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Dedicated to the memory of Zdeněk Frolík

Abstract. The cartesian closed topological hull of $\underline{\text{Met}}_n$, the category of metric spaces and nonexpansive maps, is shown to consist of those distance spaces whose pseudometric modification is positive and makes the distance lower–semicontinuos.

Keywords: metric space, cartesian closed hull

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Our aim is to describe the cartesian closed topological (shortly CCT) hull of the category $\underline{\text{Met}}_n$ of metric spaces and nonexpansive maps. Recall that a CCT category is a concrete category over $\underline{\text{Set}}$ which is topological. i.e.,

(a) each structured source has initial lift,
(b) every set carries only a set of structures,

and

(c) every constant function between two objects is a morphism,

and cartesian closed, i.e.,

(d) for arbitrary objects A and B there exists an object $[A,B]$ on the hom(A,B) such that, for each object C, morphism $h : C \times A \to B$ are precisely the functions for which $\hat{h} : C \to [A,B]$, defined by $c \mapsto h(c,-)$, is a morphism.

The CCT hull of a concrete category K is defined as the smallest CCT category L in which K is a full subcategory closed under finite products, see [**HN**]. A general construction of CCT hulls, covering all known examples, has been presented in [**ARS**]; for example, the CCT hull of the category of metric spaces and continuous mops is the category of functionally sequential topological spaces (= completely regular spaces in which every sequentially continuous real function is continuous). The CCT hull of the category of metric spaces and uniformly continuous maps has been described in [**AR**] as the category of bornological metrically generated uniform spaces.

We are going to describe the CCT hull of $\underline{\text{Met}}_n$ as a subcategory of the category of distance spaces (i.e., pseudometric spaces without the triangle inequality). A distance space is a set X together with a function $d : X \times X \to [0,+\infty]$ satisfying $d(x,x) = 0$ and $d(x,y) = d(y,x)$. The category of distance spaces and non–expansive maps (i.e., maps $f : A \to B$ with $d_A(x,y) \geq d_B(f(x),f(y))$ is denoted by $\underline{\text{Dist}}$. Observe that the full subcategory of pseudometric spaces is reflective in Dist: the reflection of (X,d) is obtained by the pseudometric modification d^* of the distance function d :

$$(1) \qquad d^*(x,y) = \inf\left\{ \sum_{i=0}^{n} d(u_i,u_{i+1}) \mid u_0,\dots,u_{n+1} \in X, u_0 = x, u_{n+1} = y \right\}.$$

The category Dist is CCT:

(a) each structured source $(X \xrightarrow{f_i} (y_i,d_i))_{i\in I}$ has an initial lift given by the following distance function on X : $d(x,x') = \sup_{i\in I} d_i(f_i(x),f_i(x'))$,
(b) every set carries only a set of distance functions,
(c) constant functions are nonexpansive,

and

(d) for arbitrary distance spaces $A = (X,d_A)$ and $B = (Y,d_B)$ the power object $[A,B]$ is $\hom(A,B)$ with the following distance function :

(2)
$$d(f,f') = \sup\{d_B(f(x)),f'(x')) \mid x,x' \in A \text{ with } d_A(x,x') < d_B(f(x),f'(x'))\}.$$

[In fact, given $C = (Z,d_C)$, a function $h : C\times A \to B$ is non–expansive iff for all $(c,a),(c',a') \in C\times A$, $d_B(h(c,a),h(c',a')) \leqq \inf\{d_C(c,c'),d_A(a,a')\}$, and this is equivalent to $d(h(c,-),h(c',-)) \leq d_C(c,c')$.]

Moreover, Dist is the quasitopos hull of $\underline{\mathrm{Met}}_n$, see [H].

Recall from [HN] that given a CCT category L and its full, concrete, finally dense subcategory K (i.e., each L-object is a final lift of some structured sink in K), then the CCT hull of K is the following full subcategory

$$(3) \qquad \mathrm{CCT}(K) = \{L \in L \mid \text{there exist an initial source } (L \xrightarrow{f_i} [A_i,B_i])_{i\in I} \text{ with } A_i,B_i \in K \text{ for all } i\}$$

of L. Since $\underline{\mathrm{Met}}_n$ is a full, concrete, finally dense subcategory of Dist (in fact, two–element pseudometric spaces are finally dense in Dist), we see that the CCT hull of $\underline{\mathrm{Met}}_n$ is the category of all distance spaces which are initial lifts of power–objects of metric spaces. We are going to describe those distance spaces explicitly :

Definition. A distance space (X,d) is called a demi–metric space provided that its pseudometric reflection (1) has the following properties :

(i) positivity: $d(x,y) > 0$ implies $d^*(x,y) > 0$,
(ii) lower semi–continuity: for arbitrary $x,y \in X$ and $K < d(x,y)$ there exists $\delta > 0$ such that $K < d(x',y')$ for all $x',y' \in X$ with $d^*(x,x') < \delta$ and $d^*(y,y') < \delta$.

Theorem. *The CCT hull of the category $\underline{\mathrm{Met}}_n$ is the category of demi–metric spaces and non–expansive maps.*

PROOF : We are to show that a distance space (X,d) is an initial lift of objects $[A,B], A,B \in \underline{\mathrm{Met}}_n$,iff it is a demi–metric space.

I. Necessity.

(a) We first prove that for metric spaces A and B, the power–object (2) is a demi–metric space.

Positivity follows from the fact that the distance function (2) majorizes the pseudometric $\rho(f,f') = \sup_{x\in A} d_B(f(x), f'(x))$: if $d(f,f') > 0$ then $f \neq f'$ and thus $\rho(f,f') > 0$, which implies $d^*(f,f') > 0$.

To verify the lower semi–continuity, let $f, f' : A \to B$ be non–expansive maps, and let $K < d(f,f')$ be given. Then, by (2), there exist $x, x' \in A$ with

$$K < d_B(f(x), f'(x')) \text{ and } d_B(f(x), f'(x') > d_A(x,x').$$

Choose any number $\delta > 0$ with $2\delta < d_B(f(x), f'(x')) - K$ and $2\delta < d_B(f(x), f'(x')) - d_A(x,x')$. Then we have the desired implication :

$$d^*(f,g) < \delta \text{ and } d^*(f',g') < \delta \text{ imply } K < d(g,g').$$

Indeed, since $\rho \leqq d$ implies $d_B(f(x), g(x)) < \delta$ and $d_B(f'(x'), g'(x')) < \delta$ and since d_B is a metric, we can make use of the triangle inequality to get

$$\begin{aligned} d_B(g(x), g'(x')) &\geqq d_B(f(x), f'(x')) - d_B(f(x), g(x)) - d_B(f'(x'), g'(x')) \\ &\geqq d_B(f(x), f'(x')) - 2\delta \\ &> d_A(x,x'). \end{aligned}$$

It follows that the pair (x,x') "counts" in the computation of $d(g,g')$, see (2). Thus,

$$d(g,g') \geqq d_B(g(x), g'(x')) \geqq d_B(f(x), f'(x')) - 2\delta > K.$$

(b) It remains to show that for each initial source $(C \xrightarrow{f_i} C_i)_I$ in Dist such that every C_i is a demi–metric space, so is C. We have $d_C(x,y) = \sup_{i\in I} d_{C_i}(f_i(x), f_i(y))$. For each i observe that $\rho_i(x,y) = d^*_{C_i}(f_i(x), f_i(y))$ is a pseudometric on C, and hence $\rho_i \leqq d_C$ implies $\rho_i \leqq d^*_C$.

The positivity of d^*_C is obvious : $d^*_C(x,y) = 0$ implies $\rho_i(x,y) = 0$ for each i, and hence $d_{C_i}(f_i(x), f_i(y)) = 0$ (by the positivity of $d^*_{C_i}$).

For the lower semi–continuity, let $K < d_C(x,y)$ be given. Then there is i with $K < d_{C_i}(f_i(x), f_i(y))$. By the lower semi–continuity of d_{C_i} there exists $\delta > 0$ such that whenever $d^*_C(x,x') < \delta$ [which, by $\rho_i \leq d^*_C$, implies $d^*_{C_i} * f_i(x), f_i(x')) < \delta$] and $d^*_C(y,y') < \delta$ [which implies $d^*_{C_i}(f_i(y), f_i(y')) < \delta$], then $K < d_{C_i}(f_i(x'), f_i(y')) \leqq d_C(x',y')$.

II. Sufficiency.

Let $A = (X, d_A)$ be a demi–metric space. For each pair $x, y \in X$ with $d_A(x,y) \neq 0$ and for each positive number $K < d(x,y)$ we will find a number $\varepsilon > 0$ and nonexpansive map

$$f : A \to [D_\varepsilon, R] (R = \text{real line}, D_\varepsilon = \{0,1\} \text{with}\, d(0,1) = \varepsilon)$$

such that the distance of $f(x)$ and $f(y)$ in $[D_\varepsilon, R]$ is larger or equal to K. It is then obvious that all those morphisms f form an initial source.

We can suppose that $d_A(x,y) > d^*_A(x,y)$ (since otherwise we can simply use the non-expansive map from A to R given by $u \mapsto \min\{d^*_A(x,u), K\}$). It follows (from the positivity) that $d^*_A(x,y) > 0$. By the lower semi–continuity there exists $\delta > 0$ such that $d^*_A(x,x') < \delta$ and $d^*_A(y,y') < \delta$ imply $K < d_A(x',y')$. We can assume without loss of generality that

$$\delta < \min\{d^*_A(x,y), K/2\}.$$

The following pseudometric

$$\rho(u,v) = \min\{d^*_A(u,v), \delta\} \qquad (u,v \in X)$$

fulfils $\rho(x,y) = \delta$. Define

$$f : A \to [D_{K-\delta}, R]$$

by the following rule:

$$(f(u))(0) = \rho(u,x) \text{ and } (f(u))(1) = K - \rho(u,y);$$

denote by d the distance in $[D_{K-\delta}, R]$, see (2).

We have to verify that (a) $f(u) \in \hom(D_{K-\delta}, R)$ for each $u \in X$, (b) f is non–expansive, i.e., $d_A(u,v) \geqq d(f(u), f(v))$ for all $u, v \in A$, and (c) $K \leqq d(f(x), f(y))$.

(a) Since $\rho \leqq \delta < \frac{K}{2}$ we have

$$\begin{aligned}|f(u)(1) - f(u)(0)| &= K - \rho(u,y) - \rho(u,x)\\ &\leqq K - \rho(x,y)\\ &= K - \delta\end{aligned}$$

and thus, $f(u)$ is non–expansive.

(b) We are to show that

$$|f(u)(i) - f(v)(i)| \leqq d_A(u,v) \qquad \text{for } i = 0,1$$

and that

$$|f(u)(1) - f(v)(0)| > K - \rho \text{ implies } |f(u)(1) - f(v)(0)| = d_A(u,v).$$

(By symmetry, the last holds with 1 and 0 switched too.) The first inequality is obvious : for $i = 0$ we have

$$|f(u)(0) - f(v)(0)| = |\rho(u,x) - \rho(v,x)| \leqq \rho(u,v) = d_A(u,v),$$

and analogously for $i = 1$. For the latter, observe that $\rho < \frac{K}{2}$ implies that the expression

$$f(u)(1) - f(v)(0) = K - \rho(u,v) - \rho(u,x)$$

is non–negative, and then, assuming then it is larger than $K - \delta$, we have $\rho(u,y) < \delta$ and $\rho(v,x) < \delta$. Consequently, $d_A^*(u,y) < \delta$ and $d_A^*(v,x) < \delta$, which implies $K < d_A(u,v)$ by the choice of δ. It follows that

$$|f(u)(1) - f(v)(0)| = K - \rho(u,y) - \rho(v,x) \leqq K < d_A(u,v).$$

(c) Since

$$|f(y)(1) - f(x)(0)| = K > K - \delta,$$

we have

$$d(f(x), f(y)) \geqq |f(y)(1) - f(x)(0)| = K. \qquad \blacksquare$$

Examples.

(1) For each real number $\varepsilon > 0$ we have the following demi–metric space R^2_ε: elements are pairs (x,y) of real numbers satisfying $|x - y| < \varepsilon$, and the distance of (x,y) and (x',y') is maximum of

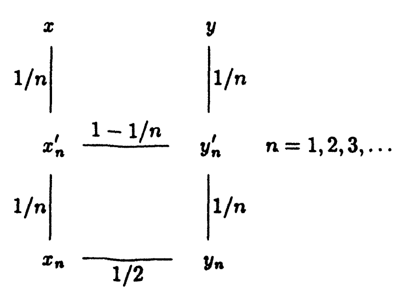
$$\begin{aligned}&|x - x'|\\ &|y - y'|\\ &|x - y'| \text{ counted only if } |x - y'| > \varepsilon, \text{ and}\\ &|x' - y| \text{ counted only if } |x' - y| > \varepsilon.\end{aligned}$$

In fact $R^2_\varepsilon \simeq [D\varepsilon, R]$, where R is the real line and $D_\varepsilon = \{0,1\}$ with $d(0,1) = \varepsilon$.

(2) Each subspace of a product $\prod_{i\in I} R^2_{\varepsilon_i}$ (with the supremum distance) is a demi–metric space.

Conversely, each demi–metric T_1 -space [i.e., such that $d(x,y) = 0$ implies $x = y$] is a subspace of a product of R^2_ε 's. This follows from the fact that R is initially dense in $\underline{\mathrm{Met}}_n$ and D_ε , $\varepsilon > 0$, are finally dense.

Remark. The semi–continuity in the definition of demi–metric cannot be considered w.r.t.d; i.e., there is a distance space A whose distance d is lower–semicontinuous w.r.t. itself but not w.r.t. d^* (and d^* is positive). In fact, consider the following space :



in which some distances are indicated and all the others are equal to 1. Here d is not lower–semicontinuous w.r.t. d^* since for the points x, y and for $K = \frac{2}{3}(< 1 = d(x,y))$ the points x_n, y_n fulfil $d^*(x, x_n) \leqq d(x, x'_n) + d(x'_n, x_n) = \frac{2}{n}$ and $d^*(y, y_n) \leqq \frac{2}{n}$ and yet, $d(x_n, y_n) < K$. However, d^* is positive, and d is lower–semicontinuous w.r.t. itself.

References

[AR] Adámek J. and Reiterman J., *Cartesian closed hull of the category of uniform spaces*, Topology and Appl. **19** (1985), 261–276.

[ARS] Adámek J., Reiterman J. and Strecker G.E., *Realization of cartesian closed topological hulls*, Manuscripta Math. **53** (1985), 1–33.

[H] Herrlich H., *Topological improvements of categories of structured sets*, Topol. and Appl. **27** (1987), 145–155.

[HN] Herrlich H. and Nel L.D., *Cartesian closed topological hulls*, Proceedings Amer. Math. Soc. **62** (1977), 215–222.

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