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SOME PROBLEMS AND LINES OF INVESTIGATION IN GENERAL TOPOLOGY

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Dedicated to Professor Miroslav Katětov on his seventieth birthday

Introduction. In this survey we are concerned with some open problems in General Topology. Some of these problems are rather old and some are of quite recent origin or are stated in print for the first time. Each problem will be discussed to some extent: I shall mention what is known about when and by whom the problem was formulated for the first time, some results related to it will be cited and references to the literature will be provided.

Some of the problems under consideration belong to relatively new (or completely new) directions of investigation. In such cases we supply the reader with necessary definitions and relevant basic facts.

There is no single rule explaining how the problems appearing in the article were chosen. One easily imagines another article of the same kind with the disjoint set of problems of no lesser importance.

Nevertheless, I consider the problems chosen to be quite interesting and difficult to solve. Many of the problems involve Lindelöf spaces. One gets an impression that we know too little about this nice class of spaces.

In Sections 1.4 and 2.8 some new results on Lindelöf spaces are established.

Notations and terminology are as in [8] and [21]. Compact means bicompact. N^+ is the set of all positive integers, all spaces are assumed to be T_1. Regularity is included in the definition of Lindelöf space.

§ 1. Problem on continuous mappings

1.1. Mappings and classification of spaces. A very general classical problem embracing many concrete interesting problems is: when every space from a given class $\mathcal{P}$ can be represented as an image of a space from a class $\mathcal{Q}$ (more narrow than $\mathcal{P}$) under a mapping satisfying certain preformulated conditions. Here is an old open problem of this kind (see [8]).

Problem 1. Is it true that each Tychonoff metacompact space Y is an image of a paracompact Hausdorff space under an open continuous mapping f such that $f^{-1}(y)$ is compact for every $y \in Y$?

A motivation for Problem 1 can be seen in the following simple fact: if $f:X \longrightarrow Y$ is an open continuous mapping with compact preimages of points such that $f(X)=Y$ and X is paracompact then Y is metacompact.

Problem 1 was for the first time formulated by myself in 1965 at Moscow Seminars on Topology. It appeared in print in [8] and was mentioned in the Nice Congress lecture in 1970.

It is known (H.H. Wicke, see [8]) that there exists a Hausdorff metacompact space Y which cannot be represented as an image of a paracompact Hausdorff space under an open continuous mapping with compact preimages of points. On the other hand every space with a uniform base is an image of a metrizable space under such a mapping (S. Hanai, A. Arhangel´skii, see [8], [21]).

The following version of Problem 1 should be also mentioned:

Problem 2. Is it true that every metacompact Tychonoff space is an image of a paracompact Hausdorff space under a pseudoopen continuous mapping with compact preimages of points ?

H. Junnilla has established a non-trivial fact: if $f:X \longrightarrow Y$ is such a mapping as in Problem 2 and $f(X)=Y$, X is a paracompact Hausdorff space and Y is Hausdorff then Y is metacompact. Thus the question is whether this theorem can be reversed.

The following question asked by A.V. Arhangel´skii at Moscow University six or seven years ago appears in print for the first time.

Problem 3. Is it true that every Lindelöf space of non-measurable cardinality can be represented as a continuous image of a Lindelöf space of countable pseudocharacter ?

The last condition means that all points are G_δ´s. It is easily seen that if X is a Lindelöf space of countable pseudocharacter, Y is a Tychonoff space and $f:X \longrightarrow Y$ is a one-to-one continuous mapping then Y is also a space of countable pseudocharacter. On the other hand the class of hereditarily Lindelöf spaces is preserved by continuous mappings (onto Tychonoff spaces). There is no known restriction on the cardinality of Lindelöf spaces of countable pseudocharacter except that it has to be a non-measurable (in Ulam's sense) cardinal (see [8]) - hence the restriction on cardinality in Problem 3.

I do not know what is the answer to Problem 3 in the case of compact Hausdorff images. For example we can consider the following

Problem 4. Can one represent the Tychonoff cube I^{τ} where τ is a non-measurable cardinal or where $\tau = 2^{\aleph_0}$) as a continuous image of a Lindelöf space of countable pseudocharacter ?

To get the positive answer to Problems 3 and 4 would be extremely difficult as we do not know whether in ZFC can exist a Lindelöf space of countable pseudocharacter and of cardinality greater than $2^{\aleph_0}$ (see in this respect [1] and [8]). To remove this cardinality obstacle we formulate

Problem 5. Is it true that every Lindelöf (every compact Hausdorff) space of cardinality not greater than $2^{\aleph_0}$ can be represented as a continuous image of a Lindelöf space of countable pseudocharacter ?

The following open problem is also published for the first time .

We start with a few motivating observations. Probably the greatest defect of Lindelöf property lies in the fact that it is not productive. On the other hand compactness being productive also enjoys the incompressibility property - in the sense that one cannot map a compact space onto a different Hausdorff space by a one-to-one continuous mapping. The idea behind the following problem is that the two properties - productivity and incompressibility - might be related to each other.

Problem 6. Let X be a Lindelöf space. Does there exist then a one-to-one continuous mapping $f:X \to Y$ such that $Y \times Y$ is Lindelöf ?

Recall that a space X is finally compact if every open covering of X contains a countable subcovering. A student of Moscow University A. Jakivčik has constructed a Hausdorff finally compact space X such that for every one-to-one continuous mapping $f:X \to Y$ onto a Hausdorff space Y the space $Y \times Y$ is not finally compact. Thus in the class of Hausdorff spaces the question similar to Problem 6 is answered negatively.

There are many interesting versions of Problem 6. Here are some of them.

Problem 7. Let X be a Lindelöf space and $\mathcal{T}$ be the product topology on $X \times X$. Can one find a topology $\mathcal{T}' \subset \mathcal{T}$ on $X \times X$ such that the set $X \times X$, topologized with $\mathcal{T}'$, is a Lindelöf space ?

Problems 6 and 7 can be formulated for arbitrary number of (possibly different) factors. For example we have

Problem 8. Let $\{(X_\alpha, \mathcal{T}_\alpha): \alpha \in A\}$ be any family (finite or infinite) of Lindelöf spaces and let $\mathcal{T}$ be the product topology on $X=\prod\{X_\alpha : \alpha \in A\}$. Is it always possible to find a topology $\mathcal{T}' \subset \mathcal{T}$ such that $(X, \mathcal{T}')$ is a Lindelöf space ?

There is another interesting unsolved problem on one-to-one continuous mappings, which involves Lindelöf spaces. One of topological properties which are opposite in many respects to Lindelöfness is pseudocompactness. This can be considered as a motivation for the following.

Problem 9. Is it true that every Tychonoff space can be mapped by a one-to-one continuous mapping onto a space which is either pseudocompact or Lindelöf ?

Let us call entightments one-to-one continuous mappings onto. Observe that only very few spaces can be entightened onto compact Hausdorff spaces (see [8]).

Problem 9 was formulated by me six or seven years ago at Moscow Seminars in General Topology. Strangely enough it remains unsolved.

Now we shall discuss Lindelöf spaces from another standpoint. It is well known that every Lindelöf space is realcompact (in the sense of E. Hewitt and L. Nachbin - see [21]). Every regular space which is a continuous image of a Lindelöf space is itself Lindelöf and hence realcompact. The problem arises whether the converse is true. Thus we have

Problem 10. Let X be a regular Tychonoff space such that every regular (Tychonoff) space which is a continuous image of X is realcompact. Is it true that X is Lindelöf ?

This problem was formulated by A.V. Arhangel´skii and O.G. Okunev in [7]. It is shown in [7] that a regular space X need not be Lindelöf if every Tychonoff image of X under one-to-one continuous mapping is a realcompact space.

1.2. Projective properties and projective Čech completeness. Let $\mathcal{P}$ be a topological property. Following [5] we say that a space X is projectively $\mathcal{P}$ or X has the property $\mathcal{P}$ projectively if for every open continuous mapping $f:X \longrightarrow Y$ where $f(X)=Y$ is a separable metrizable space, the space Y must have the property $\mathcal{P}$. In particular a space X is projectively Čech-complete if every separable metrizable image of X under an open continuous mapping is a Čech-complete space. It can be easily deduced from the known result that every Čech-complete space is projectively Čech-complete (note that Čech completeness is not preserved in general by open continuous mappings onto Tychonoff spaces - see [8]).

For a Tychonoff space X we denote by $C_p(X)$ the space of real-valued continuous functions on X in the topology of pointwise convergence (see [15], [16]). If Y is a closed subspace of X then the restriction mapping $r:C_p(X) \to C_p(Y)$, defined by $r(f)=f|Y$ for every $f \in C_p(X)$, is an open continuous mapping of $C_p(X)$ onto the subspace $C_p(Y|X)=r(C_p(X))$ of the space $C_p(Y)$ (see [15]). If Y is also countable then $C_p(Y|X)$ has a countable base. Thus if $C_p(X)$ has a property $\mathcal{P}$ projectively then $C_p(Y|X)$ has the property $\mathcal{P}$ provided Y is a countable closed subspace of X. In particular, if $C_p(X)$ is projectively Čech-complete then $C_p(Y|X)$ is Čech-complete which implies that Y is discrete. Thus if $C_p(X)$ is projectively Čech-complete then every countable closed subspace of X is discrete (see [5]). Arguments of this kind provide us with a motivation for the study of projective properties.

The space $C_p(X)$ is seldom Čech-complete - only if X is countable and discrete [5]. And when $C_p(X)$ is projectively Čech-complete ?

The answer is unknown.

Problem 11. Characterize in terms of X when $C_p(X)$ is projectively Čech-complete.

Here is an interesting concrete question related to Problem 11.

Problem 12. Is the space $C_p(\beta N)$ projectively Čech-complete ?

Observe that in the Čech-Stone compactification βN of the discrete space N of integers all closed countable subspaces are finite and hence discrete. It is evident that all projective properties are preserved by open continuous mappings. A closed subspace of a projectively Čech-complete space need not be projectively Čech-complete. For example, every pseudocompact space is projectively Čech-complete, and it is well known that each Tychonoff space can be realized as a closed subspace of a pseudocompact space (see [21]).

But very little is known on the behaviour of projective properties under products.

Problem 13. Let X and Y be projectively Čech-complete Tychonoff spaces. Is it true then that the product $X \times Y$ is projectively Čech-complete ? What if Y is a compact Hausdorff space ?

The answer to Problem 13 is unknown even in the case when Y is a metrizable compact space (or Y is the unit segment). There seems to be a good chance that the following special version of Problem 13 will be answered in the positive way.

Problem 14. Let X be a Tychonoff space such that the space $C_p(X)$ is projectively Čech-complete. Is it true then that $C_p(X) \times C_p(X)$ is projectively Čech-complete ?

Whilst the last question gives an impression of being rather special, it is closely related to the basic Problem 11.

Alongwith projective Čech-completeness it is quite tempting to investigate projective compactness, projective σ-compactness and projective finiteness. Some results of that type are mentioned in [6] and [5].

1.3. Cleavable spaces. Let $\mathcal{P}$ be a class of topological spaces. We say that a space X is cleavable with respect to $\mathcal{P}$ (or $\mathcal{P}$-cleavable) if for every subset $A \subset X$ there exists a space $Y \in \mathcal{P}$ and a continuous mapping $f:X \longrightarrow Y$ such that $f(X)=Y$ and $A=f^{-1}f(A)$. If a space X is cleavable with respect to the class of all separable metrizable spaces then X is simply called cleavable. These concepts were introduced in [9] and [6]. Again a motivation for the concept of cleavability can be found in the C_p-theory: a Tychonoff space X is cleavable if and only if for every realvalued function f on X one can find a countable family A of continuous real-valued functions on X such that f belongs to the closure of A (with respect to the topology of pointwise convergence on R^X).

It is natural to consider M-cleavable spaces - the spaces which are cleavable with respect to the class M of all metrizable spaces. Also k-cleavable spaces deserve attention, these spaces are cleavable with respect to the class of all compact Hausdorff spaces.

It is easily seen that all M-cleavable spaces and all k-cleavable spaces are Hausdorff. In M-cleavable spaces all points are G_δ's (see [6],[9]). In [9] it is shown that every M-cleavable paracompact p-space is metrizable. One of the most interesting unsolved problems concerning cleavability is the following one:

Problem 15. Let X be a cleavable (an M-cleavable) Tychonoff space. Is it true then that the diagonal $\Delta_X = \{(x,x):x \in X\}$ is a G_δ-set in $X \times X$?

In an important case the last problem was solved (see [9]): every Lindelöf cleavable space is a space with G_δ-diagonal (see 1.4). One can find many other results on cleavable spaces in [9], and [6].

It is not clear whether the following natural problem will have a nice solution:

Problem 16. Find an "inner" characterization of those Tychonoff spaces that are cleavable (M-cleavable, k-cleavable). Observe that every M-cleavable space which is Lindelöf or satisfies the countable chain condition is cleavable.

One should keep in mind that if a space X can be mapped by a one-to-one continuous mapping onto a separable metrizable space (onto a metrizable space, onto a compact Hausdorff space) then it is cleavable (M-cleavable, k-cleavable, accordingly).

For a given space X it might be an interesting problem to find out whether X is cleavable with respect to some class of spaces with much better properties than X. A general version of Problem 16 may be stated in the following way: given a class $\mathcal{P}$ of topological (Tychonoff) spaces, characterize the class $\mathcal{P}^{\#}$ of all (Tychonoff) spaces which are cleavable with respect to $\mathcal{P}$. In this direction, one can find some problems and results in [9],[6].

1.4. An approach to cleavability and a theorem on Lindelöf spaces.

Let X be a set and A - a subset of X. A family γ of subsets of X will be called a separator for A if for every $x \in A$ and every $y \in X \setminus A$ there exists $P \in \gamma$ such that $x \in P$ and $y \notin P$. In the case when X is a topological space, a separator for $A \subset X$ is said to be closed if it consists of closed sets.

Let τ be an infinite cardinal number. We shall say that a topological space X is τ-divisible if for every $A \subset X$ there exists a closed separator in X of cardinality not greater than τ. For the sake of brevity we call divisible those spaces which are $\aleph_0$-divisible.

A space X will be called strictly divisible if for every subset $A \subset X$ there exists a countable separator consisting of closed G_δ-subsets of X.

These concepts were considered by the author three or four years ago because of their close relationship to the concept of cleavability (see [6],[9]).

Recall that a family $\mathcal{E}$ of subsets of X is said to be separating points of X if for every x and y in X, where $x \neq y$, one can find $B \in \mathcal{E}$ such that $x \in B$ and $y \notin B$.

Obviously, in T_1-spaces every network and every pseudobase serves as a separator for all subsets. It is also clear that if $f: X \to Y$ is a mapping onto, $A \subset Y$ and γ is a separator for A (in Y) then the family $\{f^{-1}(P): P \in \gamma\}$ is a separator for $f^{-1}(A)$ in X. It follows that if a space X is cleavable then it is strictly divisible. A slightly more general result:

Proposition 1. If a space X is cleavable with respect to the class of all T_1-spaces with a countable closed network then it is divisible.

Another simple assertion is also quite useful.

Proposition 2. If a space X is τ-divisible then every subset A of X can be represented as the union of not more than 2^{τ} closed sets in X.

Proof. If γ is a closed separator for A in X then for every point $x \in X$ the closed set $\bigcap\{P \in \gamma : x \in P\}$ is contained in X. Put $S = \{\bigcap \mu : \mu \subset \gamma\}$ and $S^* = \{B \in S : B \subset A\}$. Then $|S^*| \leq |S| \leq 2^{\tau}$, all elements of S^* are closed sets in X and $A = \bigcup S^*$.

It is an easy fact that all points in cleavable spaces are G_δ's. Thus the theorem that every cleavable Lindelöf space has G_δ-diagonal from [9] is a straightforward corollary to the following result.

Theorem 1. If X is a strictly divisible Lindelöf space then the diagonal in $X \times X$ is G_δ.

First we shall prove

Theorem 2. Let X be a Lindelöf space of countable pseudocharacter which is $2^{\aleph_0}$-divisible. Then $|X| \leq 2^{\aleph_0}$.

Proof. Let A be a discrete subspace of X. By Proposition 2, $A = \bigcup\{A_\alpha : \alpha < \tau\}$, where $\tau = 2^{2^{\aleph_0}}$ and each A_α is closed in X. Every A_α is a discrete Lindelöf space. Hence A_α is countable for each $\alpha < \tau$ and $|A| \leq \tau = 2^{2^{\aleph_0}}$. Thus $s(X) \leq \tau = 2^{2^{\aleph_0}}$. As X is a T_1-space, by a well known formula we have (see [1],[22]):

$$|X| \leq (s(X))^{\psi(X)} \leq \tau^{\aleph_0} = 2^{2^{\aleph_0} \cdot \aleph_0} = 2^{2^{\aleph_0}} .$$

It follows that there exists a family $\mathcal{E}$ of subsets of X separating points of X and such that $|\mathcal{E}| \leq 2^{\aleph_0}$ (see [8]).

For every $B \in \mathcal{E}$ we fix a closed separator γ_B such that $|\gamma_B| \leq 2^{\aleph_0}$. Then that family $\gamma = \bigcup\{\gamma_B : B \in \mathcal{E}\}$ is separating points of X and $|\gamma| \leq 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$. Besides γ consists of closed sets. Hence $\mathcal{H} = \{X \setminus P : P \in \gamma\}$ is a pseudobase in X and $|\mathcal{H}| = |\gamma| \leq 2^{\aleph_0}$. Thus $p\,w(X) \leq 2^{\aleph_0}$.

By another well known formula (see [1],[22]) we have:

$$|X| \leq (p\,w(X))^{\psi(X) \cdot l(X)} \leq (2^{\aleph_0})^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

Theorem 2 is proved.

Theorem 3. If X is a divisible Lindelöf space of countable pseudocharacter then $p\,w(X) \leq \aleph_0$ - i.e. there exists a countable pseudobase in X.

Proof. By Theorem 2, $|X| \leq 2^{\aleph_0}$. It follows that there exists a countable family $\mathcal{C}$ of subsets of X separating points of X. For every $A \in \mathcal{C}$ choose a countable closed separator γ_A. The family $\gamma = \bigcup\{\gamma_A : A \in \mathcal{C}\}$ is countable and consists of closed sets. Obviously γ is separating points of X. Hence the family $\mathcal{H} = \{X \setminus P : P \in \gamma\}$ is a countable pseudobase of X and $p\,w(X) \leq \aleph_0$.

The following assertion is obvious.

Proposition 3. In every strictly divisible space all points are G_δ's.

Proof of Theorem 1. By Proposition 3, X is a space of countable pseudocharacter. By Theorem 2, $|X| \leq 2^{\aleph_0}$. Following the proof of Theorem 3, we can construct a countable pseudobase in X consisting of F_σ-sets. It is easy to show that every space with a countable pseudobase consisting of F_σ-sets has G_δ-diagonal.

Remark. Not every Hausdorff space with a countable base has G_δ-diagonal.

§ 2. More problems on Lindelöf spaces

2.1. The problem of D.V. Rančin. Let us say that a subspace Y of a space X is finally compact in X if every open covering γ of X *contains a countable* subfamily μ such that $Y \subset \bigcup\mu$. *If X is regular and Y is finally compact* in X we say that Y is Lindelöf in X. *These definitions belong to D. Rančin,* who has stated the *following problem.*

Problem 17. Let $Y \subset X$, and Y is Lindelöf in X. Is it true then that there exists a subspace $Z \subset X$ such that $Y \subset Z$ and Z is Lindelöf ?

Clearly if $Y \subset Z \subset X$ where X is regular and Z is Lindelöf then Y is Lindelöf in X.

Though Problem 17 is seven or eight years old and was discussed several times at seminars and conferences on Topology, it remains unsolved.

Several persons have observed that if $Y \subset X$, X is Hausdorff and Y is finally compact in X then it is not true in general that there exists $Z \subset X$ such that $Y \subset Z$, and Z is finally compact (in itself).

In one particular case Problem 17 was solved: A.V. Arhangel'skii and Hamdi M.M. Genedi (A.R.E.) have shown that if Y is pseudocompact and Y is Lindelöf in X then $\overline{Y}$ is compact.

2.2. The Lindelöf property and tightness. Let X be a space. Recall that the tightness of X is countable (notation: $t(X) \leq \aleph_0$) if for every $A \subset X$ and every $x \in \bar{A}$ there exists a countable set $B \subset A$ such that $x \in \bar{B}$. It is known that $t(X) \leq \aleph_0$ does not in general imply that $t(X \times X) \leq \aleph_0$ (see [2]). On the other hand V.I. Malychin has shown that if X is a compact Hausdorff space and $t(X) \leq \aleph_0$ then $t(X \times X) \leq \aleph_0$ (see [1]). The following question remains unanswered.

Problem 18. Is it possible to construct in ZFC a Lindelöf space X of countable tightness such that $t(X \times X) > \aleph_0$?

Problem 19. Let X be a Lindelöf Σ-space of countable tightness. Is it true then that the tightness of $X \times X$ is also countable ?

A.G. Leiderman and V.I. Malychin have shown that existence of such X, as in Problem 18 does not contradict ZFC.

2.3. Networkweight and the Lindelöf property. The following problem was formulated by myself ten or twelve years ago. Under CH the answer to it is "yes" (see [24]).

Problem 20. Find in ZFC a space X such that $X^{\aleph_0}$ is both hereditarily separable and hereditarily Lindelöf, but X does not have a countable network.

Some further material relevant to Problem 20 can be found in [27].

2.4. Lindelöf property in topological groups. In the class of all topological groups many properties behave much better than in the class of all Tychonoff spaces. For example, the first axiom of countability becomes the same as metrizability, pseudocompactness becomes productive, countable pseudocharacter is equivalent to the G_δ-diagonal property. One of common vices of a great number of very nicely looking topological properties is their unstability with respect to products: the square of a normal space need not be normal, the square of a paracompact space need not be paracompact, the square of a Lindelöf space may be a very non-Lindelöf space - it need not even be normal. So that we would enjoy very much if in the presence of a group structure a topological property would become "productive". This is exactly what happens in the case of pseudocompact topological groups (see [19]). For paracompactness, normality, countable compactness and Lindelöf property the situation is far from being clear. To construct two Lindelöf topological groups G and H such that $G \times H$ is not Lindelöf one can start with any two spaces X and Y such that X^n and Y^n are Lindelöf for all $n \in N^+$ while the product space $X \times Y$

is not Lindelöf. Under CH such spaces X and Y were constructed by E. Michael, while T. Przymysinskij has produced X and Y with the above mentioned properties just in ZFC (see [24]). Then the free topological groups F(X) and F(Y) are the groups G and H we are looking for. It turns out to be much more difficult to construct a single Lindelöf topological group G such that the square $G\times G$ is not Lindelöf. V.I. Malychin has done it under CH: he has constructed a Lindelöf topological group G such that the space $G\times G$ is not even normal (see [12]). The following question (put forward for the first time in [3]) remains open:

Problem 21. Is it possible to construct in ZFC (i.e. not using additional assumptions and tools) a Lindelöf topological group G such that the space $G\times G$ is not Lindelöf ?

Similar questions about normality, paracompactness and countable compactness also remain unsolved.

The following question is closely related to Problem 21 and is interesting in itself.

Problem 22. Let H_1 and H_2 be any Lindelöf topological groups. Is it true that one can find a Lindelöf topological group G such that H_1 and H_2 are topologically isomorphic to closed subgroups of G ?

A similar question can be asked about normality, pseudocompactness, countable compactness and many other topological properties.

Another interesting problem connected to Problem 21 in the other obvious way is as follows.

Problem 23. Is it true that every Lindelöf space is homeomorphic to a closed subspace of some Lindelöf topological group ?

The same question can be asked about normal spaces, paracompact spaces and so on.

Let us mention a concrete version of Problem 23.

Problem 24. Does there exist a Lindelöf topological group G such that the Sorgenfrey line (the "arrow" space) is homeomorphic to a closed subspace of G ?

I have formulated the last two questions at Moscow Seminars on Topology in 1978 and to Eric van Douwen at the time of Moscow International Conference on Topology in 1979. He communicated the questions to other mathematicians and they appeared in Comfort's article [19], who was doubtful to whom to attribute the questions. Problems 23 and 24 were also mentioned in [3].

Clearly Sorgenfrey line is not homeomorphic to a topological group - otherwise being first countable it would have been metrizable. But it is far from being clear what is the answer to the following question:

Problem 25. Can Sorgenfrey line be represented as a continuous image of a Lindelöf topological group?

Positive answer to the last question would imply positive solution of Problem 21.

Problems similar to the following one arise in the theory of transitive actions of topological groups on topological spaces.

Problem 26. Let Y be a first countable compact Hausdorff space which is a continuous image of a Lindelöf topological group. Is it true then that Y is metrizable ?

2.5. $C_p(X)$ and Lindelöf property. The space $C_p(X)$ of all real-valued continuous functions on a Tychonoff space X always satisfies the countable chain condition. It follows that $C_p(X)$ is paracompact if and only if $C_p(X)$ is Lindelöf. If X is a Lindelöf Σ-space (in particular if X is compact and $C_p(X)$ is normal then $C_p(X)$ is Lindelöf (E.A. Resnichenko, see [15]). But we do not know how to characterize in terms of X when $C_p(X)$ is Lindelöf.

Problem 27. Find an "inner" topological property of a Tychonoff space X necessary and sufficient for $C_p(X)$ to be Lindelöf.

It is not clear at all whether Problem 27 should have a "nice" solution. But if $\mathcal{P}$ is such a property of X as in Problem 27 and $\mathcal{P}$ is not an "artificial" topological invariant then it is natural to expect that if X is a space with property $\mathcal{P}$ then the free topological sum $X \oplus X$ of two copies of X is also a space with property $\mathcal{P}$. If this is the case then the following implication would be true: if $C_p(X)$ is Lindelöf then $C_p(X) \times C_p(X)$ is also Lindelöf.

In this way we arrive to the following problem:

Problem 28. Let $C_p(X)$ be Lindelöf. Is it true then that $C_p(X) \times C_p(X)$ is Lindelöf ?

The reasoning preceding Problem 28 reveals that whatever topological property Q of $C_p(X)$ we have which can be characterized by a natural topological property of X, there are very good chances that property Q will prove to be productive: if $C_p(X)$ has property Q then $C_p(X) \times C_p(X)$ also enjoys Q.

There are several topological properties Q for which this conclusion holds (see [15],[16]).

The following interesting problem is obviously related to Problem 28.

Problem 29. Is it true that $C_p(X)\ \ C_p(X)$ can always be represented as the continuous image of $C_p(X)$?

There exists an infinite compact Hausdorff space X (actually, $X=T(\omega_1+1)$ such that $C_p(X)\times C_p(X)$ is not homeomorphic to $C_p(X)$ (see [15]).

If instead of $C_p(X)$ in Problem 29 we take an arbitrary locally convex linear topological space L, then the answer is negative (W. Marciszewski [23]) - even when L is separable and metrizable.

An important necessary condition $C_p(X)$ to be Lindelöf is that the tightness of X^n has to be countable for every $n\in N^+$ (M. Asanov, see [10]).

The next problem at first seems a little bit strange: but actually it is not - it is just one of the ways to ask whether the class of all Tychonoff spaces X such that $C_p(X)$ is Lindelöf, is rich enough.

Problem 30. Is it true that every Tychonoff space Y can be represented as the continuous image of a Tychonoff space X such that $C_p(X)$ is Lindelöf ?

If instead of "continuous" we put "quotient" in Problem 30 then the answer will obviously be negative: the space $C_p(Y)$ will also have to be Lindelöf.

An interesting open question is connected to the following theorem (see [15],[16]): The tightness of $C_p(X)$ is countable if and only if the space X^n is Lindelöf for every $n\in N^+$. From this it follows that if X is just Lindelöf then the tightness of $C_p(X)$ need not be countable: for example if X is Sorgenfrey line then $t(C_p(X) > \aleph_0$. But what if we consider only compact parts of $C_p(X)$?

Problem 31. Let X be a Lindelöf space. Is it true then that the tightness of every compact subspace Φ of $C_p(X)$ is countable ?

Problem 31 was for the first time formulated by A.V. Arhangel'skii and V.V. Uspenskij in [17]. See also [15].

If Φ is dyadic then the answer is "yes" (see [17]). Below, a conditional solution to Problem 31 is given but I believe that the answer to Problem 30 should be "yes" just in ZFC.

2.6. A conditional solution of Problem 31 and some further discussion of problems.

Theorem 4. The following assertion does not contradict ZFC: Let Y be a Lindelöf space and Φ - a compact subspace of $C_p(Y)$. Then $t(\Phi) \le \aleph_0$.

Proof. Assume the contrary: let $t(\Phi) > \aleph_1$. Then there exists a free sequence $\{x_\alpha : \alpha < \aleph_1\}$ in Φ of the length $\aleph_1$ - see [1]. As Φ is compact, one can fix $x^* \in \Phi$ which is a point of complete accumulation for the set $\{x_\alpha : \alpha < \aleph_1\}$. Put $P_\alpha = \{x_\beta : \beta \le \alpha\}$ for each $\alpha < \aleph_1$. Then $x^* \notin \overline{P_\alpha}$ and by definition of the topology of pointwise convergence there exists a finite subset $K_\alpha \subset Y$ such that the point $x^* | K_\alpha$ is not the closure of the set $\{x|K_\alpha : x \in P_\alpha\}$ (in the space $C_p(K_\alpha)$). Let $M = \bigcup\{K_\alpha : \alpha < \aleph_1\}$ and $Z = M$. Let $H = \{x|Z : x \in \Phi\}$. Then H is a compact subspace of the space $C_p(Z)$. Clearly H is homeomorphic to a subspace of the space $C_p(M)$. As $|M| = \aleph_1$, the networkweight of $C_p(M)$ does not exceed $\aleph_1$ (see [15]). Hence the weight of H is not greater than $\aleph_1$ (see [8]).

On the other hand the point $x^*|Z$ is in the closure of the set $\{x_\alpha|Z : \alpha < \aleph_1\}$ but not in the closure of the sets $\{x_\beta|Z : \beta \le \alpha\}$ for any $\alpha < \aleph_1$ - this follows from the definition of $M \subset Z$. Hence the tightness of H is uncountable and $w(H) = \aleph_1$, $t(H) = \aleph_1$. From the reasoning of Z. Balogh and A. Dow (see [18],[20]) it follows that it does not contradict ZFC to assume that H contains a topological copy of the space $T(\omega_1+1) = \{\alpha : \alpha < \omega_1\}$ (with the usual topology).

It remains to apply the following assertion:

Proposition 4. If the space $T(\omega_1+1)$ can be embedded into the space $C_p(X)$, then the space X is not Lindelöf.

Proposition 4 is an obvious reformulation of one of the results of A.V. Arhangel'skii and V.V. Uspenskij (see [17]).

The proof of Theorem 4 is complete.

We shall formulate now a problem closely related to Problem 31.

Let us say that a space Y is colindelöf if there exists a Lindelöf space X such that Y is homeomorphic to a subspace of the space $C_p(X)$. This concept was introduced in [17]. Obviously every subspace of a colindelöf space is a colindelöf space.

Problem 32 (see [17]). Is it true that every continuous image of a colindelöf compact space is a colindelöf space ?

In every compact Hausdorff space of uncountable tightness one can easily find a closed subspace which can be mapped continuously onto the space $T(\omega_1+1)$. Proposition 4 implies that the compact space is not colindelöf.

Hence the positive solution of Problem 32 would imply the positive answer to Problem 31.

An interesting question was asked by O.G. Okunev:

Problem 33. Let X be a space such that $C_p(X)$ is a Lindelöf space and Φ - a compact subspace of $C_p(X)$. Is it true then that $t(\Phi) \leq \aleph_0$?

If $C_p(X)$ is a Lindelöf Σ -space then the answer is positive - see [15].

If the answer to Problem 33 is in the affirmative then not every space Y can be represented as a continuous image of a space X such that $C_p(X)$ is Lindelöf (see Problem 30). Indeed, to prove the last assertion it is enough to solve negatively the following problem:

Problem 34. Is it true that for every compact Hausdorff space Y there exists a space X such that $C_p(X)$ is Lindelöf and Y is homeomorphic to a subspace of $C_p(X)$? What if $Y= \beta(N)$ or $Y=I^{\tau}$, where $\tau > \aleph_0$?

Problem 35. Let X be a space such that $C_p(X)$ is a Lindelöf space and let Y be a Tychonoff space of countable pseudocharacter which is a continuous image of $C_p(X)$. Is it true then that $|Y| \leq 2^{\aleph_0}$?

Observe that if $Y=C_p(X)$ - i.e. the space $C_p(X)$ itself is of countable pseudocharacter - then $|Y|=|C_p(X)| \leq 2^{\aleph_0}$ (see [15]).

2.7. Around Veličko's problem. It was independently proved by Ph. Zenor and N.V. Veličko that $(C_p(X))^n$ is hereditarily Lindelöf for every $n \in N^+$ if and only if X^n is hereditarily separable for every $n \in N^+$, and that $(C_p(X))^n$ is hereditarily separable for each $n \in N^+$ if and only if X^n is hereditarily Lindelöf for all $n \in N^+$ (see [15],[11]).

N.V. Veličko has proved more: if $C_p(X)$ is hereditarily separable then $(C_p(X))^n$ is hereditarily separable for all $n \in N^+$ (see [11],[15]). A similar question for hereditarily Lindelöf $C_p(X)$ was formulated by him and left open:

Problem 36. Let $C_p(X)$ be hereditarily Lindelöf. Is it true then that $(C_p(X))^n$ is hereditarily Lindelöf for all $n \in N^+$?

A conditional solution of this problem was given by myself. Let us denote by SA the following assertion: every Tychonoff hereditarily separable space is hereditarily Lindelöf. It was shown by S. Todorčevič that SA is compatible with ZFC (see [27]). Let us recall that $s(Y) \leq \aleph_0$ denotes that every discrete subspace of Y is countable. Obviously if Y is hereditarily Lindelöf or hereditarily separable then $s(Y) \leq \aleph_0$. I have proved assuming SA that if $s(C_p(X)) \leq \aleph_0$ then $(C_p(X))^n$ is both hereditarily Lindelöf and hereditarily separable for every $n \in N^+$.

It is not clear whether a similar result can be proved for arbitrary cardinal τ (I venture to suggest that it cannot be proved). Thus we have the following general version of Veličko's problem:

Problem 37. Is it compatible with ZFC to assert that $hl(C_p(X)) = hl((C_p(X))^n)$ for every $n \in N^+$?

Observe that M. Asanov has shown that if X is a space with G_δ-diagonal and $C_p(X)$ is hereditarily Lindelöf then $(C_p(X))^n$ is hereditarily Lindelöf for all $n \in N^+$ - i.e. Problem 36 has in this case a positive solution (see [10]).

We should also mention here the following assertion published by R. Pol without an explicit proof in [25]:

(P) If X is a separable Tychonoff space such that $C_p(X) \times M$ is Lindelöf for every separable metrizable space M then for every $n \in N^+$ each discrete subspace of the space X^n is countable.

It seems to be unknown at the moment whether this assertion holds in ZFC - the proof R. Pol hints at does not work. One can derive Pol's assertion from SA - so that one cannot construct counterexample working in ZFC. On the other hand if (P) is true then the answer to Problem 36 is positive.

2.8. Some other problems on $C_p(X)$ and Lindelöf property. It is very easy to construct a non-Lindelöf Tychonoff space X such that $C_p(X)$ is Lindelöf. For example one can take as X the Σ-product of an uncountable family of unit segments. It is a little more difficult to describe a non-normal (pseudocompact) Tychonoff space X such that $C_p(X)$ is Lindelöf. But the following problem (stated by myself at Moscow University two years ago) remains open:

Problem 38. Let X be a Hewitt space such that $C_p(X)$ is Lindelöf. Is it true then that X is Lindelöf ? That it is at least normal ?

The next problem was formulated in my talk at the V-th Symposium on General Topology in Prague in 1981 (see [14]).

Problem 39. Let X be a Moore space (or a Tychonoff space with a uniform base in the sense of P.S. Alexandroff) such that $C_p(X)$ is Lindelöf. Is it true then that X is metrizable ?

A. Korovin has checked that if X is the Niemytzky plane or X is the Pixley-Roy space on R then $C_p(X)$ is not Lindelöf. There is a major problem in the theory of l-invariant properties which concerns Lindelöf spaces and is still open. Recall that Tychonoff spaces X and Y are said to be l-equivalent (t-equivalent) if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic (if $C_p(X)$ and

$C_p(Y)$ are homeomorphic as topological spaces).

Problem 40. Is it true that every Tychonoff space l-equivalent (t-equivalent) to a Lindelöf space is itself Lindelöff ?

This problem was formulated by myself in 1980, it appeared in print in [2], [14]. Observe that O.G. Okunev has shown that normality is not preserved by l-equivalence. For a further discussion of l-invariants see [15] and [16].

I would like to conclude this survey of problems on Lindelöf spaces with the following three questions which came to my mind just recently and the degree of difficulty of which is not quite clear to see.

Problem 41. Let X be a Hewitt space of countable spread (i.e. all discrete subspaces of X are countable). Is it true then that X is Lindelöf ? That X is normal ? What if we assume only that X is a Hewitt space in which all closed discrete subspaces are countable ?

This problem is somewhat related to Problem 38.

Problem 42. Let X be a Lindelöf space with a countable pseudobase. Is it true then that the diagonal in $X \times X$ is G_δ ?

This question is connected to results in Section 1.4.

Problem 43. Let X be a Tychonoff space with a uniform base (in the sense of P.S. Alexandroff - see [21]). Let us assume also that the Hewitt completion $\nu(X)$ of the space X is a Lindelöf space. Is it true then that X is metrizable ?

Observe that if X is a pseudocompact space with a uniform base then $\nu(X)$ is a compact and by a well known result of B. Scott and S. Watson the space X is metrizable (see [28]).

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