Hereditary κ–separability and the hereditary κ–Lindelöf property in function spaces

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Abstract. This paper is concerned with the smallest linear subspace $L_p(X)$ of $C_pC_p(X)$ containing the Tychonoff space X. It is proved that $L_p(X)$ is hereditarily κ–Lindelöf (hereditarily κ–separable, resp.) if and only if X^ω is hereditarily κ–Lindelöf (hereditarily κ–separable, resp.). Moreover, it is shown that a certain cardinal function of $L_p(X)$ called the weak pseudonet weight of $L_p(X)$ equals the net weight of X.

Keywords: Tychonoff space, function space, pointwise convergence, hereditary κ–separability, hereditary κ–Lindelöf property, weak pseudonet weight, net weight

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Throughout this article, X denotes a Tychonoff space. The symbol $C_p(X)$ stands for the algebra of all continuous real–valued functions on X, with the topology of pointwise convergence. One easily sees that the formula

$$e(x)(f) = f(x), \tag{1}$$

where $x \in X$ and $f \in C_p(X)$, defines a homeomorphic embedding of X into $C_pC_p(X)$ (cf. [**1; Proposition 3.5** (β), p.16]). Denote by $L_p(X)$ the smallest linear subspace of $C_pC_p(X)$ which contains $e(X)$ (cf. [**1; p.17**]).

It is known that $C_pC_p(X)$ is hereditary κ–Lindelöf (hereditary κ–separable, resp.) if and only if X^ω has that property (cf. [**1;Corollary 3.28**]). Our purpose is to prove that $C_pC_p(X)$ can be replaced by $L_p(X)$ in the preceding statement. Moreover, we shall show that the weak pseudonet weight of $L_p(X)$ is equal to the net weight of X.

Before proceeding to the body of the paper, let us introduce some notation and establish some useful facts.

In what follows, κ denotes an infinite cardinal number, and for simplicity, all cardinal functions will be infinite.

The smallest (infinite) cardinal number κ such that X is hereditarily κ–Lindelöf (hereditarily κ–separable, resp.) is denoted by $hl(X)$ ($hd(X)$, resp.).

As usual, $nw(X)$ denotes the net weight of X and $w(X)$ denotes the weight of X.

For each positive integer $n \geq 2$, let

$$\Delta_n = \{(x_1, \dots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j (i, j = 1, \dots, n)\}.$$

Obviously,

$$\Delta_n = \cup\{\Delta^n_{ij} : 1 \leq i < j \leq n\}, \tag{2}$$

where $\Delta^n_{ij} = \{(x_1,\dots,x_n) \in X^n : x_i = x_j\}$ (cf. [**2;Definition 10.1**]). One can readily observe that for $n \geq 2$, we have

(3) $$\Delta^n_{ij} \text{ is homeomorphic to } X^{n-1}.$$

Let $\sum_n$ be the set of all permutations of the numbers $1,\dots,n$. For $(x_1,\dots,x_n) \in X^n$, we define $q_n(x_1,\dots,x_n) = \{(x_{\sigma(1)},\dots,x_{\sigma(n)}) : \sigma \in \sum_n\}$. Let

$$X^*_n = \{q_n(x_1,\dots,x_n) : (x_1,\dots,x_n) \in X^n\}$$

be considered with the quotient topology generated by q_n. Because

$$q_n^{-1}[q_n(U_1 \times \cdots \times U_n)] = \cup\{U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} : \sigma \in \sum\nolimits_n\},$$

where $U_i \subset X$ for $i = 1,\dots,n$, then

(4) $$q_n : X^n \to X^*_n \text{ is a continuous open surjection.}$$

For $n \geq 2$, we define

(5) $$Y_n = X^*_n \setminus q_n(\Delta_n),$$

(6) $$Z_n = \{e(x_1) + \cdots + e(x_n) \in L_p(X) : (x_1,\dots,x_n) \in X^n \setminus \Delta_n\},$$

(7) $$\phi_n(e(x_1) + \cdots + e(x_n)) = q_n(x_1,\dots,x_n), \text{ where } e(x_1) + \cdots + e(x_n) \in Z_n.$$

Then since $e(X)$ is an algebraic base for $L_p(X)$ (cf. [**1; Remark 3.14**]), ϕ_n is a well-defined one-to-one mapping from Z_n onto Y_n. We shall show the following

8.Lemma. *The mapping $\phi_n : Z_n \to Y_n$ is a homeomorphism for each $n \geq 2$.*

PROOF: To begin with, we shall show that the mapping ϕ_n is continuous. Let us take $z = e(x_1) + \cdots + e(x_n) \in Z_n$ and an open neighborhood V of $\phi_n(z)$ in Y_n. Because Y_n is an open subspace of X^*_n, then V is open in X^*_n. Since $(x_1,\dots,x_n) \notin \Delta_n$, it follows from the continuity of q_n that there exist open subsets U_i of X such that $x_i \in U_i$, where $U_i \cap U_j = \emptyset$ for $i \neq j (i,j = 1,\dots,n)$ and $q_n(U_1 \times \cdots \times U_n) \subset V$. Choose functions $f_i \in C_p(X)$ such that $f_i(x_i) = 1$ and $f_i(X \setminus U_i) = \{0\}$ for $i = 1,\dots,n$. From Proposition 3.10* given in [**1;p.18**], it follows that there exist continuous linear functionals $h_i : L_p(X) \to R$ such that $h_{i|e(X)} = f_i \circ e^{-1}$ for $i = 1,\dots,n$. Put

$$W = \cap\{h_i^{-1}(R \setminus \{0\}) : i = 1,\dots,n\}.$$

Because $h_i(z) = h_i[e(x_1)] + \cdots + h_i[e(x_n)] = f_i(x_1) + \cdots + f_i(x_n) = 1$ for $i = 1,\dots,n$, then $z \in W$. To complete the proof of continuity of ϕ_n it suffices to check that $\phi_n(W \cap Z_n) \subset q_n(U_1 \times \cdots \times U_n)$.

Let $z' \in W \cap Z_n$ and $z' = e(x'_1) + \cdots + e(x'_n)$, where $(x'_1,\dots,x'_n) \in X^n \setminus \Delta_n$. Then $f_i(x'_1) + \cdots + f_i(x'_n) = h_i(z') \neq 0$, so $f_i(x'_j) \neq 0$ for some $j = 1,\dots,n$. Hence, for each $i = 1,\dots,n$ there exists $j = 1,\dots,n$ such that $x'_j \in U_i$ so that $(x'_1,\dots,x'_n) \in \cup\{U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} : \sigma \in \sum_n\}$. Thus, $\phi_n(z') = q_n(x'_1,\dots,x'_n) \in q_n(U_1 \times \cdots \times U_n)$.

Now we shall show that ϕ_n is an open mapping. Let U be an open set in $L_p(X)$ and $y \in \phi_n(U \cap Z_n)$. Take a point $z = e(x_1) + \cdots + e(x_n) \in U \cap Z_n$ such that $y = \phi_n(z)$. We can choose open sets $U_i \subset X$ such that $x_i \in U_i$, where $U_i \cap U_j = \emptyset$ for $i \neq j (i, j = 1, \ldots, n)$ and $e(U_1) + \cdots + e(U_n) \subset U \cap Z_n$. Put $V = \phi_n(e(U_1) + \cdots + e(U_n))$. Because $V = q_n(U_1 \times \cdots \times U_n)$, then by (4), V is open in Y_n. Moreover, $y \in V \subset \phi_n(U \cap Z_n)$, so ϕ_n is an open mapping. ■

Now we are a position to prove the main theorems of this paper.

9.Theorem. *If X is a Tychonoff space, then $hl[L_p(X)] = hl(X^\omega)$.*

PROOF: Denote $\kappa = hl[L_p(X)]$. By virtue of [**6; Theorem 3**,p.177], to prove that $hl(X^\omega) \leq \kappa$, it suffices to show that $hl(X^n) \leq \kappa$ for each positive integer n.

Clearly, $hl(X) \leq \kappa$. Assume that $hl(X^{n-1}) \leq \kappa$ for some $n \geq 2$. Because $Z_n \subset L_p(X)$, then $hl(Z_n) \leq \kappa$; hence, Lemma 8 yields that $hl(Y_n) \leq \kappa$. Let U be an open subset of $X^n \setminus \Delta_n$. Then, by (4), the mapping $p_n = q_{n|U}$ is open, continuous and, moreover, all fibers of p_n have cardinalities less than or equal to $n!$. For $i = 1, 2 \ldots n!$, put $V_i = \{y \in p_n(U) : p_n^{-1}(y) = i\}$ and $U_i = p_n^{-1}(V_i)$. Applying [3; **Proposition 2.1.4**, p.95] and the theorem given in [3; **Problem 4.4D(b)**, p.358], we deduce that $p_{n|U_i}$ is a perfect mapping from U_i onto V_i, so U_i is κ-Lindelöf for $i = 1, \ldots, n!$ (cf. [**3; Theorem 3.8.9**, p.248]). As $U = \cup_{i=1}^{n!} U_i$, then U_i is κ-Lindelöf. Hence, $hl(X^n \setminus \Delta_n) \leq \kappa$. Now, using (2)–(3) and our assumption that $hl(X^{n-1}) \leq \kappa$, we deduce that $hl(\Delta_n) \leq \kappa$; thus $hl(X^n) \leq \kappa$.

As $L_p(X) \subset C_pC_p(X)$, the inequality $hl[L_p(X)] \leq hl(X^\omega)$ is an immediate consequence of [**6; Theorem 6**,p.178] ■

10.Theorem. *If X is a Tychonoff space, then $hd[L_p(X)] = hd(X^\omega)$.*

PROOF: Put $\kappa = hd[L_p(X)]$ and fix a positive integer $n \geq 2$. Similarly as in the proof of Theorem 9, we show that $hd(Y_n) \leq \kappa$. Let $M \subset X^n \setminus \Delta_n$ and let A_0 be a dense subset of $q_n(M) \subset Y_n$ such that $|A_0| \leq \kappa$; then by (4) we have that

$$\overline{q_n^{-1}(A_0)} \cap q_n^{-1}[q_n(M)] = q_n^{-1}(\overline{A_0}) \cap q_n^{-1}[q_n(M)] = q_n^{-1}[q_n(M)]$$

(cf. [**3; Exercise 1.4.C**,p.57]); so $q_n^{-1}(A_0)$ is dense subset of $q_n^{-1}[q_n(M)]$. Put $D_0 = q_n^{-1}(A_0)$ and $M_1 = \overline{D_0 \cap M} \cap M$. Let us choose a set $A_1 \subset q_n(M \setminus M_1)$ such that A_1 is dense in $q_n(M \setminus M_1)$ and such that $|A_1| \leq \kappa$. Put $D_1 = D_0 \cup q_n^{-1}(A_1)$ and $M_2 = \overline{D_1 \cap M} \cap M$. We can inductively define sets A_i, D_i, M_i such that $|A_i| \leq \kappa, A_i$ is a dense subset of $q_n(M \setminus M_i), D_i = D_{i-1} \cup q_n^{-1}(A_i)$ and $M_i = \overline{D_{i-1} \cap M} \cap M$. The set $D = D_{n!-1} \cap M$ is of cardinality $\leq \kappa$. To show that D is dense in M, we need the following notation:

For $x = (x_1, \ldots, x_n) \in X^n, U = U_1 \times \cdots \times U_n \subset X^n$ and $\sigma \in \sum_n$, let $x^\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ and $U^\sigma = U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}$. Denote by $\mathcal{B}(x)$ the family of all open sets of the form $U_1 \times \cdots \times U_n \subset X^n$ which contain x. Finally, let $id \in \sum_n$ be the identity permutation.

Suppose that the set D is not dense in M. Then there exists $x \in M$ and $U \in \mathcal{B}(x)$ such that

(d_0) $\qquad U \cap D = \emptyset.$

Because $x \in M \setminus M_{n!-1}$ and the set $q_n^{-1}(A_{n!-1})$ is dense in $q_n^{-1}[q_n(M \setminus M_{n!-1}]$, then there exist ${}_1y, \sigma_1 \in \sum_n$ and ${}_1V \in \mathcal{B}({}_1y)$ such that

(a_1) ${}_1V \subset U$,
(b_1) ${}_1y \in q_n^{-1}(A_{n!-1})$,
(c_1) ${}_1y^{\sigma_1} \in M \setminus M_{n!-1} \subset M \setminus M_{n!-2}$,
(d_1) ${}_1V^{\sigma_1} \cap D_{n!-2} \cap M = \emptyset$.

It follows from (a_1), (b_1) and (d_0) that $\sigma_1 \neq id$. As the set $q_n^{-1}(A_{n!-2})$ is dense in $q_n^{-1}[q_n(M \setminus M_{n!-2})]$, then we obtain from (c_1) that there are ${}_2y, \sigma_2 \in \sum_n$ and ${}_2V \in \mathcal{B}({}_2y)$ such that

(a_2) ${}_2V \subset {}_1V$,
(b_2) ${}_2y^{\sigma_1} \in q_n^{-1}(A_{n!-2})$,
(c_2) ${}_2y^{\sigma_2} \in M \setminus M_{n!-2} \subset M \setminus M_{n!-3}$,
(d_2) ${}_2V^{\sigma_2} \cap D_{n!-3} \cap M = \emptyset$.

It follows from $(d_0), (a_1), (d_1), (a_2), (b_2)$ that $\sigma_2 \in \sum_n \setminus \{id, \sigma_1\}$. Arguing similarly, in the i–th step, we obtain ${}_iy, \sigma_i \in \sum_n$ and ${}_iV \in \mathcal{B}({}_iy)$ such that

(a_i) ${}_iV \subset {}_{i-1}V$,
(b_i) ${}_iy^{\sigma_{i-1}} \in q_n^{-1}(A_{n!-i})$,
(c_i) ${}_iy^{\sigma_i} \in M \setminus M_{n!-i} \subset M \setminus M_{n!-i-1}$,
(d_i) ${}_iV^{\sigma_i} \cap D_{n!-i-1} \cap M = \emptyset$.

From all the conditions (d_0), (a_j), (d_j) $(1 \leq j \leq i)$ and from $(a_i), (b_i)$ it follows that $\sigma_i \in \sum_n \setminus \{id, \sigma_1, \ldots, \sigma_{i-1}\}$. As ${}_{n!-1}y^{\sigma_{n!-1}} \in M$ and the set $q_n^{-1}(A_0)$ is dense in $q_n^{-1}[q_n(M)]$, then there is ${}_{n!}y$ such that ${}_{n!}y^{\sigma_{n!-1}} \in_{n!-1} V^{\sigma_{n!-1}} \cap q_n^{-1}(A_0)$. For some $\sigma_{n!} \in \sum_n$, we have that ${}_{n!}y^{\sigma_{n!}} \in M$. From the conditions $(a_i), (d_i)$ it follows that $\sigma_{n!} \neq \sigma_i$ for $1 \leq i \leq n!-1$. Because $\sigma_i \neq \sigma_j$ for $i \neq j (1 \leq i, j \leq n!-1)$, then $\sigma_{n!} = id$ which contradicts (d_0). Hence, the set D is dense in M and $hd(X^n \setminus \Delta_n) \leq \kappa$. Arguing as in the proof of Theorem 9, we can inductively show that $hd(X^n) \leq \kappa$ for each positive integer n. By applying Theorem 3* of [**6;p.177**], we obtain that $hd(X^\omega) \leq \kappa$.

The inequality $hd[L_p(X)] \leq hd(X^\omega)$ follows from [**6; Theorem 6**, p.179] ∎

The preceding two theorems imply

11.Corollary. *If X is a Tychonoff space, then $hl[C_pC_p(X)] = hl(X^\omega)$ and $hd[C_pC_p(X)] = hd(X^\omega)$.*

The above corollary is a slight modification of Corollary 3.28 given in [**1; p.38**], but with a different proof.

12.Corollary. *If X is a Tychonoff space, then $hl[L_p(X)] = hl[C_pC_p(X)]$ and $hd[L_p(X)] = hd[C_pC_p(X)]$.*

13.Corollary. *If X is a locally compact Hausdorff space, then $hl[L_p(X)] = w(X)$.*

PROOF: If ωX is the one–point compactification on X, then $hl(X \times X) = hl(\omega X \times \omega X) \geq \Delta(\omega X) = w(\omega X) = w(X)$ where $\Delta(\omega X)$ denotes the diagonal degree of ωX (cf. [**4; p.16 and Corollary 7.6**, p.27]). This implies that $hl(X \times X) = w(X)$, so that $hl(X^\omega) = w(X)$. Theorem 9 completes the proof. ∎

14. Corollary. *If X is a locally compact Hausdorff space, then $hl[C_pC_p(X)] = w(X)$.*

15. Corollary. *If X is a locally compact Hausdorff space, then $hd[C_p(X)] = w(X)$.*

PROOF: Inasmuch as $nw[C_p(X)] = nw(X) = w(X)$ (cf. [1; **Theorem 1.1**, p.24] and [3; **Theorem 3.3.5**, p.197]), then $hd[C_p(X)] \leq w(X)$.

The inequality $w(X) \leq hd[C_p(X)]$ follows from Corollary 3.26 given in [1; **p.37**] and the fact that $hl(X^2) = w(X)$. ■

16. Definition. A family ε of subsets of a space X will be called a *weak κ-pseudonet* in X if and only if $|\varepsilon| \leq \kappa$. and for each open set $U \subset X$, there exists a set $A \subset U$ such that $|A| \leq \kappa$ and for each $x \in U \setminus A$, there exists $E \in \varepsilon$ such that $x \in E$ and $|E \setminus U| \leq \kappa$. The cardinal number.

$$wpn(X) = \min\{\kappa \geq \omega : \text{ there exists a weak}\kappa\text{-pseudonet in } X\}$$

will be called the weak pseudonet weight of X.

17. Theorem. *If X is a Tychonoff space, then $wpn[L_p(X)] = nw(X)$.*

PROOF: Because $nw(X) = nw[C_pC_p(X)]$ (cf. [**Theorem 1.1**, p.24], it follows that $wpn[L_p(X)] \leq nw(X)$.

Denote $\kappa = wpn[L_p(X)]$ and consider the spaces Y_2, Z_2 defined in (5)-(6). By Lemma 8, Y_2 is homeomorphic to Z_2, so $wpn(Y_2) = wpn(Z_2) \leq \kappa$. Let ε be a weak κ-pseudonet in Y_2. It is easy to observe that $|q_2^{-1}(B)| \leq \kappa$ for each $B \subset Y_2$ such that $|B| \leq \kappa$. This-together with (4)-implies that the family $\{q_2^{-1}(E) : E \in \varepsilon\}$ is a weak κ-pseudonet in $X^2 \setminus \Delta_2$; hence $wpn(X^2 \setminus \Delta_2) \leq \kappa$. As Δ_2 is homeomorphic to X, we have $wpn(\Delta_2) = wpn(X) \leq wpn[L_p(X)] = \kappa$ so that $wpn(X^2) \leq \kappa$. As $nw(X) = wpn(X^2)$ (cf. [5; **Corollary 1.12**]), we have $nw(X) \leq \kappa$. ■

18. Corollary. *If X is a Tychonoff space, then $wpn[C_pC_p(X)] = nw(X)$.*

We do not know if Corollary 18 can be strengthened to say that $wpn[C_p(X)] = nw(X)$.

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