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AROUND KATĚTOV'S METRIZATION THEOREM

Heikki J.K. Junnila

Dedicated to Professor Miroslav Katětov on his seventieth birthday

ABSTRACT. We discuss some results related to the theorem of Katětov that a compact Hausdorff space X is metrizable if, and only if, X^3 is hereditarily normal, and we prove that X is metrizable if, and only if, X^ω is hereditarily metanormal.

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1. *INTRODUCTION.*

In 1948, M. Katětov [K] proved that a compact Hausdorff space X is metrizable if X^3 is hereditarily normal. Even though this theorem is not very difficult to prove, it is a remarkable result: it provides a topological characterization of metrizability which involves no explicit countability condition whereas the earlier topological metrization theorems required the existence of some kind of a countable structure.

In his paper, Katětov raised the question whether hereditary normality of X^2, for a compact Hausdorff space X, is enough to make X metrizable. In 1978, P.J. Nyikos [N1] showed that it is consistent with ZFC that the answer to Katětov's question is "no": using the assumption that Martin's Axiom and the negation of the Continuum Hypothesis hold, he provided an example of a non-metrizable compact Hausdorff space X such that the space X^2 is hereditarily normal; recently, G. Gruenhage has obtained an example of such a space under the assumption that the Continuum Hypothesis holds. Both examples are described in detail in [GN]; this paper also contains information on the following problem, which still remains open.

Problem 1 Is it consistent with ZFC that every compact space whose square is hereditarily normal is metrizable?

After Katětov's result,there have appeared other results which take the form: if X is a compact Hausdorff space and X^n has hereditarily some given property, then X is metrizable. Even some earlier results can be presented in that form: Šneĭder proved in 1945 [Š] that a compact Hausdorff space X is metrizable if the diagonal is a G_δ-set in X^2, and it follows from this result that X is metrizable provided that X^2 is hereditarily Lindelöf.The last–mentioned result was strengthened in 1984 by Gruenhage [G], who proved that a compact Hausdorff space is metrizable provided that X^2 is hereditarily paracompact; under Martin's Axiom and the negation of the Continuum Hypothesis, hereditary paracompactness can be weakened to hereditary collectionwise normality, as was shown by Nyikos in 1981 [N2]. In 1971 P. Zenor [Z] had proved that a compact Hausdorff space X is metrizable provided that X^3 is hereditarily countably paracompact.

Other properties of a compact Hausdorff space X, besides metrizability, can be characterized in terms of hereditary properties of some powers of X. For example, Gruenhage [G] proved that Eberlein compactness of X is equivalent with X^2 being hereditarily σ–metacompact and Corson compactness of X is equivalent with X^2 being hereditarily meta–Lindelöf. There is still much work to do on this area: for example, the following problem, raised in [G], remains open.

Problem 2 Characterize those compact Hausdorff spaces whose square is hereditarily metacompact.

Hereditary metacompactness of X^2 does not characterize Eberlein compact spaces: N.N. Yakovlev [Y] has shown that for the one–point compactification $A(\omega_1)$ of the discrete space on ω_1, the (Eberlein compact) space $A(\omega_1)^\omega$ is not hereditarily metacompact.

2. *A METRIZATION THEOREM.*

As mentioned in the introduction, a compact Hausdorff X space is metrizable provided that X^3 is either hereditarily normal or hereditarily countably paracompact. In this section we shall show that metrizability of X follows if the infinite power X^ω satisfies hereditarily a property significantly weaker than normality and countable paracompactness. The property to be considered was introduced by E.K. van Douwen.

Definition 1 [vD] A topological space X is *metanormal* provided that for every discrete family $\{F_n : n \in \omega\}$ of closed subsets of X there exists a family $\{L_n : n \in \omega\}$ of G_δ-subsets of X such that $\bigcap_{n\in\omega} L_n = \emptyset$ and, for each $n \in \omega$, $F_n \subset L_n$.

Note that, besides all normal spaces, all countably metacompact spaces are metanormal.

To prove that certain infinite powers are not hereditarily metanormal, we introduce the following concept.

Definition 2 Let F be a closed subset of a topological space X and let A be an uncountable subset of $X \setminus F$. We say that F *attracts* A provided that every uncountable subset B of A contains a countable subset C such that $Cl(C) \cap F \neq \emptyset$.

If F attracts some subset of X, then we say that F is an *attractive subset* of X.

If the attractive set F consists of one point x, then we say that x is an *attractive point* of X.

Remark For later use, we mention the following alternative characterization of an uncountable set A attracted to a closed set F: whenever I is an uncountable set and $\{V_i : i \in I\}$ is a family of neighborhoods of the set F, then there are only countably many elements $a \in A$ such that the set $\{i \in I : a \in V_i\}$ is countable.

Typical attractive subsets of a compact space are exhibited in the following lemma.

Lemma 1 *The following hold for a subset A of a compact Hausdorff space X:*

1° *If A is uncountable and relatively discrete, then A is attracted by the set $F = Cl(A) \setminus A$.*

2° *Assume that $A = \{x_\alpha : \alpha \in \omega_1\}$ and the following conditions hold:*

(a) $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$,

(b) A contains no uncountable relatively discrete subset, and

(c) for each $\beta \in \omega_1$, the set $\{x_\alpha : \alpha \leq \beta\}$ is open in A.

Then A is attracted to the set $F = \bigcap_{\beta\in\omega_1} Cl\{x_\alpha : \alpha \geq \beta\}$.

Proof. 1° follows from the observation that, under the assumption made in 1°, we have that $Cl(B) \cap F \neq \emptyset$ for every infinite $B \subset A$.

2° Since X is compact, we have that $F \neq \emptyset$. Note that it follows from (b) and (c) that A is hereditarily separable. Let C be a countable dense subset of A. Then $F \subset Cl(A) = Cl(C)$ and hence $Cl(C) \cap F \neq \emptyset$. A similar argument shows that every uncountable set $B \subset A$ contains a countable set D with $Cl(D) \cap F \neq \emptyset$. □

Proposition 1 *A compact Hausdorff space X is hereditarily Lindelöf if, and only if, X contains no attractive set.*

Proof. Necessity of the condition follows directly, since every closed subset of a regular hereditarily Lindelöf space is a G_δ–set.

To prove sufficiency, assume X is not hereditarily Lindelöf. Then X has a subset $A = \{x_\alpha : \alpha \in \omega_1\}$ such that $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$ and, for each $\beta \in \omega_1$, the set $\{x_\alpha : \alpha \leq \beta\}$ is open in A. If A contains an uncountable relatively discrete set, then by part 1° of Lemma 1, X contains an attractive set. On the other hand, if every relatively discrete subspace of A is countable, then it follows from part 2° of Lemma 1 that X again contains an attractive set. □

Now we show that a space can have no attractive points if its countably infinite power is hereditarily metanormal.

Proposition 2 *Let p be an attractive point of a topological space X. Then the subspace $X^\omega \setminus \{p\}^\omega$ of X^ω is not metanormal.*

Proof. Denote the subspace in question by Y. For each $k \in \omega$, let

$$S_k = \{y \in X^\omega : y(k) \neq p \text{ and } y(m) = p \text{ for every } m \neq k\}.$$

Note that $\{S_k : k \in \omega\}$ is a discrete family of closed subsets of Y. Partition the set ω into infinite pieces A_n, $n \in \omega$; for every $k \in \omega$, let $n_k \in \omega$ be defined by the condition that $k \in A_{n_k}$. For every $n \in \omega$, let $F_n = \bigcup_{k \in A_n} S_k$. It follows from the corresponding property of the family $\{S_k : k \in \omega\}$ that $\{F_n : n \in \omega\}$ is a discrete family of closed subsets of Y. We show that there is no family of G_δ–sets in Y as required in the definition of a metanormal space.

For every $n \in \omega$, let L_n be a G_δ-subset of Y such that $F_n \subset L_n$. We show that $\bigcap_{n\in\omega} L_n \neq \emptyset$. For every $n \in \omega$, let $G_{n\,i}$, $i \in \omega$, be open subsets of Y such that $\bigcap_{i\in\omega} G_{n\,i} = L_n$ and, for every $i \in \omega$, $G_{n\,i+1} \subset G_{n\,i}$.

Let B be an uncountable subset of $X \setminus \{p\}$ which is attracted to p. For all $b \in B$ and $k \in \omega$, denote by $y_{b\,k}$ that point of Y, whose k^{th} coordinate is b and all the other coordinates equal p. Note that $y_{b\,k} \in S_k \subset F_{n_k} \subset G_{n_k\,k}$ and hence there exist neighborhoods $V_{b\,k\,j}$, $j \neq k$, of p in X such that

$$\left[\prod_{j<k} V_{b\,k\,j}\right] \times \{b\} \times \left[\prod_{j>k} V_{b\,k\,j}\right] \subset G_{n_k\,k}.$$

Let us now construct a point y which belongs to the set $\bigcap_{n\in\omega} L_n$. By induction on $k \in \omega$, we define points $y_k \in B$ and uncountable sets $E_{k\,j} \subset B$, for $j > k$, as follows.

By the remark following Definition 2, there exists $y_0 \in B$ such that, for every $j > 0$, the set $D_{0\,j} = \{b \in B : y_0 \in V_{b\,j\,0}\}$ is uncountable. Note that, for every $j > 0$, the set $E_{0\,j} = D_{0\,j} \cap V_{y_0\,0\,j}$ is uncountable.

Let $k > 0$ be such that the points y_l and the uncountable sets $E_{l\,j}$ have already been defined for all $l < k$ and $j > l$. Then there exists $y_k \in E_{k-1\,k}$ such that, for each $j > k$, the set $D_{k\,j} = \{b \in E_{k-1\,j} : y_k \in V_{b\,j\,k}\}$ is uncountable. For each $j > k$, the set $E_{k\,j} = D_{k\,j} \cap V_{y_k\,k\,j}$ is uncountable, and this completes the inductive step. Note that it follows from the inductive construction that $E_{k\,j} \subset E_{l\,j}$ whenever $l < k < j$.

We show that the point $y = \langle y_k \rangle_{k\in\omega}$ belongs to the set $\bigcap_{n\in\omega} L_n = \bigcap_{n\in\omega \& i\in\omega} G_{n\,i}$. Let $n \in \omega$ and $i \in \omega$. Then there exists $m \in A_n$ such that $m > i$. For each $k < m$, we have that $y_m \in E_{m-1\,m} \subset E_{k\,m}$ and hence that $y_k \in V_{y_m\,m\,k}$. For each $k > m$, we have that $y_k \in F_{k-1\,k} \subset E_{m\,k} \subset V_{y_m\,m\,k}$. By the foregoing, we have that

$$y = \langle y_k \rangle_{k\in\omega} \in \left[\prod_{k<m} V_{y_m\,m\,k}\right] \times \{y_m\} \times \left[\prod_{k>m} V_{y_m\,m\,k}\right] \subset G_{n_m\,m} = G_{n\,m} \subset G_{n\,i}. \quad \square$$

Remark A straightforward extension of the above proof shows that the space $Y = X^\omega \setminus \{p\}^\omega$ not only fails to be metanormal, but it does not even have the following weaker property (which should be called *orthonormality*, to conform with van Douwen's terminology): for every discrete family $\{F_n : n \in \omega\}$ of closed subsets and for all open sets

O_n, $n \in \omega$, such that $F_n \subset O_n$ for every $n \in \omega$, there exist G_δ-sets L_n, $n \in \omega$, such that the set $\bigcap_{n\in\omega} L_n$ is open and, for every $n \in \omega$, $F_n \subset L_n \subset O_n$. Note that, besides metanormal spaces, all countably orthocompact spaces satisfy this weaker property.

We shall now indicate a slight extension of Proposition 2.

Corollary *Let K be a compact attractive subset of a topological space X. Then the subspace $X^\omega \setminus K^\omega$ of X^ω is not metanormal.*

Proof. Let Z be the space obtained from X by identifying the set K to a point p, and let f be the corresponding quotient mapping. Note that p is an attractive point of Z; hence it follows from Proposition 2 that the subspace $Z^\omega \setminus \{p\}^\omega$ of Z^ω is not metanormal.

Since K is compact, f is a perfect mapping. By a theorem of Z. Frolík [F] and N. Bourbaki [B], the "product mapping" $\bar{f} = \langle f, f, \ldots \rangle$ from X^ω onto Z^ω is a perfect mapping. We have that

$$\bar{f}^{-1}(Z^\omega \setminus \{p\}^\omega) = X^\omega \setminus K^\omega,$$

and it follows that the restriction of $\bar{f}$ to the subspace $X^\omega \setminus K^\omega$ of X^ω is a perfect mapping onto $Z^\omega \setminus \{p\}^\omega$. It is easy to see that metanormality is preserved under closed mappings. It follows, since $Z^\omega \setminus \{p\}^\omega$ is not metanormal, that neither is $X^\omega \setminus K^\omega$. □

Using the above corollary, we can easily prove our main result.

Theorem 1 *A compact Hausdorff space X is metrizable if, and only if, the space X^ω is hereditarily metanormal.*

Proof. Necessity of the condition is trivial. To prove sufficiency, assume that X^ω is hereditarily metanormal. Since X^ω is homeomorphic with $(X^2)^\omega$, the latter space is hereditarily metanormal, and it follows from the previous corollary that X^2 contains no attractive set. By Proposition 1, X^2 is hereditarily Lindelöf. It follows by Šneĭder's metrization theorem [Š] that X is metrizable. □

Corollary *A compact Hausdorff space is metrizable if, and only if, the space X^ω is hereditarily countably metacompact.*

We close this section with some more consequences of Proposition 2 and its proof.

Note that, in the proof of Proposition 2, we actually showed that if the space X contains an attractive point p, then $X^{\omega} \setminus \{p\}^{\omega}$ contains a family $\{F_n : n \in \omega\}$ of subsets which is closed and discrete in the relative product topology but for which there exist no family $\{L_n : n \in \omega\}$ of G_{δ}-sets in the relative box topology such that $\bigcap_{n\in\omega} L_n = \emptyset$ and, for every $n \in \omega$, $F_n \subset L_n$; in particular, the box topology of X^{ω} is not hereditarily metanormal.

Since the one-point compactification $A(\omega_1)$ of the discrete space on ω_1 has the compactifying point as an attractive point, we get the following.

Example $A(\omega_1)^{\omega}$ is not hereditarily metanormal either in the product topology or in the box topology.

Since $A(\omega_1)^{\omega}$ (in the product topology) is an Eberlein compact space of weight ω_1, it follows from results of D. Amir and J. Lindenstrauss (see [D], Chapter 5) that $A(\omega_1)^{\omega}$ can be embedded in the sequence-space $c_0(\omega_1)$, when the latter space is equipped with the topology of pointwise convergence. On the other hand, if X is any non-ccc compact Hausdorff space, then we can embed $c_0(\omega_1)$ in the space $C_p(X)$ (the set $C(X)$ equipped with the topology of pointwise convergence). As a consequence, we have the following result.

Proposition 3 *If X is a non-ccc compact Hausdorff space, then the space $C_p(X)$ contains a non-metanormal subspace with compact closure.*

In connection with the above result, we should recall the result of H.P. Rosenthal [R] that if X is a ccc compact Hausdorff space, then every compact subspace of $C_p(X)$ is metrizable.

3. *THE NON-COMPACT CASE.*

The metrization theorems of Katětov and Zenor mentioned in the introduction do not remain true for non-compact spaces. However, for general spaces, the following result obtains.

Theorem [K],[Z] *The following conditions are mutually equivalent for every topological space X:*

1° X^ω *is perfectly normal.*

2° X^ω *is hereditarily normal.*

3° X^ω *is hereditarily countably paracompact.*

The above result might lead one to conjecture that a topological space X is perfect provided that X^ω is hereditarily metanormal or hereditarily countably metacompact. However, this result does not hold even for Lindelöf spaces, as we shall now indicate.

Let $L(\omega_1)$ be the "one-point Lindelöfization" of the discrete space on ω_1, with the ordinal ω_1 as the "Lindelöfying" point. Then the neighborhoods of ω_1 in $L(\omega_1)$ are the sets with countable complement while the points $\alpha < \omega_1$ are isolated. The subset $\{\omega_1\}$ is not a G_δ-set in $L(\omega_1)$ and hence $L(\omega_1)$ is not a perfect space.

We shall show that $L(\omega_1)^\omega$ satisfies a base property which implies that $L(\omega_1)^\omega$ is hereditarily metacompact. Recall that a base $\mathcal{B}$ of a space X is *of subinfinite rank* provided that for every infinite subfamily $\mathcal{C}$ of $\mathcal{B}$, if $\bigcap \mathcal{C} \neq \emptyset$, then $\mathcal{C}$ contains two distinct sets which are related by inclusion. The base $\mathcal{B}$ is *Noetherian* provided that the poset $(\mathcal{B}, \subset)$ has no infinite increasing chains.

Proposition 4 $L(\omega_1)^\omega$ *has a Noetherian base of subinfinite rank.*

Proof. Let n, f and α be such that $n \in \omega$, $\alpha \in \omega_1$, f is a mapping with $Dom(f) \subset \{0, ..., n\}$ and $Im(f) \subset \omega_1$, and $\alpha > \max Im(f)$. Then define

$$V_{n,f,\alpha} = \{x \in L(\omega_1)^\omega : f \subset x \text{ and } x(k) \geq \alpha \text{ for every } k \in \{0, ..., n\} \setminus Dom(f)\}.$$

It is easy to see that the collection of all such $V_{n,f,\alpha}$ forms a base for the topology of $L(\omega_1)^\omega$; we denote this base by $\mathcal{B}$.

Note that, for all n, f, α and m, g, β, we have that $V_{n,f,\alpha} \subset V_{m,g,\beta}$ if, and only if, $m \leq n$, $g \subset f$ and $\beta \leq \alpha$. Using this observation, it is easy to see that $\mathcal{B}$ is Noetherian.

To show that $\mathcal{B}$ is of subinfinite rank, let $x \in L(\omega_1)^\omega$ and n_k, f_k and α_k, $k \in \omega$, be such that $x \in \bigcap_{k \in \omega} V_{n_k, f_k, \alpha_k}$. Then we can find $i \in \omega$ and $j \in \omega$ such that $i < j$, $n_i \leq n_j$, $\alpha_i \leq \alpha_j$ and $\max Im(f_i) \leq \max Im(f_j)$. We show that $V_{n_j, f_j, \alpha_j} \subset V_{n_i, f_i, \alpha_i}$.

Let us first show that $Dom(f_i) \subset Dom(f_j)$. Let $n \in Dom(f_i)$. Since $f_i \subset x$, we have that $f_i(n) = x(n)$. It follows that $x(n) \leq \max Im(f_i) \leq \max Im(f_j) < \alpha_j$, and it follows further, since $n \leq n_i \leq n_j$ and $x \in V_{n_j,f_j,\alpha_j}$, that $n \in Dom(f_j)$. We have shown that $Dom(f_i) \subset Dom(f_j)$. Since $x \in V_{n_i,f_i,\alpha_i} \cap V_{n_j,f_j\alpha_j}$, we have that $f_i \cup f_j \subset x$, and it follows, since $Dom(f_i) \subset Dom(f_j)$, that $f_i \subset f_j$. Now the inclusion $V_{n_j,f_j,\alpha_j} \subset V_{n_i,f_i,\alpha_i}$ follows by the observation made in the preceding paragraph of this proof. We have shown that $\mathcal{B}$ is of subinfinite rank. □

W.F. Lindgren and Nyikos [LN] have observed that every space with a Noetherian base of subinfinite rank is (hereditarily) metacompact. Hence the following result obtains.

Corollary *$L(\omega_1)^\omega$ is hereditarily metacompact.*

Without proof, we mention another hereditary covering property enjoyed by the space $L(\omega_1)^\omega$: it is not difficult to show that this space is also hereditarily screenable.

Using a result of K. Alster, we obtain the following generalization of Proposition 4: if X is a scattered P–space of weight ω_1, then X^ω has a Noetherian base of subinfinite rank. This follows from Proposition 4, because Alster showed in [A] that any Lindelöf space X with the stated properties can be embedded in $L(\omega_1)^\omega$, and an examination of Alster's proof shows that this result holds also for non–Lindelöf spaces.

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