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On Baire approximations of normal integrands

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Abstract. Let $D \subset T \times X$, where T is a measurable space and X a metric space. We give conditions on T, X and D such that every upper semicontinuous in x and meausarable function $f : D \to \overline{R}$ is the limit of a decreasing sequence of measurable functions which are continuous in x.

Keywords: Normal integrand, Baire approximation

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1. Problem. The well known theorem of Baire states that every upper semicontinuous function on a metric space is the limit of a decreasing sequence of continuous functions. In this note we prove an analogue of this theorem for an upper semicontinuous functions which depends measurably on the parameter.

Throughout the paper $(T, \mathcal{T})$ is a measurable space, (X, d) a metric space, and D a nonempty subset of $T \times X$. The set D is always considered with the trace σ–field $\{D \cap A | A \in \mathcal{T} \otimes \mathcal{B}(X)\}$, where $\mathcal{B}(X)$ stands for the Borel σ–field on X. By D_t we denote t–sections of D, i.e. $D_t = \{x | (t, x) \in D\}, t \in T$.

We shall deal with the following classes of extended real–valued functions on D:
$F_1(D) = \{f : D \to \overline{R} | f$ is measurable and for each $t \in T, f(t, \cdot)$ is upper semicontinuous on $D_t\}$,
$F_2(D) = \{f : D \to \overline{R} |$ there is a sequence of measurable functions $f_n : D \to \overline{R}$ such that $f_{n+1} \le f_n, f_n(t, \cdot)$ is continuous on D_t and $\lim_{n\to\infty} f_n(t, x) = f(t, x), \quad (t, x) \in D\}$.

Functions from $F_1(D)$ are called normal integrands. They are extensively studied in mathematical programming. The class $F_2(D)$ also appears in optimization, especially, in stochastic dynamic programming ([**7**]–[**9**]).

It is obvious that $F_2(D) \subset F_1(D)$. We give sufficient conditions for the equality $F_1(D) = F_2(D)$, and point out its application.

Remark. If X is separable and $g : T \times X \to \overline{R}$ is a Carathéodory map (i.e. measurable in t and continuous in x), then g is measurable. Hence, in the definition of $F_2(T \times X)$ it suffices to require the measurability of $f_n(\cdot, x)$ instead of the product–measurability of f_n.

2. Auxiliary results. In this section we quote some set–theoretical and topological results which will be used in the proof of the main theorem.

Let $\mathcal{R}$ be a family of sets. By $S(\mathcal{R})$ we denote the family of all sets obtained from $\mathcal{R}$ by the Souslin operation. If $S(\mathcal{R}) = \mathcal{R}$, we say $\mathcal{R}$ is a Souslin family (cf. [**2**], Chap.2; [**12**], §2).

The σ–field $\mathcal{T}$ on T is a Souslin family provided one of the following conditions holds:

(i) $\mathcal{T}$ is complete with respect to a σ–finite measure,
(ii) $\mathcal{T}$ is the family of all μ–measurable sets, where μ is an outer measure on T,
(iii) $\mathcal{T}$ is the σ–field of universally measurable sets for some σ–field on T,
(iv) T is a topological space, and $\mathcal{T}$ is the family of all sets with the Baire property.

A metric space X is called Souslin if it is a continuous image of a Polish space. In such a space the family of all Souslin subsets coincides with $S(\mathcal{B}(X))$ and $S(\mathcal{F})$, where $\mathcal{F}$ is the set of all closed sets in X.

For a subset $Z \subset T \times X$ we denote by $pr_T Z$ its projection on T, i.e. $pr_T Z = \{t | (t,x) \in Z \text{ for some } x \in X\}$. The following projection theorem is very useful.

Theorem 2.1. *(cf. [2], Th.1.3). Suppose $\mathcal{T}$ is a Souslin family and X is a Souslin space. Then $pr_T Z \in \mathcal{T}$ for every $Z \in S(\mathcal{T} \otimes \mathcal{B}(X))$.*

We shall also use the following results.

Theorem 2.2. *([4]). Let T and X be Polish spaces, A a Souslin subset of $T \times X$, and B a subset of A such that B is Borel in A, and B_t is open in $A_t, t \in T$. Then*

$$B = A \cap \bigcup_{n \in N} C_n \times G_n,$$

where C_n is Borel in T and G_n is open in X, $n \in N$.

Theorem 2.3. *([5], Th.2.3; [6], Th.5). Let X with the topology $\mathcal{G}$ be a separable and metrizable space. Then for any sequence $\{B_n\}$ of Borel sets in X there is a separable and metrizable topology $\mathcal{G}'$ on X such that $\mathcal{G} \subset \mathcal{G}'$, each $B_n \in \mathcal{G}'$, and the σ-fields generated by $\mathcal{G}$ and $\mathcal{G}'$ are the same.*

3. Main theorem. The following theorem is the main result of the paper.

Theorem 3.1. *Let X be a metrizable space and $D \in S(\mathcal{T} \otimes \mathcal{B}(X))$. Then $F_1(D) = F_2(D)$, provided one of the conditions holds:*

(i) $\mathcal{T}$ is a Souslin family and X is a Souslin space,
(ii) T and X are Polish spaces, and $\mathcal{T} = \mathcal{B}(T)$.

Moreover, for each $f \in F_1(D)$ there is $g \in F_2(T \times X)$ such that $g|_D = f$.

PROOF : It suffices to prove that each $f \in F_1(D)$ can be extended to a function from $F_2(T \times X)$. Since there is an increasing homeomorphism of $\overline{R}$ and $[-1,1]$, we may assume $|f(t,x)| \le 1, (t,x) \in D$.

We start with the proof under the assumption (i). By Theorem 2.1, the set $P = pr_T D$ belongs to $\mathcal{T}$. Similarly, as in the proof of the theorem of Baire (see e.g. [1], p.390), we define the functions $f_n : P \times X \to (-\infty, 1]$ and $g_n : P \times X \to (-1, 1]$ by the formulae

$$f_n(t,x) = \sup\{f(t,y) - n\, d(x,y) | y \in D_t\},$$
$$g_n(t,x) = \max\{f_n(t,x), -1 + \frac{1}{n}\}, n \in N.$$

These functions are measurable in t. Indeed, for any $x \in X$ and $r \in R$ we have

$$\{t | f_n(t,x) > r\} = \{t | f(t,x) - n\, d(x,y) > r \text{ for some } y \in D_t\} = \\ = pr_T\{(t,y) \in D | f(t,y) - n\, d(x,y) > r\} \in \mathcal{T},$$

because of the measurability of f and the projection theorem. Being decreasing, the sequence $\{g_n\}$ is convergent to a function $g : P \times X \to [-1,1]$. From the proof of the theorem of Baire we know that $g_n(t,\cdot)$ are continuous, and $g|_D = f$. Since g_n is a Carathéodory map on $P \times X$, it is measurable. Hence, $g \in F_2(P \times X)$. It is clear that g has an extension to a function from $F_2(T \times X)$.

For the second part of the proof we assume that T and X are Polish spaces. Since $f \in F_1(D)$, for each rational $r \in (-1,1]$ the set $A^r = f^{-1}([-1,r))$ is Borel in D, and its t–section is open in $D_t, t \in T$. In virtue of Theorem 2.2,

$$A^r = D \cap \bigcup_{n \in N} C_n^r \times G_n^r,$$

where $C_n^r \in \mathcal{B}(T)$ and G_n^r is open in $X, n \in N$. Denote by $\mathcal{G}$ the Polish topology of T. By Theorem 2.3, there is a stronger separable and metrizable topology $\mathcal{G}'$ on T, such that $C_n^r \in \mathcal{G}'$ for each $n \in N$ and each rational $r \in (-1,1]$, and the σ–fields generated by $\mathcal{G}$ and $\mathcal{G}'$ are the same. Now we regard $T \times X$ with the new product topology, where T is equipped with $\mathcal{G}'$. In this topology each A^r is open in D and, consequently, f is upper semicontinuous. We refer to the proof of the theorem of Baire again. Let $f_n : T \times X \to (-\infty, 1]$ and $g : T \times X \to (-1,1]$ be defined as

$$f_n(a) = \sup\{f(b) - n\rho(a,b) | b \in D\},$$
$$g_n(a) = \max\{f_n(a), -1 + \frac{1}{n}\}, n \in N,$$

where ρ is a metric of $T \times X$. The decreasing sequence of continuous functions $\{g_n\}$ is convergent to a function $g : T \times X \to [-1,1]$ such that $g|_D = f$. Since the σ–fields generated by $\mathcal{G}$ and $\mathcal{G}'$ are the same and the topology of X does not change, each g_n is Borel–measurable with respect to the original topology of $T \times X$, and continuous in x. Thus $g \in F_2(T \times X)$, which completes the proof. ∎

Remarks. 1. Related problems were studied by Dynkin ([**3**], Lemma 3.2), Schäl ([**8**], §11; [**10**]) and Ślęzak ([**11**], Lemma 1). Our theorem generalizes corresponding results from [**10**] and [**11**].
2. It would be interesting to know if the inclusion $F_1(T \times X) \subset F_2(T \times X)$ holds for an arbitrary σ–field $\mathcal{T}$ and a Polish space X.

4. Application. The assumption that a function f belongs to the class $F_2(D)$ appears in some theorems in optimization ([**7**]-[**9**]). Using Theorem 3.1, we can replace this assumption by an easier verified condition $f \in F_1(D)$. As an example we give one of such results.

Let $f : T \times X \to \overline{R}$ be measurable and bounded from above in t, and $q : \mathcal{T} \times X \to [0,1]$ a transition probability from X to T, i.e. $q(\cdot, x)$ is a probability measure on $\mathcal{T}$ for each $x \in X$, and $q(A, \cdot)$ is measurable for each $A \in \mathcal{T}$. Then the function

$$v(x) = \int_T f(t,x)q(dt,x), \quad x \in X$$

is well defined. In stochastic optimization it is interesting to know under which assumptions v is upper semicontinuous.

We say that the transition probability q is s–continuous, if for each bounded and measurable $u : T \to R$ the function

$$x \to \int_T u(t)q(dt,x), \quad x \in X$$

is continuous.

The following result is an immediate consequence of Prop.14.1 of Schäl [8] and our Theorem 3.1.

Theorem 4.1. *Suppose either (i) T is a Souslin family and X is a Souslin space, or (ii) T and X are Souslin subsets of Polish spaces and $\mathcal{T} = \mathcal{B}(T)$. If f belongs to $F_1(T \times X)$ and is bounded from above, and q is s–continuous, then v is upper semicontinuous.*

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