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On paracompact locales and metric locales

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Abstract. In the paper, some new results on paracompact locales and metric locales are obtained. In particular, we prove that the existence of σ–locally finite refinement of all covers is equivalent to the paracompactness in a regular locale which answers an open question posed by A.Pultr.

Keywords: paracompact locale, metric, Boolean locale

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Paracompact locales and metric locales were first investigated by J.Isbell ([**I**]). A full discussion of paracompact locales and of metric locales can be found in Dowker–Strauss ([**DS**]) and in Pultr ([$\mathbf{P}_1$]) respectively. Some questions remain open (see [$\mathbf{P}_1$] or [$\mathbf{P}_2$]). In this paper, we shall provide some further results and answer some questions.

Following Dowker–Strauss [**DS**], if L is a locale, we say that a family of elements of L is a cover if its join is the top element; the family is locally finite (discrete) if there is a cover each element of which meets at most finitely many elements of the given family. A family $\{c_r : r \in \Gamma\}$ is said to refine the family $\{a_\alpha : \alpha \in \Lambda\}$ if for each r there is some α such that $c_r \leq a_\alpha$. Now a paracompact locale is defined as classical topology (i.e., for each cover there is a locally finite cover which refines it). For $\alpha \in L$, let $\neg a$ denote the pseudocomplement of a, i.e., $\neg a = \bigvee\{b \in L : b \wedge a = 0\}$.

In [$\mathbf{P}_1$], A.Pultr posed the following question:
Is the existence of σ–locally finite refinement of all covers equivalent to the paracompactness in a (regular) locale; in particular, is a metrizable locale paracompact? ([$\mathbf{P}_1$], Remark 3.2) The following theorem shall answer it in the affirmative. First we need a lemma.

Lemma 1. *Let $\{x_i : i \in J\}$ be locally finite and let $x_i \leq y_i$ for all $i \in J$. Then we have $\bigvee\{x_i : i \in J\} \leq \bigvee\{y_i : i \in J\}$, where $b \leq a$ denotes $\neg b \bigvee a = 1$.*

PROOF: Let C be a cover such that for each $c \in C, c \wedge x_i \neq 0$ only for $i \in K(c)$, where $K(c)$ is a finite subset of J. Take a $c \in C$, put $K = K(c)$, $I = J \setminus K$. Then we have

$$c \wedge \bigvee\{x_i : i \in I\} = 0 \text{ and hence } c \leq \neg \bigvee\{x_i : i \in I\}.$$

Since K is finite, so we have $\bigvee\{x_i : i \in K\} \leq \bigvee\{y_i : i \in K\}$. Thus

$$c \wedge (\neg \bigvee\{x_i : i \in J\} \vee \bigvee\{y_i : i \in J\}) \geq$$
$$\geq c \wedge ((\neg \bigvee\{x_i : i \in I\} \wedge \neg \bigvee\{x_i : i \in K\}) \vee \bigvee y_i) =$$
$$= c \wedge (\neg \bigvee\{x_i : i \in K\} \vee \bigvee y_i) = c \wedge 1 = c.$$

That is, $c \leq \neg \bigvee x_i \vee \bigvee y_i$ and since C was a cover, so we have $\bigvee x_i \leq \bigvee y_i$. ∎

Recall that a locale L is said to be regular if for each $a \in L$, we have $a = \bigvee\{b \in L : b \leq a\}$.

Theorem 1. *Let L be a regular locale such that for each cover A there is a σ-locally finite cover which refines A. Then L is paracompact.*

PROOF: It suffices to show that each σ–locally finite cover of L has a locally finite refinement. Now let $A = \bigcup_n A_n$ be a σ–locally finite cover of L, where each A_n is locally finite and $A_n \subseteq A_{n+1}, n = 1, 2 \ldots,$

For each $a \in A$, let $n(a)$ be the smallest number with $a \in A_{n(a)}$. Put $A'_m = \{a \in A : n(a) = m\}$. Then

$$\bigcup_{m=1}^{\infty} A'_m = A \text{ and } \bigcup_{m=1}^{n} A'_m = A_n \quad A'_m \cap A'_n = \emptyset, m \neq n, m, n = 1, 2, \ldots;$$

hence each A'_m is also locally finite.

For each m, write $A'_m = \{a_i \in A : i \in J_m\}$ with $J_m \cap J_n = \emptyset$ and consider a well ordering $\prec$ on $J = \bigcup_{m=1}^{\infty} J_m$ such that for each $i \in J_m, j \in J_n$, we have $i \prec j$ whenever $m < n$.

Now we consider another family

$$D = \{d \in L : (\exists a \in A)(d \leq a)\}.$$

It easily follows from the regularity of L that D is a cover of L too. So there is a σ–locally finite cover $B = \bigcup_{n=1}^{\infty} B_n$ which refines D, where each B_n is locally finite. Hence for each $b \in B$, there is an $i(b) \in J = \bigcup J_n$, say $i(b) \in J_m$, such that $b \leq a_{i(b)}$.

Now we set

$$e_{n,i} = \bigvee\{b \in \bigcup_{j=1}^{n} B_j : i(b) = i\} \text{ and } E_{n,m} = \{e_{n,i} : i \in \bigcup_{k=1}^{m} J_k\}.$$

Then we have $e_{n,i} \leq a_i$ for each $i \in J$ and each n by Lemma 1, and $E_{n,m} \subseteq E_{n,m+1}$.

It follows readily from the local finiteness of A_m that $E_{n,m}$ is locally finite for each pair m and n, and that $\bigcup_{m=1} \bigcup_{n \leq m} E_{n,m}$ is a cover of L.

Now, for each $i \in J_n \subseteq J$, put

$$c_{i_0} = a_{i_0} \wedge \neg w_{i_0},$$

where $w_{i_0} = \bigvee\{e_{m,i} : i \prec i_0, m = 1, 2, \ldots n\}$.

We shall check that the family $C = \{c_i : i \in J\}$ is as required.

(i) C is locally finite: In fact, for each m, $\bigcup_{k=1}^{m} E_{k,m}$ and A_m are locally finite. So there is a cover D_m such that for each $d \in D_m$, d meets at most finitely many elements of $\bigcup_{k=1}^{m} E_{K,m}$ and of A_m.

Write $D_m^* = \{d \wedge e : d \in D_m, e \in \bigcup_{k=1}^{m} E_{k,m}\}$, then we have $\bigvee D_m^* = \bigvee \bigcup_{k=1}^{m} E_{k,m}$; hence $D^* = \bigcup_{m=1}^{\infty} D_m^*$ is a cover of L.

For each $z \in D^*$, there is an m with $z \in D_m^*$, moreover, there is a $k \leq m$ and an $e_{k,i} \in E_{k,m}$ with $z \leq e_{k,i}$. So for each $n > m$, $i \in J_n$, we have $z \leq w_i$; hence $z \wedge \neg w_i = 0$. On the other hand, z meets at most finitely many elements of A_n; hence z meets at most finitely many elements of C. Thus we have shown (i).

(ii) C is a cover of L. For each $z \in D_m^*$, write $I = \{i \in \bigcup_{k=1}^{m} J_k : (\exists n \leq m)(z \wedge e_{n,i} \neq 0)\}$. Then I is finite since z meets only finitely many elements of $\bigcup_{n=1}^{m} E_{n,m}$. Thus, we have $z \wedge (\bigvee_{n \leq m} e_{n,i}) = 0$ for each $i \in \bigcup_{k=1}^{m} J_k \setminus I$; in particular, $z \wedge w_{i_0} = 0$, where $i_0 = \min\{i \in I\}$; hence $z \leq \neg w_{i_0}$. Now we can show $z \leq \{c_i : i \in I\}$: in fact,

$$\begin{aligned}
\bigvee\{c_i : i \in I\} &= \bigvee\{a_i \wedge \neg w_i : i \in I\} \\
&= \bigvee\{a_{1,i} \wedge a_{2,i} : i \in I\}, \text{ where } a_{1,i} = a_i, a_{2,i} = \neg w_i \\
&= \bigwedge_{f \in \Pi D_i} \bigvee\{a_{f(i),i} : i \in I\} \text{ where } D_i = D = \{1,2\} \text{ for each } i \in I.
\end{aligned}$$

It suffices to show $\bigvee_i a_{f(i),i} \geq z$ for each $f \in \prod_{i \in I} D_i$. Write $\bar{i} = \min\{i \in I : f(i) = 2\}$. If $\bar{i} = i_1 0$, then $\neg w_{i_0} = a_{f(i_0),i_0} \leq \bigvee a_{f(i),i}$, hence we have $z \leq \bigvee a_{f(i),i}$. If $\bar{i} = i_1 \succ i_0$, say $i_1 \in J_{n'}, n' \leq m$, then $\bigvee\{a_i : i \prec i_1\} = \bigvee\{a_{f(i),i} : i \prec i_1\} \leq \bigvee a_{f(i),i}$. On the other hand,

$$\begin{aligned}
z \wedge \neg w_{i_1} &= z \wedge \neg w_{i_0} \wedge \neg(\bigvee\{e_{k.i} : i \in I, i \prec i_1, k \leq n'\}) = \\
&= z \wedge \neg(\bigvee\{e_{k,i} : i \in I, i \prec i_1, k \leq n'\}),
\end{aligned}$$

hence, we have

$$\begin{aligned}
& z \wedge (\bigvee a_{f(i),i}) \geq (z \wedge \neg w_{i_1}) \vee (z \wedge \bigvee\{a_i : i \in I, i \prec i_1\}) = \\
&= (z \wedge \neg(\bigvee\{e_{k.i} : i \in I, i \prec i_1, k \leq n'\})) \vee (z \wedge \bigvee\{a_i : i \in I, i \prec i_1\}) \\
&= z \wedge (\neg(\bigvee\{e_{k.i} : i \in I, i \prec i_1, k \leq n'\}) \vee (\bigvee\{a_i : i \in I, i \prec i_1\})) \\
&= z \wedge 1 = z.
\end{aligned}$$

since $\bigvee\{e_{k,i} : k \leq n'\} \leq a_i$ for each i that is $z \leq \bigvee a_{f(i),i}$, hence $z \leq \bigvee\{c_i : i \in I\} \leq \bigvee C$. Thus we have shown that C is a cover of L. ■

Corollary 1. *Regular Lindelöf locales are paracompact.*

Remark. This corollary also follows easily from the work of Madden and Vermeer who showed that "regular Lindelöf" is equivalent to "realcompact".

Since each metrizable locale has a σ–discrete base (see [$\mathbf{P}_1$]), we have answered the problem from [$\mathbf{P}_2$] (p.459) as

Corollary 2. *Each metrizable locale is paracompact.*

The next theorem is a counterpart of the following classical result:

"Every regular paracompact space is collectionwise normal"

which also generalizes a result of Pultr in [$\mathbf{P}_1$].

A locale L is said to be collectionwise normal if for each co-discrete system $\{x_i : i \in J\}$ there is a discrete system $\{y_i : i \in J\}$ such that $x_i \vee y_i = 1$ for each $i \in J$, where $B \subseteq L$ is said to be co-discrete (co-locally finite), if there is a cover D such that for each $d \in D, d \not\leq x_i$ for at most one (finitely many) element(s) of B.

Theorem 2. *Each regular paracompact locale is collectionwise normal.*

PROOF: Let A be a regular paracompact locale and let $B = \{b_r : r \in J\}$ be a co-discrete system. Then there is a cover C such that for each $c \in C$, $c \leq b_r$ for all but at most one element $r \in J$. By regularity, we see that

$$D = \{d \in A : (\exists c \in C)(d \leq c)\}$$

is a cover of A. By paracompactness, D has a locally finite refinement Z which covers A. For each $z \in Z$ we can assign a $c(z) \in C$ such that $z \leq c(z)$. Write

$$z_c = \bigvee\{z \in Z : z \leq c(z) = c\}.$$

By lemma 1, we see that $z_c \leq c$ and that $Z_0 = \{z_c : c \in C\}$ is also locally finite and a cover of A.

For each $r \in J$, we write

$$z_r = \bigvee\{z_c \in Z_0 : c \leq b_r\}.$$

Again by Lemma 1, we have $z_r \leq b_r$. Now it remains to show that

$$\widetilde{B} = \{z_r : r \in J\}$$

is co-discrete. In fact, for each $z_c \in Z_0$, where $c \in C$, if $z_c \not\leq z_{r_0} = \bigvee\{z_{c'} \in Z_0 : c' \leq b_{r_0}\}$; then $c \not\leq b_{r_0}$. Thus $c \leq b_r$ for all $r \neq r_0$; hence $z_c \leq z_r = \bigvee\{z_{c'} \in Z_0 : c' \leq b_r\}$ for all $r \neq r_0$.

Furthermore, $\neg\widetilde{B} = \{\neg z_r : r \in J\}$ is discrete and $\neg z_r \vee b_r = 1$. ∎

Corollary ([$\mathbf{P}_1$], Theorem 2.5). *Metric locales are collectionwise normal.*

In [$\mathbf{P}_1$], Pultr established the following metrizability criteria:

(i) L is metrizable;
(ii) $L = L_{\mathcal{A}}$ for a countable $\mathcal{A}$;
(iii) L is regular and has a σ-discrete base;

By modifying his proof, we can add into the list above a more general statement:

(*) L is regular and has a σ-locally finite base.

We omit the details of the proof.

Now we turn our attention to Boolean locales. By Stone's representation theorem, every Boolean locale can be regarded as a regular–open–set lattice $RO(X)$ of a regular space X(up to isomorphism). So it suffices to disccuss those Boolean locales of the form $RO(X)$ for a regular space X.

The following lemmas is useful but easy.

Lemma 2. *For each (regular) space X, let D be a dense subset of X. Then $RO(X)$ is isomorphic to $RO(D)$.*

Lemma 3. *Let X be a T_3 space, then each prime element in $RO(X)$ is also prime in $O(X)$; hence it is of the form $X \setminus \{x\}$, where x is an isolated point in X.*

Theorem 3. *Let X be a T_3 space; then $RO(X)$ is spatial iff X has a dense subset of isolated points in X.*

PROOF: $\Leftarrow$. By lemma 3, it is clear.
$\Rightarrow$. Let D be a subset of isolated points in X. By Lemma 2, we have

$$\begin{aligned}&\cap\{p \subseteq X : p \text{ is prime in } RO(X)\}\\ &= \cap\{X \setminus \{x\} : x \in D\} = X \setminus D.\end{aligned}$$

So

$$\bigwedge\{p \in RO(X) : p \text{ is prime}\} = X \setminus \overline{D};$$

hence, by the assumption that $RO(X)$ is spatial, $X \setminus \overline{D}$ must be empty. ∎

Next, we shall characterize the metrizability of Boolean locales. Recall that a family $\mathcal{B}$ of non–empty open subsets of a space X is called π–base if for each non–empty open subset U of X there is a $V \in \mathcal{B}$ such that $V \subseteq U$.

We say a family $\mathcal{B}$ of subsets of X is almost locally finite (discrete) if $\mathcal{B}$ is locally finite (discrete) with respect to an open dense subset D of X, i.e., for each $d \in D$ there is a neighbourhood U_d which meets at most finitely many (one) members of $\mathcal{B}$. X is called π–metrizable if X is regular and has a σ–almost locally finite π–base.

Theorem 4. *Let X be a T_3 space; then $RO(X)$ is metrizable iff X is π–metrizable.*

PROOF: It suffices to note that a family of regular open subsets is a cover of $RO(X)$ iff the union is dense in X (and our metrizability criterion (*)). ∎

Remark. In particular, for each regular space X which has a countable π–base, $RO(X)$ is metrizable, for example, the real line, the Sorgenfrey line. But not all $RO(X)$ are metrizable.

Lemma 4. *For a metrizable locale L, the following conditions are equivalent.*

(i) *L is c.c.c. (i.e., each disjoint family is countable).*
(ii) *L has a countable base.*

PROOF:
(i)$\Rightarrow$(ii) Each σ–discrete family is countable.
(ii)$\Rightarrow$(i) Clear. ∎

Example 1. Let X be the space D^k, where $D = \{0,1\}$ with discrete topology and $k = 2^{\omega_0}$. It is well known that X is c.c.c. and $\pi w(X) = k(\pi w(X)) = \min\{$the cardinality of $\mathcal{B}$: $\mathcal{B}$ is a π–base for $X\}$. Thus $RO(X)$ is also c.c.c. If $RO(X)$ is metrizable, then by lemma 4, $RO(X)$ has a countable base; equivalently, X has a countable π-base which is impossible.

(As usual, a space X is said to be c.c.c.–to satisfy the countable chain condition–if each family of disjoint open sets of X is countable, equivalently, for each family $\mathcal{U}$ of open sets there is a countable subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\overline{\cup\mathcal{U}_0} = \overline{\cup\mathcal{U}}$.)

Theorem 5. *For each Boolean locale L, L is c.c.c. iff $RO(X)$ is Lindelöf.*

PROOF: Let L be of the form of $RO(X)$ for a regular space X.

Let $B = \{b_r \in RO(X) : r \in J\}$ be a cover of $RO(X)$; then $\cup\{b_r \subseteq X : r \in J\}$ is open dense in X. By c.c.c., there is a countable subfamily B_0 of B whose union is dense in X, i.e., B_0 is a cover of $RO(X)$.

$\Leftarrow$. For each family of disjoint regular open subsets of X, by the Zorn lemma, we can find a maximal family of disjoint regular open subsets which contains it; moreover, by the regularity its union is dense in X; hence this family is a cover of $RO(X)$. Thus it must be countable by Lindelöfness. ∎

The following result may be known to those who work with complete Boolean algebras.

Proposition 6. *Every Boolean locale is paracompact.*

PROOF: It suffices to show that for each regular space X the $RO(X)$ is paracompact. In fact, we shall do a little more.

Let $B = \{b_r \in RO(X) : r \in J\}$ be a cover of $RO(X)$. We consider the poset $\mathcal{S} = \{D \subseteq RO(X) : D$ refines B and D is s disjoint$\}$. Again by Zorn lemma, we have a maximal element $\mathcal{V}$ in $\mathcal{S}$ whose union is also dense in X. In fact, if $X \setminus \overline{\cup\mathcal{V}} \neq \emptyset$, we can find an element U in B and a $V \in RO(X)$ such that $\emptyset \neq V \subseteq U \cap (X \setminus \overline{\cup\mathcal{V}}$ since $\cup B$ is dense in X which contradicts with the maximality of $\mathcal{V}$. Thus we have shown that every cover of a Boolean locale has a discrete refinement. ∎

Remark. This fact is closely related to the fact that the Axiom of Choice holds in the topos of sheaves on a complete Boolean algebra ([J]. Theorem 5.39).

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