Commentationes Mathematicae Universitatis Carolinae

Enayet U, Tarafdar

Five equivalent theorems on a convex subset of a topological vector space

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 2, 323--326

Persistent URL: http://dml.cz/dmlcz/106750

Five equivalent theorems on a convex subset of a topological vector space

E. TARAFDAR

Abstract. Recently we have proved a fixed point theorem of a set valued mapping and have shown that this fixed point theorem is equivalent to Fan–Knaster–Kuratowski–Mazurkiewicz theorem. In this note we have extended three other theorems originality due to Fan, and have shown that these extended versions are equivalent to the fixed point theorem. In other words, we have shown that all these five theorems are equivalent

Keywords: fixed point, convex set

Classification: 47H10

In [6] we have given an independent proof of the following fixed point theorem 1 and have shown that it is equivalent to Fan–Knaster–Kuratowski–Mazurkiewicz's theorem [3] (here Theorem 2) which is a generalization of the classical Knaster–Kuratowski–Mazurkiewicz theorem [4].

The object of this note is to prove that the following five theorems are true and equivalent. In fact, we have shown that theorems 3,4 and 5 are equivalent to the fixed point theorem 1.

Throughout this paper E will denote a Hausdorff topological (real) vector space and X a nonempty convex subset of E. For a subset A of X, A^c will denote the complement of A in X, i.e., $A^c = X \setminus A$ and $\overline{A}$ will denote the closure of A in E.

Theorem 1. *(Fixed point theorem).*
Let $F : X \to 2^X$ be a set valued mapping such that

(i) *for each $x \in X, F(x)$ is a nonempty convex subset of X;*
(ii) *for each $y \in X, F^{-1}(y) = \{x \in X : y \in F(x)\}$ contains a relatively open subset 0_y of X (0_y may be empty for some y);*
(iii) $\bigcup_{x \in X} 0_x = X$;
and
(iv) *there exists a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset X_1 of X and the set $D = \bigcap_{x \in X_0} 0_x^c$ is compact (D could be empty), 0_x^c being the complement of 0_x in X (see the above notations). Then there exists a point $x_0 \in X$ such that $x_0 \in F(x_0)$.*

This fixed point theorem generalizes an earlier fixed point theorem of ours [7].

Theorem 2. *(Fan–Knaster–Kuratowski–Mazurkiewicz' s theorem).*

Let $\emptyset \neq Y \subset X$. For each $y \in Y$, let $F(y)$ be a relatively closed subset of X such that the convex hull of each finite subset $\{y_1, y_2, \ldots, y_n\}$ of Y is contained in the corresponding union $\bigcup_{i=1}^{n} F(y_i)$. Then for each nonempty subset Y_0 of Y such that Y_0 is contained in a compact convex Y_1 of X, $\bigcap_{y \in Y_0} F(y) \neq \emptyset$. If, in addition, the set $\bigcap_{y \in Y_0} F(y)$ is compact, then $\bigcap_{y \in Y} F(y) \neq \emptyset$.

Theorem 3.

Let $A \subset X \times X$ be a subset such that

(i) *$(x, x) \in A$ for each $x \in X$;*
(ii) *for each $x \in X$, the set $\overline{\{y \in X : (x, y) \in A\}}$ is a subset of X;*
(iii) *for each $y \in X$, the set $\{x \in X : (x, y) \notin A\}$ is convex or empty; and*
(iv) *X has a nonempty subset X_0 contained in a compact convex subset X_1 of X such that the set $B = \bigcap_{x \in X_0} \overline{\{y \in X : (x, y) \in A\}}$ is compact. Then the set $\bigcap_{x \in X} \overline{\{y \in X : (x, y) \in A\}}$ is a nonempty subset of B.*

Theorem 4.

Let f and g be two real valued functions defined on $X \times X$ such that

(a) *$f(x, y) \leq g(x, y)$ for all $x, y \in X$;*
(b) *$g(x, x) \leq 0$ for all $x \in X$;*
(c) *for each $y \in X$, the subset $\overline{\{x \in X : f(x, y) \leq 0\}}$ is a subset of X;*
(d) *for each $x \in X$, the set $\{y \in X : g(x, y) > 0\}$ is convex or empty; and*
(e) *X has a nonempty set X_0 contained in a compact convex subset X_1 of X such that the set*

$$C = \bigcap_{y \in X_0} \overline{\{x \in X : f(x, y) \leq 0\}}$$

is compact.

Then the set $\bigcap_{y \in X} \overline{\{x \in X : f(x, y) \leq 0\}}$ is a nonempty compact subset of C.

Theorem 5.

Let $g : X \times X \to R$ be a function satisfying

(a)' *$g(x, x) \leq 0$ for each $x \in X$;*
(b)' *for each $y \in X$, the set $\overline{\{x \in X : g(x, y) \leq 0\}}$ is a subset of X;*
(c)' *fir each $x \in X$, the set $\{y \in X : g(x, y) > 0\}$ is convex or empty; and*
(d)' *X has a nonempty subset X_0 contained in a compact convex subset X_1 of X such that the set*

$$L = \bigcap_{y \in X_0} \overline{\{x \in X : g(x, y) \leq 0\}}$$

is compact.

Then the set $\bigcap_{y \in X} \overline{\{x \in X : g(x,y) \leq 0\}}$ *is a nonempty compact subset of* L.

PROOF :

In [6] we have shown the equivalence of Theorem 1 and Theorem 2. The rest of the proof is given in the following five steps.

Step 1.

Top begin with we prove that Theorem 1 implies Theorem 3. Let the conditions of Theorem 3 hold. We need only to show that the set $S = \bigcap_{x \in X} \overline{\{y \in X : (x,y) \in A\}}$ $\neq \emptyset$ as S being closed subset of the compact set B is compact. On the contrary, suppose that $S = \emptyset$. Then clearly the set $\bigcap_{x \in X} H(x) = \emptyset$ where $H(x) = \{y \in X : (x,y) \in A\}$. Hence for each $y \in X$, the set $F(y) = \{x \in X : y \notin H(x)\} = \{x \in X : (x,y) \notin A\}$ is nonempty and, therefore, convex by (iii). Now for each $x \in X, F^{-1}(x) = \{y \in X : x \in F(y)\} = \{y \in X : (x,y) \notin A\} = \{y \in X : (x,y) \in A\}^c \supset \left[\overline{\{y \in X : (x,y) \in A\}}\right]^c = 0_x$, say. Thus for each $x \in X, F^{-1}(x)$ contains a relatively open set 0_x. Also by assumption $S = \emptyset$, i.e. $X = \bigcup_{x \in X} \left[\overline{\{y \in X : (x,y) \in A\}}\right]^c = \bigcup_{x \in X} 0_x$. Lastly $\bigcap_{x \in X_0} 0_x^c = \bigcap_{x \in X_0} \overline{\{y \in X : (x,y) \in A\}} = B$ is a compact set. Thus the set valued mapping $F : X \to 2^X$ satisfies all the conditions of Theorem 1 and, therefore, there exists a point $x_0 \in F(x_0)$, i.e. $(x_0, x_0) \notin A$ which contradicts the condition (i). Hence $S \neq \emptyset$.

Step 2. In this step we prove that Theorem 3 implies Theorem 1. Let $F : X \to 2^X$ be a set valued mapping satisfying the conditions of Theorem 1. Assume that F has no fixed point. We define the set $A \subset X \times X$ by

$$A = \{(x,y) \in X \times X : y \notin F^{-1}(x)\}.$$

Then for each $x \in X, (x,x) \in A$ as $x \notin F^{-1}(x)$ for each $x \in X$ by assumption. Thus the condition (i) of Theorem 3 holds. Now for each $x \in X$, the set $\overline{\{y \in X : (x,y) \in A\}} = \overline{\{x \in X : y \notin F^{-1}(x)\}} = \left[\overline{\{y \in X : y \in F^{-1}(x)\}^c}\right] = [F^{-1}(x)]^c \subset \overline{0_x^c} = 0_x^c$ which is a subset of X. Thus the condition (ii) of Theorem 3 is fulfilled.

Also for each $y \in X$, the set $\{x \in X : (x,y) \notin A\} = \{x \in X : y \in F^{-1}(x)\} = F(y)$ is a nonempty convex set and, therefore, the condition (iii) of Theorem 3 is fulfilled. Finally the set $B = \bigcap_{x \in X_0} \overline{\{y \in X : (x,y) \in A\}} \subset \bigcap_{x \in X_0} 0_x^c = D$ is compact being a closed subset of the compact set D. Thus the condition (iv) of Theorem 3 is also fulfilled. Hence by Theorem 3, $\bigcap_{x \in X} \overline{\{y \in X : (x,y) \in A\}}$ is nonempty and, therefore, $\bigcap_{x \in X} 0_x^c$ is nonempty which contradicts the condition (iii) of Theorem 1. Thus F must have a fixed point.

Step 3.

To see that Theorem 3 implies Theorem 5, it suffices to define

$$A = \{(x, y) \in X \times X : g(y, x) \leq 0\}.$$

Step 4.

Theorem 5 implies theorem 4. Indeed, the assumptions of theorem 4 can easily be seen to imply the assumptions of Theorem 5 and hence $\bigcap_{y \in X} \overline{\{x \in X : g(x, y) \leq 0\}} \neq \emptyset$, which together with (a) gives the assertion of Theorem 4.

Step 5.

To see that Theorem 4 implies Theorem 3, it suffices to define

$$f(x, y) = g(x, y) = \begin{cases} 0 \text{ for } (y, x) \in A \\ 1 \text{ for } (y, x) \notin A. \end{cases}$$

∎

Remark. Note that the conditions (ii) of Theorem 3, (c) of Theorem 4 and (b)' of Theorem 5 are automatically fulfilled if the set X is closed.

The above results include many wellknown results as special cases e.g. the minimax inequality of Fan [3], Lin [5] and Tarafdar [8].

The author gratefully acknowledges the valuable suggestions of the referee.

References

[1] Allen G., *Variational inequalities, complementarity problems, and duality theorems*, J.Math. Anal.Appl. **58** (1977), 1-10.

[2] Fan K., "A minimax inequality and applications, Inequalities,Vol III, (Ed.O.Shisha)," New York, London, Academic Press 13, 1972, pp. 103-113.

[3] Fan K., *Some properties of convex sets related to fixed point theorems*, Math.Ann. **266** (1984), 519-537.

[4] Knaster B., Kuratowski C., Mazurkiewicz S., *Ein Beweis des Fixpunktsatzes für n−dimensionale Simplexe*, Fund.Math. **14** (1929), 132-137.

[5] Lin T.C., *Convex sets, fixed points, variational and minimax inequalities*, Bull.Austral.Math. Soc. **34** (1986), 107-117.

[6] Tarafdar E., *A fixed point theorem equivalent to Fan−Knaster−Kuratowski−Mazurkiewicz's theorem*, J.Math.Anal.Appl. **128** (1987), 475-479.

[7] Tarafdar E., *On nonlinear variational inequalities*, Proc.Amer.Math.Soc. **67** (1977), 95-98.

[8] Tarafdar E., *Variational problems via a fixed point theorem*, Indian Journal of Mathematics **28** (1986), 229-240.

Department of Mathematics, University of Queensland, St. Lucia, Queensland, 4067, Australia

(*Received* December 19,1988, *revised* March 3,1989)