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Estimates on the eigenvalues for some nonlinear ordinary differential operators

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Dedicated to the memory of Svatopluk Fučík

Abstract. We obtain some detailed information about the location of the eigenvalues for a class of nonlinear Sturm–Liouville operators.

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We consider the eigenvalue problem

$$\begin{cases} Lu + f(x,u) = \lambda u \\ u(a) = u(b) = 0 \end{cases} \tag{1}$$

on the compact interval $[a,b]$ of the real line $\mathbf{R}$, where $Lu := -(p(x)u')' + q(x)u$ is a regular Sturm–Liouville operator with real coefficients $p \in C^1([a,b]), p > 0, q \in C[a,b]$, and $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ is continuous and such that $f(x,0) = 0$ for all $x \in [a,b]$.

Clearly, $u = 0$ is a solution of (1) for all $\lambda \in \mathbf{R}$, and we are interested in the existence and location of *eigenvalues* of (1), namely values of λ for which there exists a nontrivial solution (an *eigenfunction*) of (1).

It is well–known that the linear problem corresponding to (1), namely

$$\begin{cases} Lu = \lambda u \\ u(a) = u(b) = 0 \end{cases} \tag{2}$$

has an infinite sequence of eigenvalues $\lambda_n^0 (n = 1,2,\dots)$ such that $\lambda_1^0 < \lambda_2^0 < \dots$ and $\lambda_n^0 \to +\infty$. It is also well–known that $\lambda_n^0 - \lambda_{n-1}^0 \to \infty$ as $n \to \infty$; therefore, given any $C > 0$, we have $\lambda_n^0 + C < \lambda_{n+1}^0$ for all large enough n.

In this note, we aim at discussing the following statement of "quasi discrete" spectrum for (1):

Theorem. *Assume that f satisfies*

H1) $0 \leq f(x,s)s \leq \alpha s^2$ *for some* $\alpha \geq 0$ *and all* $(x,s) \in [a,b] \times \mathbf{R}$;

H2) $f(x,-s) = -f(x,s)$ *and* $|f(x,s)| \leq a|s|^p$ *for some* $p > 1$, *some* $a \geq 0$ *and all* $(x,s) \in [a,b] \times \mathbf{R}$;

H3) *$f(x,s)s \geq 2F(x,s) \geq \alpha s^2 - \beta|s|$ for some $\beta \geq 0$ and all $(x,s) \in [a,b] \times \mathbf{R}$, where α is as in H1)and*

$$F(x,s) = \int_0^s f(x,t)\,dt.$$

Then the following conclusions hold:

1) *(1) has no eigenvalues outside the intervals $[\lambda_n^0, \lambda_n^0 + \alpha]$ $(n = 1,2,\ldots;$ we assume without loss of generality that $\lambda_{n+1}^0 - \lambda_n^0 > \alpha$ for all $n)$;*
2) *for each $n = 1,2,\ldots$, (1) has a one-parameter family $\lambda_n = \lambda_n(r)$, $r > 0$, of eigenvalues (with corresponding eigenfunctions $u_n(r)$ satisfying $\int_a^b u_n^2(r) = r^2$) such that*

$$2_i) \qquad \lambda_n^0 \leq \lambda_n(r) \leq \lambda_n^0 + \alpha \quad \textit{for all} \quad r > 0,$$
$$2_{ii}) \qquad \lambda_n(r) \to \lambda_n^0\,(r \to 0); \lambda_n(r) \to \lambda_n^0 + \alpha(r \to \infty).$$

The proof will be divided into three steps; we first show (Lemma 1) the nonexistence part by means of the spectral properties of the selfadjoint operator in $L^2(a,b)$ associated with L; next we use the Liusternik–Schnirelmann (LS) theory to prove in Lemma 2 the existence of eigenvalues, and finally in Lemma 3 we carry out the required estimates; these are obtained by means of elementary bounds on the "energy functional" of (1) restricted to the manifold $[u \in \overset{\circ}{W}{}^{1,2}(a,b) : \int_a^b u^2 = r^2]$.

We draw the reader's attention to the fact that H1) and H3) above force f keep uniformly close to the linear function αs in the following sense: on setting $h(x,s) := f(x,s) - \alpha s$, then $|h(x,s)| \leq \alpha|s|$ and $|h(x,s)| \leq \beta$ for all $(x,s) \in [a,b] \times \mathbf{R}$, as is easily checked.

Note in particular that the assumptions of the theorem are all satisfied by $f \equiv 0$, in which case the above inequalities boil down to $\lambda_n(r) = \lambda_n^0$ for all r and all n, as due. On the other hand, if $f \equiv 0$ the assumption $p > 1$ in H2) is necessary for the validity of the result; an immediate counterexample is given by $f(x,s) = \alpha s$ itself $(\alpha \neq 0)$. Indeed, we then have again a linear problem, whose eigenvalues are $\lambda_n(r) = \lambda_n^0 + \alpha\,(r > 0,\ n = 1,2,\ldots)$ so that the first assertion in $2_{ii})$ is not satisfied.

To exhibit a nontrivial example, let us take

$$f(x,u) = f(u) = C\frac{u^3}{1+u^2}$$

with $C > 0$; then f satisfies H1) and H2) with $\alpha = a = \beta = C$ and $p = 3$. To check H3), observe that

$$2F(x,u) = 2F(u) = C[u^2 - \log(1+u^2)]$$

while

$$f(u)u = C[u^2 - \frac{u^2}{1+u^2}]$$

Therefore, on setting $g(u) := f(u)u - 2F(u)$, we see that $g'(u) = \frac{2Cu^3}{(1+u^2)^2} > 0$ for $u > 0$; since $g(0) = 0$, this gives $g(u) \geq 0$ for all u (notice g is even).

The second inequality in H3) follows similarly on setting $h(u) := 2F(u) - C(u^2 - |u|)$ and observing that $h'(u) = C(1 - \frac{2u}{1+u^2}) > 0$ for $u > 0$.

Lemma 1. *Assume f satisfies H1). Then, if $\lambda < \lambda_1^0$ or $\lambda \in]\lambda_n^0 + \alpha, \lambda_{n+1}^0[(n \geq 1)$, problem (1) has only the trivial solution $u = 0$.*

PROOF : Take $\lambda \in]\lambda_n^0 + \alpha, \lambda_{n+1}^0[$ (the case $\lambda < \lambda_1^0$ can be treated similarly) and write (1) as

$$\begin{cases} Lu - (\lambda - \frac{\alpha}{2}) = \frac{\alpha}{2}u - f(x,u) =: g(x,u) \\ u(a) = u(b) = 0 \end{cases} \tag{3}$$

where, due to H1), g satisfies:

$$-\frac{\alpha}{2} \leq g(x,s)/s \leq \frac{\alpha}{2} \qquad (s \neq 0) \tag{4}$$

i.e. $|g(x,s)| \leq \frac{\alpha}{2}|s|$ for all $(x,s) \in [a,b] \times \mathbf{R}$.

Let now $H = L^2(a,b)$ equipped with the usual scalar product and norm $\|u\|^2 = \int_a^b u^2(t)\,dt$, and let T denote the realization in H of the differential operator L; T is a selfadjoint operator in H with domain

$$D(T) = \{u \in H | u', u'' \in H \quad \text{and} \quad u(a) = u(b) = 0\}$$

and spectrum $\sigma(T) = \{\lambda_n^0 : n = 1, 2, \dots\}$.

Let moreover G denote the Nemytskii operator by the function g : $G(u)(x) = g(x, u(x))$ for $u \in H$ and $x \in [a,b]$; from (4) we have $\|G(u)\| \leq \frac{\alpha}{2}\|u\|$ for all $u \in H$.

Then (3) is equivalent to

$$Tu - \mu u = G(u), \qquad u \in D(T)$$

where $\mu = \lambda - \frac{\alpha}{2}$, so that $\lambda_n^0 + \frac{\alpha}{2} < \mu < \lambda_{n+1}^0 - \frac{\alpha}{2}$. This in turn can be written as

$$u = (T - \mu I)^{-1} G(u) \tag{5}$$

where I is the identity map in H. Due to the selfadjointness of T, we have $\|(T - \mu I)^{-1}\| = [\text{dist}(\mu, \sigma(T))]^{-1}$; therefore, if u is a solution of (5), then

$$\|u\| \leq \|(T - \mu I)^{-1}\| \, \|G(u)\| \leq \frac{\alpha}{2}[\text{dist}(\mu, \sigma(T))]^{-1} \|u\|$$

But $\text{dist}(\mu, \sigma(T)) = \min[\lambda_{n+1}^0 - \mu, \mu - \lambda_n^0] > \frac{\alpha}{2}$; the last inequality then shows that necessarily $u = 0$, proving the Lemma. ■

To prove the existence of the eigenvalues for (1) we make use of the LS critical point theory: standard references for this are e.g. [3] or [4].

We shall hence forth consider weak solution of (1), namely $u \in \overset{\circ}{W}{}^{1,2}(a,b)$ such that

$$\int (pu')v' + \int quv + \int f(x,u)v = \lambda \int uv \tag{6}$$

for all $v \in \overset{\circ}{W}{}^{1,2}(a,b)$; $\int$ stands for $\int_a^b$.

Let us set further

$$\varphi_0(u) = \frac{1}{2}\int p(u')^2 + \frac{1}{2}\int qu^2 \tag{7}$$

$$\varphi(u) = \varphi_0(u) + \int F(x,u) \tag{8}$$

φ_0 and φ are the "energy functionals" associated with (2), (1) respectively. For $r > 0$, let moreover

$$Mr := [u \in \overset{\circ}{W}{}^{1,2}(a,b) : \int u^2 = r^2]$$

and for each $n = 1, 2, \dots$ set

$$K_n(r) = \{K \subset M_r : K \text{ compact, symmetric }, \gamma(K) = n\}$$

where $\gamma(K)$ denotes the genus of K. Finally, introduce the "LS critical levels"

$$C_n(r) = \inf_{K_n(r)} \sup_K 2\varphi(u). \tag{9}$$

With these notations, as a special case of theorem 6 in [1] we obtain the following result:

Lemma 2. *Assume f satisfies H2) and suppose further that $F(x,u) \geq 0$ for all $x \in [a,b]$ and all $u \in \mathbb{R}$. Then given $r > 0$, there exists a sequence $u_n(r)$ of (weak) eigenfunctions of (1) belonging to M_r and such that*

$$2\varphi(u_n(r)) = C_n(r) \qquad (n = 1, 2, \dots) \tag{10}$$

where $C_n(r)$ is as in (9); the eigenvalue $\lambda_n(r)$ corresponding to $u_n(r)$ satisfies

$$\lambda_n(r)r^2 = 2\varphi_0(u_n(r)) + \int f(x, u_n(r))u_n(r). \tag{11}$$

Moreover, $\lambda_n(r) \to +\infty$ as $n \to \infty$ (for each $r > 0$) and $\lambda_n(r) \to \lambda_n^0$ as $r \to 0$ (for each $n = 1, 2, \dots$).

Remark 1. To obtain (11), just put $u = v = u_n(r)$ in (6) and use the normalization condition $\int u_n^2(r) = r^2$ together with the definition of φ_0.

Remark 2. The behaviour of $\lambda_n(r)$ as $n \to \infty$ or $r \to 0$ can be formulated in more precise terms (see [1]): in particular one gets $\lambda_n^2(r) \sim n^2 (n \to \infty)$ for each $r > 0$ (as in the linear case) and $\lambda_n(r) = \lambda_n^0 + O(r^{p-1}) (r \to 0)$ for each $n \in \mathbb{N}$. Related results can be found in a very recent paper by Shibata ([5]).

Remark 3. If $f \equiv 0$ (i.e. if the problem is linear) the LS procedure gives exactly the eigenvalues λ_n^0 of (2); we have in this case

$$r^2\lambda_n^0 = \inf_{K_n(r)} \sup_K 2\varphi_0(u) \tag{12}$$

which is nothing but a reformulation of the classical Courant's minimax principle in terms of the sets $K_n(r)$; see e.g. [2].

The conclusions of the Theorem will now be a consequence of the following result.

Lemma 3. *Assume f satisfies H1), H2), H3) and let $\lambda_n(r)$ be the eigenvalues of (1) as given by Lemma 2. Then, for each $n = 1, 2, \ldots$, we have:*

$$\begin{aligned} &i) \quad \lambda_n^0 \leq \lambda_n(r) \leq \lambda_n^0 + \alpha \, (r > 0); \\ &ii) \quad \lambda_n(r) \to \lambda_n^0 + \alpha \quad \text{as } r \to \infty. \end{aligned}$$

PROOF : First note that H1) implies

$$0 \leq F(x, s) \leq \frac{\alpha}{2} s^2$$

for all $(x, s) \in [a, b] \times \mathbf{R}$; this implies (see the definitions (7), (8) of φ_0 and φ)

$$\varphi_0(u) \leq \varphi(u) \leq \varphi_0(u) + \frac{\alpha}{2} r^2$$

for all $u \in M_r$. Using the definition (9) of $C_n(r)$ and Remark 3 above we get therefore

$$r^2\lambda_n^0 \leq C_n(r) \leq r^2\lambda_n^0 + \alpha r^2 \tag{13.}$$

Moreover, the same assumption H1) together with one half of H3) gives

$$2F(x, s) \leq f(x, s)s \leq \alpha s^2$$

for all $(x, s) \in [a, b] \times \mathbf{R}$; therefore

$$2\varphi(u) \leq 2\varphi_0(u) + \int f(x, u)u \leq 2\varphi(u) + \alpha r^2$$

for all $u \in M_r$ (we have also used the inequality $\varphi_0 \leq \varphi$ already seen above).

Writing this for $u = u_n(r)$ and using (10) and (11) we get

$$C_n(r) \leq r^2\lambda_n(r) \leq C_n(r) + \alpha r^2. \tag{14}$$

Using together (13) and (14) yields

$$\lambda_n^0 \leq \lambda_n(r) \leq \lambda_n^0 + 2\alpha$$

and the inequality i) now follows on considering that (1) has no eigenvalues outside $[\lambda_n^0, \lambda_n^0 + \alpha]$, as shown in Lemma 1 (and assuming for convenience that $\lambda_{n+1}^0 - \lambda_n^0 > 2\alpha$ for all n).

To prove ii) we use the second half of H3), namely

$$2F(x,u) \geq \alpha u^2 - \beta|u|$$

which gives, on integrating and using Schwarz's inequality,

$$2\int F(x,u) \geq \alpha r^2 - \widehat{\beta} r$$

for all $u \in M_r(\widehat{\beta} = \beta(b-a)^{1/2})$. Therefore,

$$2\varphi(u) \geq 2\varphi_0(u) + \alpha r^2 - \widehat{\beta} r \qquad (u \in M_r)$$

and so, again from the definition of $C_n(r)$ and Remark 3,

$$c_n(r) \geq r^2\lambda_n^0 + \alpha r^2 - \widehat{\beta} r.$$

Together with the inequality $C_n(r) \leq r^2\lambda_n(r)$ proved in (14), this shows that

$$\lambda_n(r) \geq \lambda_n^0 + \alpha - \frac{\widehat{\beta}}{r} \qquad (r > 0)$$

and ii) now follows on looking at i) and taking the limit as $r \to \infty$. ■

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