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# Representation of semigroups by products of simple graphs

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*Abstract.* We construct representations of every countable commutative semigroup by products of simple graphs which are formed by means of simple operations (sum and product) from graphs with very simple structure (namely from paths and also from circuits).

*Keywords:* Simple graph, direct product, representation of semigroups

*Classification*: Primary 05C99, Secondary 20M30

## 1. Notations and definitions.

We denote by $\omega$ the first infinite cardinal, i.e. the set of all non–negative integers.

Let $n, m \in \omega$. By $n \bmod m$ we shall denote the unique integer $r$ such that $n = m \cdot q + r$ where $q \in \omega$ and $0 \leq r < m$. Further we put $[n,m] = \{i \in \omega; n \leq i \leq m\}$.

By $\omega^M$ we denote the semigroup of all functions on $M$ with values in $\omega$ endowed with the pointwise addition, and by $\exp \omega^M$ the semigroup of all subsets of $\omega^M$ with the addition defined by $A + B = \{f + g; f \in A \text{ and } g \in B\}$.

If $A$ is an arbitrary set then $|A|$ denotes the cardinality of $A$.

In this paper, all graphs will be simple, i.e. undirected and without loops and multiple edges. The set of vertices of a graph $G$ is denoted by $V(G)$ while $E(G)$ is the set of edges. A *sequence* $S$ in a graph $G$ is any finite series $(v_0, v_1, \ldots, v_n)$ of vertices in $G$ such that consecutive vertices are connected by an edge. The vertices $v_0$ and $v_n$ are called the *endpoints* of $S$ and the number $n$ is the *length* of $S$. A sequence $(v_0, v_1, \ldots, v_n)$ is called a *path* if $v_i \neq v_j$ whenever $i \neq j$. The *distance* between vertices $x$ and $y$ in a graph $G$ is denoted by $d^G(x,y)$ or simply by $d(x,y)$.

Let $G = (V,E)$ and $G' = (V',E')$ be graphs. A mapping $f : V \to V'$ is said to be *compatible* if the following condition holds: $\{x,y\} \in E \Rightarrow \{f(x), f(y)\} \in E'$.

Now let us recall some notations from category theory. Let $\mathcal{K}$ be a category. We write $A \cong B$ if $A$ and $B$ are *isomorphic* objects in $\mathcal{K}$. The *product* of a family of objects $\{A_i; i \in I\}$ in $\mathcal{K}$ is denoted by $\prod A_i$ and $\sum A_i$ denotes the coproduct of this family.

Recall that simple graphs and compatible mappings constitute a category with products and coproducts. The products are usually called *direct products*. The coproducts are *disjoint sums*.

## 2. Introduction.

Let $\mathcal{K}$ be a category with finite products and let $(S,+)$ be a commutative semigroup. A collection $\{r(s); s \in S\}$ of objects of $\mathcal{K}$ is called a *productive representation of* $(S,+)$ *in* $\mathcal{K}$ if

(1) $r(s+s')$ is isomorphic to $r(s) \times r(s')$ for all $s, s' \in S$ and
(2) $r(s)$ is not isomorphic to $r(s')$ whenever $s \neq s'$.

Which semigroups can be represented in which categories? In answering this question the following theorem proved to be very useful.

**Theorem 1 (Trnková, [6]).** *For every commutative semigroup $S$ there is a homomorphism $h : S \to \exp \omega^{\aleph_0 \cdot |S|}$ such that*

*(1) $h(s) \cap h(s') = \emptyset$ for $s \neq s'$ (hence $h$ is one–to–one) and*
*(2) every $f \in r(s)$ is distinct from the constant zero and $|h(s)| = \aleph_0 \cdot |S|$ for every $s \in S$.*

A lot of theorems on productive representations of commutative semigroups in a category has been proved on the basis of Theorem 1. For example, in [1],[3],[5], [6] and [7], Theorem 1 is used to prove a number of results on representations of commutative semigroups by products of graphs. In particular, the following theorem has been proved.

**Theorem 2.** *Every commutative semigroup $S$ has a productive representations in the category of simple graphs and compatible mappings.*

For constructing a representation of a commutative semigroup by products of graphs, Theorem 1 is applied as follows:

Let $S$ be a commutative semigroup and let $h : S \to \exp \omega^{\aleph_0 \cdot |S|}$ be a homomorphism, fulfilling the conditions (1) and (2) of Theorem 1. Put $\varkappa = \aleph_0 \cdot |S|$. Now, let a set $\{G_\lambda; \lambda < \varkappa\}$ of graphs be given. For every $f \in \omega^\varkappa \setminus \{\mathbf{O}\}$, where $\mathbf{O}$ denotes the constant zero, put

$$G(f) = \prod \{G_\lambda^{f(\lambda)}; \lambda < \varkappa \text{ and } f(\lambda) \neq 0\}$$

(Further we shall write $\prod\limits_{\lambda<\varkappa} G_\lambda^{f(\lambda)}$ instead of $\prod \{G_\lambda^{f(\lambda)}; \lambda < \varkappa \text{ and } f(\lambda) \neq 0\}$.) Then clearly $G(f) \times G(g) \cong G(f+g)$. For $s \in S$ define $r(s)$ as a coproduct (disjoint sum) of $\varkappa$ copies of $\sum\limits_{f \in h(s)} G(f)$, i.e.

$$r(s) = \sum_{\lambda<\varkappa} \Big( \sum_{f \in h(s)} G(f) \Big)_\lambda .$$

Since $r(s)$ contains each of its components in $\varkappa$ copies, $r(s+s') \cong r(s) \times r(s')$.

The non–isomorphism of graphs $r(s)$ and $r(s')$ for $s \neq s'$ is forced by a suitable choice of the starting set $\{G_\lambda; \lambda < \varkappa\}$. The choice is always made such that the following condition (C) is fulfilled.

(C) If $G(f)$ is a summand of $r(s)$, then necessarily $f \in h(s)$.

Since $h(s) \neq h(s')$ for $s \neq s'$, the above condition implies that also $r(s) \not\cong r(s')$ whenever $s \neq s'$.

Condition (C) is obviously fulfilled whenever the following condition (R) holds.

**Condition (R).** *Every $f \in \omega^{\varkappa} \setminus \{O\}$ can be fully recognized from one (distinguished) component of $G(f)$.*

The aim of many papers dealing with representations of commutative semigroups by products of graphs is to construct representations by graphs having some special additional properties. For example, in [5], Trnková proved the following strengthening of Theorem 2:

Given a graph $G$, every commutative semigroup $S$ has a productive representation $\{r(s); s \in S\}$ in the category of simple graphs and compatible mappings such that every $r(s)$ contains $G$ as a full subgraph.

In order to prove this theorem, Trnková constructed an appropriate set $\{G_\lambda; \lambda < \varkappa\}$ of graphs fulfilling the condition (R) such that all graphs $G_\lambda$ contain $G$. To achieve this, the structure of the graphs $G_\lambda$ and consequently the structure of $r(s)$ had to be considerably complicated. On the contrary, in [1], Adámek and Koubek tried to give representations of commutative semigroups by products of graphs as simple as possible. They proved, for example, that every *countable* commutative semigroup can be represented by products of bipartite graphs of diameter 3. (By a diameter of a graph $G$ we understand the least number $n$ such that any two vertices in a component of $G$ can be connected by a path of length at most $n$). On the other hand, they showed that no non–trivial group can be represented by products of bipartite graphs of diameter 2. (Bipartite graphs of diameter 2 are exactly sums of complete bipartite graphs.)

Similarly, in [3], Koubek, Nešetřil and Rödl showed that no non–trivial group can be represented by products of simple graphs of diameter 1. (Graphs of diameter 1 are exactly sums of complete graphs.) On the other hand, they showed that every set $\{G_\lambda; \lambda < \varkappa\}$ of complete graphs with at least three vertices fulfills condition (R). So, they gave another representation $\{r(s); s \in S\}$ of every commutative semigroup by products of graphs. It is true, that the structure of graphs $r(s)$ they obtained is complicated, but these $r(s)$ are formed by means of simple operations (sums and products) from graphs with simple structure (namely from complete graphs $G_\lambda$).

The above result of Koubek, Nešetřil and Rödl motivates the aim of the presented paper. Given a set $\varkappa = \{G_\lambda; \lambda < \varkappa\}$ of graphs, is it true that $\varkappa$ fulfills condition (R)? We show that it is true for the set of all paths of length greater than 1, for the set of all odd circuits and for the set of all even circuits. On the other hand, it is not true for the set of all circuits.

## 3. Results.

Let us begin with a definition.

**Definition 1.** Let $\sigma$ be an infinite cardinal. We say that a set $\{G_\lambda; \lambda < \varkappa\}$ of graphs is *$\sigma$–productively independent* if for every two functions $f, g : \varkappa \to \sigma$ the following holds:

$$\prod_{\lambda<\varkappa} G_\lambda^{f(\lambda)} \cong \prod_{\lambda<\varkappa} G_\lambda^{g(\lambda)} \Rightarrow f = g.$$

(We write $\prod_{\lambda<\varkappa} G_\lambda^{f(\lambda)}$ instead of $\prod\{G_\lambda^{f(\lambda)}; \lambda < \varkappa \text{ and } f(\lambda) \neq 0\}$.)

We say that a proper class $\mathcal{H}$ of graphs is ***productively independent*** if every subset of $\mathcal{H}$ is $\sigma$–productively independent for every cardinal $\sigma$.

**Remark.** Let $\mathcal{H} = \{G_\lambda; \lambda < \varkappa\}$ be a set of graphs. If all products $\prod_{\lambda<\varkappa} G_\lambda^{f(\lambda)}$, where $f \in \omega^\varkappa \setminus \{O\}$, are connected then the following statements are equivalent:

(1) $\mathcal{H}$ is $\omega$–productively independent.
(2) $\mathcal{H}$ fulfills condition (R).

Obviously, the product of each collection of complete graphs with at least three vertices is connected. Hence, the following theorem implies that every set of complete graphs with at least three vertices fulfills condition (R).

**Theorem 3 (Koubek, Nešetřil, Rödl, [3]).** *The class of all complete graphs with at least three vertices is productively independent.*

Now we shall prove the result of the paper.

**Theorem 4.** *The following sets of simple graphs fulfil condition (R) ( and consequently they are $\omega$–productively independent):*

($\alpha$) *the set $\{P_n; n \geq 2\}$ of all paths of length greater than 1,*
($\beta$) *the set $\{C_{2k+1}; k \geq 2\}$ of all circuit of odd length,*
($\gamma$) *the set $\{C_{2k}; k \geq 1\}$ of all circuits of even length.*

In the proof of Theorem 4 we shall often use the following simple proposition (see [2] or [4]).

Suppose that $\{G_i; i \in I\}$ is a collection of bipartite graphs and $G = \prod_{i\in I} G_i$ is the direct product of this collection. Let $x$ and $y$ be vertices in $G$. Denote $x = (x_i)_{i\in I}, y = (y_i)_{i\in I}$ and $d_i = d(x_i, y_i)$. Then the following holds:

$d(x, y) < \infty$ if and only if the collection $\{d_i; i \in I\}$ is bounded and moreover all $d_i$'s are odd or all $d_i$'s are even.

Further, if $d(x, y) < \infty$ then $d(x, y) = \sup\{d_i; i \in I\}$.

PROOF of Theorem 4:

**Case ($\alpha$).** Denote by $P_n = ([0, n], \{\{i, i+1\}; i \in [0, n-1]\})$ the path of length $n$. Let $f : \omega \setminus \{0, 1\} \to \omega$ be an arbitrary function and let $G = \prod_{n=2}^{\infty} P_n^{f(n)}$. We show that $f$ can be recognized from a distinguished component of $G$.

For this, let $C$ be an arbitrary component of $G$ containing a vertex of degree 1. Clearly all such component are pairwise isomorphic, so $C$ is a distinguished component of $G$. We show that $f$ can be recognized from $C$. Choose for this purpose arbitrary but fixed vertex $x$ of degree 1 in the graph $C$.

(I) We determine by induction the values $f(2k)$ for $k = 1, 2\ldots$

So, let us suppose that the values $f(2j)$ for $j \in [1, k-1]$ are known; we are going to determine $f(2k)$.

Let $M_x$ be the set of all vertices in $C$ of degree 1 having a distance of $2k$ from the vertex $x$. If $y \in M_x$ then all projections of $y$ have degree 1 and moreover $y$

can differ from $x$ only in the projections into paths $P_{2j}$ where $j \in [1,k]$. (Indeed, if the vertex $y$ of degree 1 differs from $x$ in the projection into a certain path $P_s$, where $s > 2k$, then $d(x,y) > 2k$ and if it differs from $x$ in the projection into a certain path $P_{2j+1}$, where $j \in [1,k-1]$, then $d(x,y) = \infty$.) Moreover $x$ necessarily differs from $y$ in at least one of the projections into paths $P_{2k}$ because otherwise $d(x,y) < 2k$. Thus

$$|M_x| = (2^{f(2k)} - 1) \cdot \prod_{j=1}^{k-1} 2^{f(2j)}$$

which enables us to calculate $f(2k)$ from the numbers $f(2j)$, where $j = 1, 2, \ldots, k-1$, and from the structure of the graph $C$.

(II) Let $k$ be a positive integer. We determine $f(2k+1)$.

(A) First suppose that $f(2k) \neq 0$.

Let $N_x$ be the set of all vertices $y$ in $C$ such that the following conditions hold (in C):

(1) $d(y) = 2$
(2) $d(y,x) = 2k$
(3) $d(z,y) = 2 \Rightarrow d(z,x) \leq 2k$
(4) $d(z) = 1 \Rightarrow d(z,y) \geq 2k$

Let $y \in N_x$. Then the following holds:

(a) The vertex $y$ has exactly one co–ordinate of degree 2, whereas all other co–ordinates have degree 1.

We show that

(b) The co–ordinate $y_i$ of $y$, which has degree 2, is a projection of $y$ into a certain path $P_{2k+1}$ and has a distance of $2k$ from the projection $x_i$ of the vertex $x$ into this path.

First we show that the co–ordinate $y_i$ of degree 2 has a distance of 2k from the corresponding projection $x_i$. For this, suppose that $d(x_i, y_i) < 2k$. Then the co–ordinate $y_j$ such that $d(y_j, x_j) = 2k$ (such a co–ordinate exists by (2)) has degree 1 (i.e. $y_j$ is an endpoint of some path $P_{2k}$). Thus, if we define vertex $z$ by $z_i = x_i$ and $z_r = y_r$ for $r \neq i$, then $d(z) = 1$ and $d(z,y) = d(x_i, y_i) < 2k$ (since $d(x,y)$ is even, we also have that $d(x_i, y_i) = d(z_i, y_i)$ is even and so $d(z,y) < \infty$). But the existence of vertex $z$ contradicts condition (4).

Now we show that the co–ordinate $y_i$ of degree 2 is a projection into a certain path $P_{2k+1}$. Indeed, by the above, $y_i$ is a projection into a certain path $P_r$ where $r \geq 2k+1$. Suppose that $r > 2k+1$. Then there is a vertex $z_i$ in the path $P_r$ such that $d(z_i, y_i) = 2$ and $d(z_i, x_i) = 2k+2$. Thus, if $z$ denotes the vertex having the i–th co–ordinate equal to $z_i$ and identical with $y$ in all other co–ordinates then $d(z,y) = 2$ and $d(z,x) > 2k$ which contradicts condition (3).

Further,

(c) the vertex $y$ coincides with $x$ in all projections into paths $P_r$ where $r \geq 2k+1$ excepting the co–ordinate $y_i$, and also in all projections into paths $P_{2j+1}$ where $j \in [1, k-1]$.

(Otherwise $d(x,y) > 2k$ or $d(x,y) = \infty$.)

On the contrary, every vertex $y$ which fulfills conditions (a),(b),(c) belongs to $N_x$.

The conditions (1) – (3) are obvious, let us prove (4). If $z$ is a vertex in $C$ and $d(z) = 1$ then either $d(z_i, y_i) = 2k$ (and hence $d(z,y) \geq 2k$) or $d(z_i, y_i) = 1$. We show that the latter case leads to a contradiction. Indeed, since $f(2k) \neq 0$, then we can consider the projections $z_j$ and $y_j$ of the vertices $z$ and $y$ into a certain path $P_{2k}$. Vertices $z_j$ and $y_j$ have degree 1, so $d(z_j, y_j)$ is an even number. Thus $d(z,y) = \infty$, so $z$ is not in C, a contradiction.

(B) Now suppose that $f(2k) = 0$.

Similarly as in case (A) we show that the set $\widehat{N}_x$ defined by conditions (1) – (3) coincides with the set defined by conditions (a) – (c). Indeed, some co–ordinate $y_i$ of the vertex $y$ has a distance of 2k from the corresponding co–ordinate $x_i$ of $x$. Clearly $y_i$ is the projection into a certain path $P_r$ where $r \geq 2k+1$, $y_i$ has degree 2 and all other co–ordinates of $y$ have degree 1. Further we proceed analogously as in case (A).

The sets $N_x$ (in case (A)) and $\widehat{N}_x$ (in case (B)) are described by conditions (a) – (c) which follows that

$$f(2k+1) \cdot \prod_{j=1}^{k} 2^{f(2j)} = \begin{cases} |N_x| & \text{if } f(2k) \neq 0 \\ |\widehat{N}_x| & \text{if } f(2k) = 0. \end{cases}$$

Thus $f(2k+1)$ can be determined from the numbers $f(2j)$, where $j = 1, 2, \ldots, k$, and the structure of the graph C.

**Case ($\beta$).** Denote by $C_k = ([0, k-1], \{\{i, i+1\}; i \in [0, k-1]\} \cup \{\{0, k-1\}\})$ the circuit of length $k$. Let $f : \omega \setminus \{0\} \to \omega$ be an arbitrary function and let $G = \prod_{k=1}^{\infty} C_{2k+1}^{f(k)}$. We show that $f$ can be recognized from a distinguished component of $G$. Since all components of $G$ are isomorphic we choose an arbitrary component $C$ and show that $f$ can be recognized from $C$.

So, choose an arbitrary but fixed vertex $x^0$ in C and a vertex $x^1$ adjacent to $x^0$. Suppose without loss of generality that $x^0 = (0, 0, \ldots)$ and $x^1 = (1, 1, \ldots)$. Define by induction a sequence $\{x^i\}_{i=0}^{\infty}$ of vertices in C such that $\{x^i, x^{i+1}\} \in E(C)$ and moreover there exists exactly one path of length 2 between vertices $x^{i+1}$ and $x^{i-1}$. It can be easily seen that the sequence $\{x_i\}_{i=0}^{\infty}$ is uniquely defined (for given vertices $x^0$ and $x^1$) and that the projection of $x^i$ into $C_{2k+1}$ is equal to $i \bmod 2k+1$.

If $C_k$ is the circuit of length $k$, $x$ and $y$ are two vertices in $C_k$ with a distance $d$ and $i$ is a non–negative integer then by $N(k, d, i)$ we denote the number of sequences from $x$ to $y$ of length $i$ in the graph $C_k$. Clearly, if $k$ *is odd* then $N(k, d, d) = 1, N(k, k-d, d) = 1$ for $d \neq k$ and $N(k, 0, k) = 2$.

Obviously, if $x$ and $y$ are two vertices in $C$, $\{p_j; j \in J\}$ is the set of all projections of the product $\prod_{k=1}^{\infty} C_{2k+1}^{f(k)}$ and $k_j$ is the length of the circuit $p_j(G)$ then the number of all sequences of length $i$ from $x$ to $y$ is equal to

$$\prod_{j \in J} N(k_j, d^{p_j(G)}(p_j(x), p_j(y)), i).$$

In particular, for $k \geq 1$, the number of all sequences from $x^0$ to $x^{2k+1}$ of length $2k+1$ is equal to

$$2^{f(k)} \cdot \prod_{j \in \hat{J}} N(k_j, d^{p_j(G)}(p_j(x^0), p_j(x^{2k+1})), 2k+1),$$

where $\hat{J} \subseteq J$ and $\{p_j; j \in \hat{J}\}$ is the set of all projections of the product $\prod_{k=1}^{\infty} C_{2k+1}^{f(k)}$ into circuits of length $\leq 2k-1$. (Note that the empty product of natural numbers, we put to be equal to 1.) But the value of the product

$$\prod_{j \in \hat{J}} N(k_j, d^{j(G)}(p_j(x^0), p_j(x^{2k+1}), 2k+1) =$$
$$\prod_{j \in \hat{J}} N(k_j, d^{C_{k_j}}(0, 2k+1 \bmod k_j), 2k+1)$$

depends only on $k$ and on the numbers $f(1), f(2), \ldots, f(k-1)$. Since moreover this value is different from 0, we can determine the number $f(k)$ from the numbers $f(1), f(2), \ldots, f(k-1)$ and the structure of the graph C.

**Case** $(\gamma)$. Let $f : \omega \setminus \{0,1\} \to \omega$ be an arbitrary function, and denote $G = \prod_{k=2}^{\infty} C_{2k}^{f(k)}$. We show that $f$ can be recognized from a distinguished component of $G$. We perform the proof in a similar way as in the proof of Case $(\beta)$.

(I) First we suppose that $f(2) = 0$.

We define the sequence $\{x^i\}_{i=0}^{\infty}$ of vertices in the same way as in $(\beta)$, as well as the numbers $N(k,d,i)$. Obviously, $N(2k,d,d) = 1$ for $d < k$ and $N(2k,k,k) = 2$. Thus, for $k \geq 2$ the number of all sequences from $x^0$ to $x^k$ of length $k$ is equal to

$$2^{f(k)} \cdot \prod_{j \in \hat{J}} N(k_j, d^{p_j(G)}(p_j(X^0), p_j(x^k), k),$$

where $\hat{J} \subseteq J$ is defined in the same way as in Case $(\beta)$. The rest of the proof is the same as in Case $(\beta)$.

(II) If $f(2) \neq 0$ then we must modify the construction of the sequence $\{x^i\}_{i=0}^{\infty}$. This construction can be done as follows.

Define $x^0$ and $x^1$ as in Case $(\beta)$. Before we define vertices $x^i$, where $i \geq 2$, we determine the number $f(2)$ from the structure of $C$. Denote by $O(x^1)$ the set of all vertices connected by an edge with $x^1$. If $y \in O(x^1)$ and $y \neq x^0$ then clearly $d(y, x^0) = 2$. Denote by $k(y)$ the number of all paths of length 2 between $y$ and $x^0$. It can be easily seen that

$$\min\{k(y); y \in 0(x^1)\} = 2^{f(2)},$$

which immediately enables us to calculate $f(2)$.

Now by induction we define $x^{i+1}$ to be arbitrary vertex such that $\{x^i, x^{i+1}\} \in E(C)$ and moreover there exist exactly $2^{f(2)}$ paths of length 2 between $x^{i+1}$ and $x^{i-1}$.

The rest of the proof is the same as in (I). ∎

**Example.** The set $\{C_k; k \geq 3\}$ of all circuits is not $\omega$-productively independent because, for example, the equality $K_2 \times C_{2k+1} \cong K_{2(2k+1)}$, which can be easily seen, implies that

$$K_2 \times C_{2j+1} \times C_{2k+1} \cong C_{2(2j+1)} \times C_{2k+1} \cong C_{2j+1} \times C_{2(2k+1)}.$$

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