^{2\pi} \frac{1}{u_{j_1}(\xi_1)-z} \Big\{ 2\, i \sum_{\nu_2=0}^{m_{j_2}} \sum_{s_2=0}^{p_{j_2\nu_2}-1} \begin{vmatrix} K_{j_1j_1}(\xi_1,\xi_1)K'_{j_1j_2}(\xi_1,\psi_{j_2s_2}^{\nu_2}) \\ K_{j_2j_1}(\psi_{j_2s_2}^{\nu_2},\xi_2)K_{j_2j_2}(\psi_{j_2s_2}^{\nu_2}, \psi_{j_2s_2}^{\nu_2}) \end{vmatrix} \times$$

$$\times \frac{1}{u'_{j_2}(\psi_{j_2s_2}^{\nu_2})} +$$

$$+ \int_{\Gamma_\delta} \begin{vmatrix} K_{j_1j_1}(\xi_1,\xi_1)K_{j_1j_2}(\xi_1,\xi_2) \\ K_{j_2j_1}(\xi_2,\xi_1)K_{j_2j_2}(\xi_2,\xi_2) \end{vmatrix} \frac{d\xi_2}{u_{j_2}(\xi_2)-z} \Big\}\, d\xi_2 \quad =$$

$$= (2\pi i)^2 \sum_{\nu_1=1}^{m_{j_1}} \sum_{\nu_2=1}^{m_{j_2}} \sum_{s_2=0}^{p_{j_1\nu_1}-1} \sum_{s_2=0}^{p_{j_2\nu_2}-1}$$

$$\left|\begin{matrix} K_{j_1 j_2}(\psi_{j_1 s_1}^{\nu_1}, \psi_{j_1 s_1}^{\nu_1}) K_{j_1 j_2}(\psi_{j_1 s_1}^{\nu_1}, \psi_{j_2 s_2}^{\nu_1}) \\ K_{j_2 j_1}(\psi_{j_2 s_2}^{\nu_2}, \psi_{j_1 s_1}^{\nu_1}) K_{j_2 j_2}(\psi_{j_2 s_2}^{\nu_2}, \psi_{j_2 s_2}^{\nu_2}) \end{matrix}\right| \times \frac{1}{u'_{j_1}(\psi_{j_1 s_1}^{\nu_1}) u'_{j_2}(\psi_{j_2 s_2}^{\nu_2})} +$$

$$+ 2\pi i \sum_{\nu_1=1}^{m_{j_1}} \sum_{s_1=0}^{p_{j_1\nu_1}-1} \int_{\Gamma_\delta} \times \left|\begin{matrix} K_{j_1 j_1}(\psi_{j_1 s_1}^{\nu_1}, \psi_{j_1 s_1}^{\nu_2}) K_{j_1 j_2}(\psi_{j_1 s_1}^{\nu_1}, \xi_2) \\ K_{j_2 s_1}(\xi_2, \psi_{j_1 s_1}^{\nu_1}) K_{j_2 j_2}(\xi_2, \xi_2) \end{matrix}\right| \times$$

$$\times \frac{d\xi_2}{u'_{j_1}(\psi_{j_1 s_1}^{\nu_1})(u_{j_2}(\xi_2) - z)} + 2\pi i \sum_{\nu_2=1}^{m_{j_2}} \sum_{s_2=0}^{p_{j_2\nu_2}-1}$$

$$\int_{\Gamma_\delta} \left|\begin{matrix} K_{j_1 j_1}(\xi_1, \xi_1) K_{j_1 j_2}(\xi_1, \psi_{j_2 s_2}^{\nu_2}) \\ K_{j_2 j_1}(\psi_{j_2 s_2}^{\nu_2}, \xi_1) K_{j_2 j_2}(\psi_{j_2 s_2}^{\nu_2}, \psi_{j_2 s_2}^{\nu_2}) \end{matrix}\right| \frac{d\xi_1}{(u_{j_1}(\xi_1) - z) u'_{j_2}(\psi_{j_2 s_2}^{\nu_2})} +$$

$$+ \int_{\Gamma_\delta} \int_{\Gamma_\delta} \left|\begin{matrix} K_{j_1 j_1}(\xi_1, \xi_1) K_{j_1 j_2}(\xi_1, \xi_2) \\ K_{j_2 j_1}(\xi_2, \xi_1) K_{j_2 j_2}(\xi_2, \xi_2) \end{matrix}\right| \frac{d\xi_1\, d\xi_2}{(u_{j_1}(\xi_1) - z)(u_{j_2}(\xi_2) - z)} .$$

Using analogical considerations as in the case of $d_j(z)$, $j = 1, 2, \ldots, n$ it follows from the representation (19) that the function $d_{j_1 j_2}(z)$ which is regular on the upper half-plane, has an analytical continuation into $V'_\varepsilon(A')$, through the segments $(A' - \varepsilon, A')$ and $(A', A' + \varepsilon)$. Both these analytical continuations coincide. From Proposition 4 it follows that this continuation is a multi-valued function with the branching point $z = A'$.

Now we investigate the function $d_{j_1 j_2 \ldots j_q}(z)$, defined by the formula

$$d_{j_1\dots j_q}(z)=\int_0^{2\pi}\dots\int_0^{2\pi}\begin{vmatrix} K_{j_1j_1}(\xi_1,\xi_1)\dots K_{j_1j_q}(\xi_1,\xi_q) \\ \dots\dots\dots\dots\dots\dots \\ K_{j_qj_1}(\xi_q,\xi_1)\dots K_{j_qj_q}(\xi_q,\xi_q)\end{vmatrix}\times$$

$$\times\frac{d\xi_1\dots d\xi_q}{(u_{j_1}(\xi_1)-z)\dots)u_{j_q}(\xi_q)-z)}$$

(which is regular on the upper half-plane). Finally, using the mentioned "theorem about residua" repeatedly many times we can write down the following expression:

(20) $$d_{j_1\dots j_q}(z)=(2\pi i)^q\sum_{\nu_1,\nu_2,\dots,\nu_q=1}^{m_{j_1},m_{j_2},\dots m_{j_q}}\ \sum_{s_1=0}^{P_{j_1,\nu_1}-1}\dots\sum_{s_q=0}^{P_{j_q\nu_q}-1}$$

$$\begin{vmatrix} K_{j_1j_1}(\psi_{j_1s_1}^{\nu_1},\psi_{j_1s_1}^{\nu_1})K_{j_1j_2}(\psi_{j_1s_1}^{\nu_1},\psi_{j_2s_2}^{\nu_2})\dots K_{j_1j_q}(\psi_{j_1s_1}^{\nu_1},\psi_{j_qs_q}^{\nu_q})) \\ K_{j_1j_1}(\psi_{j_2s_2}^{\nu_2},\psi_{j_1s_1}^{\nu_1})K_{j_2j_2}(\psi_{j_2s_2}^{\nu_2},\psi_{j_2s_2}^{\nu_2})\dots K_{j_2j_q}(\psi_{j_2s_2}^{\nu_2},\psi_{j_qs_q}^{\nu_q}) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ K_{j_qj_1}(\psi_{j_qs_q}^{\nu_q},\psi_{j_1s_1}^{\nu_1})K_{j_qj_2}(\psi_{j_qs_q}^{\nu_q},\psi_{j_2s_q}^{\nu_2})\dots K_{j_qj_q}(\psi_{j_qs_q}^{\nu_q},\psi_{j_qs_q}^{\nu_q})\end{vmatrix}\times$$

$$\times\frac{1}{u'_{j_1}(\psi_{j_1s_1}^{\nu_1})u_{j_2}(\psi_{j_2s_2}^{\nu_2})\dots u'_{j_q}(\psi_{j_qs_q}^{\nu_q})}+\dots+$$

$$+(2\pi i)^{q-q'}\sum_{\nu_1=1}^{m_{j_1}}\dots\sum_{\nu_{q-q'}=1}^{m_{j_{q-q'}}}\ \sum_{s_1=0}^{P_{j_1\nu_1}-1}\dots\sum_{s_{q-q'}=0}^{P_{j_{q-q'}\nu_{q-q'}}-1}\overbrace{\int_{\Gamma_\sigma}\dots\int_{\Gamma_\sigma}}^{q'}\times$$

$$
\times\begin{vmatrix}
K_{j_1j_1}(\psi^{\nu_1}_{j_1s_1},\psi^{\nu_1}_{j_1s_1})\dots K_{j_1j_{q'}}(\psi^{\nu_1}_{j_1s_1},\psi^{\nu_{q'}}_{j_{q'}s_{q'}})K_{j_1j_{q'+1}}(\psi^{\nu_1}_{j_1s_1},\xi_{q'+1})\dots \\
\dots K_{j_1j_q}(\psi^{\nu_1}_{j_1j_q},\xi_q) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
K_{j_{q'}j_1}(\psi^{\nu_{q'}}_{j_{q'}s_{q'}},\psi^{\nu_1}_{j_1s_1})\dots K_{j_{q'}j_{q'}}(\psi^{\nu_{q'}}_{j_{q'}s_{q'}},\psi^{\nu_{q'}}_{j_{q'}s_{q'}})K_{j_{q'}j_{q'+1}}(\psi^{\nu_{q'}}_{j_{q'}s_{q'}},\xi_{q'+1})\dots \\
\dots K_{j_{q'}j_q}(\psi^{\nu_{q'}}_{j_{q'}s_{q'}},\xi_q) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
K_{j_{q'+1}j_1}(\xi_{q'+1},\psi^{\nu_1}_{j_1s_1})\dots K_{j_{q'+1}j_{q'}}(\xi_{q'+1},\psi^{\nu_{q'}}_{j_{q'}s_{q'}})K_{j_{q'+1}j_{q'+1}} \\
(\xi_{q'+1},\xi_{q'+1})\dots K_{j_{q'+1}j_q}(\xi_{j_{q'+1}},\xi_q) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
K_{j_qj_1}(\xi_q,\psi^{\nu_1}_{j_1s_1})\dots K_{j_qj_{q'}}(\xi_q,\psi^{\nu_{q'}}_{j_{q'}s_{q'}})K_{j_qj_{q'+1}}(\xi_q,\xi_{q'+1})\dots \\
\dots K_{j_qj_q}(\xi_q,\xi_q)
\end{vmatrix}\times
$$

$$
\times\frac{1}{u'_{j_1}(\psi^{\nu_1}_{j_1s_1})\dots u'_{j_{q'}}(\psi^{\nu_{q'}}_{j_{q'}s_{q'}})(u_{j_{q'+1}}(\xi_{q'+1})-z)\dots(u_{j_q}(\xi_q)-z)}+\dots+
$$

$$
+\int_{\Gamma_\sigma}\dots\int_{\Gamma_\sigma}\begin{vmatrix}
K_{j_1j_1}(\xi_1,\xi_1)\dots K_{j_1j_q}(\xi_1,\xi_q) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
K_{j_qj_1}(\xi_q,\xi_1)\dots K_{j_qj_q}(\xi_q,\xi_q)
\end{vmatrix}\frac{d\xi_1\dots d\xi_q}{(u_{j_1}(\xi_j)-z)\dots(u_{j_q}(\xi_q)-z)}.
$$

Using the same arguments as above we infer from (20) that the function has an analytical continuation into the $\varepsilon$-neighborhood $V'_\varepsilon(A')$ of $z = A'$ through the intervals $(A'-\varepsilon, A')$ and $(A', A'+\varepsilon)$. We denote by $d^*_{j_1\dots j_q}(z)$ the analytical continuation of $d_{j_1\dots j_q}(z)/\mathbb{C}^1_+$ into the region $V'_\varepsilon(A')$. From the proposition 4 it follows that the function is, generally speaking, a multivalued function with the branching point $z = A'$.

Let us denote by $d_q^*(z)$, $q = 1,2,\ldots$ the function

$$(21) \quad d_q^*(z) = \sum_{j_1,j_2,\ldots,j_q=1}^{n} d^*_{j_1 j_2 \ldots j_q}(z), \quad z \in V_\varepsilon'(A')$$

which is again a multivalued one on $V_\varepsilon'(A')$, with a branching point $z = A'$. From the Puiseux theorem and the expression (20) it follows that the function $d_q^*(z)$, $z \in V_\varepsilon'(A')$ expands into the following series:

$$d_q^*(z) = \sum_{s=-q}^{\infty} F_{s,A'}(K)(Z-A')^{s/p}, \quad z \in V_\varepsilon'(A')$$

Thus the proof of Lemma 2 is completed.

Proof of Lemma 1. Denote by

$$(22) \quad \Delta^*(z) = \sum_{q=1}^{\infty} \frac{1}{q!} d_q^*(z), \quad z \in V_\varepsilon'(A').$$

Then from (21) for $d_q^*(z)$, $q = 1,2,\ldots$ and from the Hadamard inequality for determinants (see [4]) it follows that the series (22) converges absolutely in $V_\varepsilon'(A')$ and defines a multivalued analytical function with a unique branching point $z = A'$. Therefore, from the Puiseux theorem (see [6]) and the expansion (22) used for $d_q^*(z)$, $q = 1,2,\ldots$ we obtain the statement of Lemma 1.

Proof of Theorem 2. Following Proposition 2 and the Fredholm theorem a point $z \in \mathbb{C}^1 \setminus \Sigma_{cont}(H)$ is an eigenvalue of H iff $\Delta(z) = 0$.

Because of the self-adjointness of H, it is sufficient to show that the function $\Delta(z)$ has only a finite number of real zeros not belonging to the continuous spectrum. We shall only show that $\Delta(z)$ may have a finite number of real zeros greater than A, $A = \sup_{\lambda \in \Sigma_{cont}(H)} \lambda$. The remaining intervals of the complement of the continuous spectrum may be investigated in an analogous way. It follows from Lemma 1 that the function $\Delta(z)$ can be expressed in the $\varepsilon$-neighborhood $V_\varepsilon'(A')\setminus(-\infty,A]$ of $z = A'$ by the following series:

$$\Delta(z) = \sum_{s=-\hat{q}}^{\infty} F_{s,A}(K)(z-A)^{s/p}, \quad z \in V_\varepsilon'(A')\setminus(-\infty,A] .$$

Therefore, A cannot be a limit point for the set $\{z \in R^1 : \Delta(z) = 0, z > A\}$. On the other hand, the function $\Delta(z)$ is regular in $\mathbb{C}^1 \setminus \Sigma_{cont}(H)$ and $\Delta(z) \to 1$ for $|z| \to \infty$, $\mathrm{Im}\, z = 0$, and thus it has only a finite number of zeros belonging to $(A + \varepsilon, \infty)$ for any $\varepsilon > 0$.

## R e f e r e n c e s

[1] FRIEDRICHS K.O.: Über die Spektralzerlegung eines Integraloperators, Math.Ann. 115(1938), 249-272.

[2] LAKAJEV S.N.: About the discrete spectrum of a generalized Friedrichs model, Dokl.Akad.Nauk.Uzbek.SSR,No. 4, 1979, 9-10.

[3] ACHIEZER N.I., GLAZMAN I.M.: Theory of linear operators in Hilbert space, Nauka,Moscow, 1965 (in Russian).

[4] LOVITT W.V.: Linear integral equations, New York, 1924.

[5] EVGRAFOV M.A.: Analytical functions, Nauka ,1968(in Russian).

[6] VAINBERG M.M., TRENOGIN V.A.: Theory of branching of solutiona of nonlinear equations,Nauka,1969(in Russian).

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