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INDEPENDENT SET and CLIQUE problems in intersection–defined classes of graphs

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Dedicated to the memory of Zdeněk Frolík

Abstract. We study the computational complexity of INDEPENDENT SET and CLIQUE problems restricted to intersection graphs of segments in the plane, and to some subclasses of this class of graphs.

Keywords: Graph, independent set, clique, computational complexity

Classification: 05C99

1. Preliminary.

In this paper we study the computational complexity of the following two problems

INDEPENDENT SET.
Input: A graph G and a positive integer k.
Question: Does there exist a set of k vertices of G no two of which are joined by an edge?
(A set of vertices of G which contains pair–wise nonadjacent vertices is called *independent* and the maximum size of an independent set is denoted by $\alpha(G)$.)

CLIQUE.
Input: A graph G and a positive integer k.
Question: Does there exist a set of k vertices of G with any two of its members joined by an edge?
(A maximal set of pairwise adjacent vertices is called a CLIQUE and the maximum size of a clique in G is denoted by $\omega(G)$.)

It is possible to say that these are two of basic combinatorial optimization problems. They were studied intensively and they are, in general, known to be NP–complete. Here we study these problems restricted to various classes of graphs which are induced by geometric configurations in the plane. The complexity of these problems is nearly completely answered.

Intersection graphs of geometric objects in the plane belong to popular and studied classes of graphs. The most general (if we restrict ourselves to arc–connected sets in the plane) is the class of *string graphs*, i.e. intersection graphs of curves in the plane [EET], [KGK], [Kral], [S]. Although INDEPENDENT SET and CLIQUE restricted to string graphs are NP–complete problems, they are polynomially solvable in various subclasses (interval graphs, permutation graphs, chordal graphs, circle graphs [Gav]). We will be concerned with intersection graphs of straight line segments in the plane, and in order to obtain sharper results, we consider the following classes

Definition. Let k be a positive integer. Denote by

SEG the class of intersection graphs of straight line segments in the plane;

k–DIR the class of intersection graphs of segments lying in at most k directions in the plane;

PURE–k–DIR the class of intersection graphs of segments lying in at most k directions in the plane, with the additional condition that any two parallel segments are disjoint.

(Let us state explicitely that a graph G is said to be an intersection graph of objects of some type, if there exists a collection R of some of these objects such that G is isomorphic to the intersection graph of R, i.e. to the graph $I(R) = (R, \{rs \mid r, s \in R, r \neq s \text{ and } r \cap s \neq \emptyset\})$. Then R is called a representation of G. In the sequel, we will say "a SEG graph" and "a SEG representation" instead of "a graph belonging to SEG" and "a representation by straight line segments", respectively (and similar for k–DIR and PURE–k–DIR graphs).)

Remark. These definitions generalize some particular cases studied earlier. For instance 1–DIR are exactly interval graphs. Note also that PURE–1–DIR are discrete graphs, with the exception of these two all the other classes are NP–complete or NP–hard to recognize [Kra2], [KM2].

Obviously k–DIR $\subset$ $(k+1)$–DIR, PURE–k–DIR $\subset$ PURE–$(k+1)$–DIR and PURE–k–DIR $\subset$ k–DIR hold true for every $k \geq 1$. We will also use the fact that every PURE–2–DIR graph is bipartite later on.

2. INDEPENDENT SET problem.

Theorem 1. *INDEPENDENT SET is polynomially solvable in 1–DIR and PURE–2–DIR, but it is NP-complete when restricted to 2–DIR and PURE–3–DIR.*

PROOF : INDEPENDENT SET problem is known to be solvable in polynomial time for interval (i.e. 1–DIR) graphs and for bipartite (and hence PURE–2–DIR) graphs [GJ]. The main point of the theorem lies in its NP–complete part.

Suppose G is a planar graph with maximum degree four (INDEPENDENT SET problem is NP–complete for such graphs [GJ]). Fix an embedding of G in a grid such that edges of G are piecewise linear curves following the grid lines (such an embedding in a linear sized grid always exists and is constructible in polynomial time [V]). For every edge e of G, let k_e denote the number of linear pieces the drawing of e consists of. Let G' be a graph obtained from G by subdividing every edge e with $2[(k_e+1)/2]+2$ vertices (i.e. by replacing every edge e by a path of length $2[(k_e+1)/2]+3$). It follows that

$$\alpha(G') = \alpha(G) + \sum_{e \in E(G)} ([(k_e+1)/2]+1)$$

and thus to complete the proof it suffices to show that $G' \in$ 2–DIR$\cap$PURE–3–DIR.

Having the grid embedding of G, construct a representation of G' as follows (see schematic Figure 1). Replace every vertex of G by a short horizontal segment. For an edge e, consider every linear piece of the drawing of e to be a segment of the representation. The pieces which were incident with the vertices of G are slightly shortened so that they do not intersect the short horizontal segments, and slightly shifted so that vertical segments lie on distinct lines. Finally, short segments are added (either overlapping in the case of 2–DIR or in the third direction in the case of PURE–3–DIR) near the segments representing the vertices of G so that the number of $2[(k_e+1)/2]+2$ segments on the path joining the endpoints of e is reached. (Note that this would not be possible to arrange without overlapping or using the third direction, since G' is in PURE–2–DIR iff G is bipartite.) ∎

3. How to describe a SEG representation.

Before we proceed to the CLIQUE problem we are going to discuss several ways in which a system of segments in the plane can be described. This is necessary because our polynomial algorithm for finding a clique number of a k–DIR graph G depends on knowing some facts about a k–DIR representation of G. We suggest 3 approaches:

1. Complete description. Here we describe the segments of the representation. We can do it formally as follows:

Given a graph $G = (V, E)$, we introduce functions $f_i\colon V \longrightarrow Z$, $i = 1,2,3,4$ so that a vertex $u \in V$ is represented by the segment with endpoints $[f_1(u), f_2(u)]$, $[f_3(u), f_4(u)]$. (Thus we describe the segments by giving the coordinates of their endpoints. Without loss of generality we restrict ourselves to segments with endpoints located in integer points.)

The size of such a description, i.e. the number of bits necessary to give is then

$$L = \sum_{u\in V} \sum_{i=1}^{4} \lceil \log_2 f_i(u) \rceil \leq 4n \lceil \log_2 x \rceil,$$

where $x = \max\{|f_i(u)| : i = 1,2,3,4,\ u \in V\}$ and $n = |V|$.

The advantage of this description is that given f_i, $i = 1,2,3,4$, one can easily decide (in time polynomial in n and L) whether the system of segments actually represents G, i.e. whether for every pair of distinct vertices $u, v \in V$, the segments $\overline{[f_1(u), f_2(u)][f_3(u), f_4(u)]}$ and $\overline{[f_1(v), f_2(v)][f_3(v), f_4(v)]}$ are disjoint if and only if $uv \notin E$.

On the other hand the disadvantage of the complete description is the fact, that there are graphs with $O(n^2)$ vertices which can be represented by segments, but every SEG representation of which requires a complete description of size at least 2^n [KM].

2. Partial description. Here we give a full description of the representation from the topological point of view, i.e. in the description we omit the fact that the segments are straight. We may consider the partial description as a result of the following procedure:

We take the representation R and consider all lines in the plane carrying at least one segment of R (we call such a line "a line determined by the representation"). Then we apply a homeomorphism of the plane which maps (injectively) the crossing points of the lines into the vertices of a linear sized (in $n = |V| = |R|$) grid. This is possible even with the images of the parts of the lines between the crossing points being straight since we can intérpret the lines determined by R as a drawing of a planar graph, and every planar graph is known to have a Fáry embedding in a polynomial sized grid [**FFP**]. Thus the partial description consists of

a) a set C of curves in the plane,
b) a mapping φ which assigns to every vertex $u \in V$ a segment $\varphi(u)$ on some curve $c \in C$.

(Note that b) is equivalent with

ba) saying which segment appears on which line and
bb) saying the orderings of the endpoints of the segments on the lines.)

The disadvantage of this partial description is that it is NP–hard to decide whether it actually describes an intersection graphs of straight line segments (i.e. whether the curves of the description can be stretched). This follows from the result of Shor [personal communication] on the stretchability of pseudoline arrangements.

On the other hand, every SEG representation of a SEG graph has a partial description of size $O(n \log n)$.

3. Discrete partial description. For our purpose it is not necessary to know the lines of the representation (nor the curves of the partial description), only the incidencies are important. So we consider two sets of points – the set of crossing points of the lines determined by R (denoted by $C(R)$) and the set of the endpoints of the segment (denoted by $E(R)$). Then the discrete partial description consists of

c) a set X $(=C(R) \cup E(R))$,
d) a set $\mathcal{A}$ of subsets of X (the lines determined by R),
e) a mapping ψ which assigns to every vertex $u \in V$ a subset $\psi(u)$ of X (the endpoints and crossing points which appear on the segment representing the vertex u).

Note that $|C(R)| \leq \binom{n}{2}$ and $E(R) \leq 2n$ where $n = |V|$, and $\psi(u) \geq 2$ for every $e \in V$ (we do not consider a single point to be a segment). It is also straightforward that $\psi(u) \cap \psi(v) \neq \emptyset$ if and only if $u = v$ or $uv \in E$, provided R was a representation of $G = (V, E)$.

The disadvantage of the discrete partial description is that it is NP–hard to recover the partial description from it.

The main advantage of this description we see in its general form (cf. the following section). It grasps the properties of SEG representations on which our algorithm for finding a maximum clique in a k–DIR graph is based. Thus though we relate our result mainly to intersection graphs of segments, the algorithm introduced in the following section computes the clique number in polynomial time for considerably wider class of graphs.

4. CLIQUE problem.

Throughout this section we suppose that $G = (V, E)$ is a graph with n vertices.

If G is in PURE–k–DIR, $\omega(G) \leq k$ and one can find $\omega(G)$ in time $\leq O(n^k)$, provided k is fixed. In the sequel, we are going to present a polynomial algorithm that determines $\omega(G)$ for k–DIR graphs. (Note that in this case there is no upper bound for $\omega(G)$ except of $\omega(G) \leq n$.) In fact our algorithm works for a larger class of graphs.

Definition. We say that (X, f) where X is a finite set and f: $V \longrightarrow \exp X$ assigns to every vertex of G a subset of X is a *finite (intersection) representation* of G if for every pair of distinct vertices u, $v \in V$, $f(u) \cap f(v) \neq \emptyset$ if and only if $uv \in E$.

A triple $(X, \mathcal{A}, f)$ where (X, f) is a finite representation of G and $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ is a set of some subsets of X is called a *special representation* of G if

i) $|A_i \cap A_j| \leq 1$ for every $1 \leq i < j \leq r$,
ii) for every $u \in V$ there is exactly one i such that $f(u) \subset A_i$,
iii) for every i, the set $U_i = \{f(u)\colon f(u) \subset A_i,\ u \in V\}$ is nonepmty and has Helly number 2, i.e. for every $U \subset U_i$ such that $\cap U = \emptyset$ there exist u, $v \in V$ with $f(u)$, $f(v) \in U$ and $f(u) \cap f(v) = \emptyset$.

In addition $(X, \mathcal{A}, f)$ is called *k-special* if

iv) for every $M \subset \{1, 2, \ldots, r\}$ such that $A_i \cap A_j \neq \emptyset$ for all i, $j \in M$, M has size at most k (i.e. every subset of $\mathcal{A}$ of size $k+1$ contains two disjoint A_i's).

Theorem 2. *Let k be a fixed positive integer and let $(X, \mathcal{A}, f)$ be a k-special representation of G. Then one can compute $\omega(G)$ in time $\leq O(rmn + kr^k(km + k^2 n)) \leq O(n^k(m+n))$, where $m = |X|$ and $r = |\mathcal{A}|$.*

PROOF : For $1 \leq i \neq j \leq r$, denote by I_{ij} the intersecting point of A_i and A_j providing the intersection $A_i \cap A_j$ is nonempty (if A_i and A_j are disjoint I_{ij} remains undefined).

For $i = 1, 2, \ldots, r$, put

$$\omega_i = \max\{|U|\colon U \subset V,\ \bigcup_{u \in U} f(u) \subset A_i,\ f(u) \cap f(v) \neq \emptyset \text{ for all } u,\ v \in U\}.$$

Claim 1. *Suppose $C \subset V$ is a clique in G, i.e. C is a maximal set of vertices such that the sets $f(u)$, $u \in C$ have pairwise a nonempty intersection. Put $M = \{i\colon \exists u \in C$ with $f(u) \subset A_i\}$. Then either $|M| = 1$ and $|C| \leq \omega_i$ for some i, or $|M| > 1$ and then M is such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in M$ and $C = \bigcup_{i \in M} U_{iM}$, where*

$$U_{iM} = \{u\colon u \in V,\ f(u) \subset A_i \text{ and } \{I_{ij}\colon j \in M,\ i \neq j\} \subset f(u)\}.$$

PROOF of Claim 1: The case $|M| = 1$ is straightforward. If $|M| > 1$, then $\emptyset \neq f(u) \cap f(v) \subset A_i \cap A_j \subset \{I_{ij}\}$ whenever $f(u) \subset A_i$, $f(v) \subset A_j$ and $u, v \in C$. Hence $A_i \cap A_j \neq \emptyset$ and $I_{ij} \in f(u) \cap f(v)$. ∎

Now the algorithm is clear

Algorithm.

1. For every $i = 1,2,\ldots,r$ compute ω_i and put $\Omega_1 = \max_{i=1,2,\ldots,r} \omega_i$.

2. For every $M \subset \{1,2,\ldots,r\}$ such that $2 \leq |M| \leq k$ and such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in M$ compute $\omega_M = \sum_{i \in M} |U_{iM}|$. Put $\Omega_2 = \max \omega_M$.

3. Set $\omega(G) = \max\{\Omega_1, \Omega_2\}$.

To justify the upper bound on the running time of the algorithm we have the following two claims

Claim 2. *For every i, ω_i can be computed in $O(mn)$ steps.*

PROOF of Claim 2: Due to the Helly property iii), $\omega_i = \max_{x \in A_i} \omega_{ix}$, where $\omega_{ix} = |\{u : u \in V, x \in f(u) \subset A_i\}|$. ∎

Claim 3. *For every $M \subset \{1,2,\ldots,r\}$, $|M| \geq 2$, ω_M can be computed in $O(m|M| + n|M|^2) \leq O(km + k^2 n) = O(m+n)$ steps.*

PROOF of Claim 3: First we find the crossing points I_{ij}, $i \neq j \in M$ (in time $\leq O(m|M|) \leq O(mk) = O(m)$). Then we check which $f(u)$'s contain all of them (in time $O(n|M|^2) \leq O(nk^2) = O(n)$). ∎

Since it follows from ii) that $r \leq n$, step 1 of the algorithm requires running time $O(rmn) \leq O(mn^2)$, while step 2 requires time $\leq kr^k \cdot O(km + k^2 n) \leq O(k^3 n^k (m + n))$. This concludes the proof of Theorem 2. ∎

Remark 4.1. Note that given k and a special representation $(X, \mathcal{A}, f)$, one can check in time $\leq O(mn^{k+1})$ whether $(X, \mathcal{A}, f)$ is k-special. Also it is not necessary to save the lines $\mathcal{A}$ in the description. If we suppose that $|f(u) \cap f(v)| \geq 2$ whenever $f(u)$, $f(v)$ lie on the same line (and are not disjoint), we can recover the lines $\mathcal{A}$ just from (X, f).

Let us now return to the intersection graphs of segments in the plane.

Lemma 4.2. *Let $(X, \mathcal{A}, \psi)$ be a discrete partial description of SEG representation R of a graph G. Then $(X, \mathcal{A}, \psi)$ is a special representation of G. Moreover, if R is a k-DIR representation then $(X, \mathcal{A}, \psi)$ is k-special.*

PROOF :

i) is straightforward.
ii) follows form i) (note that $|\psi(u)| \geq 2$ for every $u \in V$).
iii) It is well known that a system of intervals on a line has Helly number 2, and the same is true for intervals in a discrete linearly ordered set.
iv) is again clear since the lines parallel with the same direction are either identical or disjoint.

∎

Corollary 4.3. *Given a discrete partial description of a k-DIR graph G, one can compute $\omega(G)$ in $O(k^3 n^{k+2})$ steps.*

Remarks 4.4. A more careful investigation shows that given a k-special representation encoded in a suitable data structure one can compute the clique number in time $O(n^{k+1})$.

4.5. We know that the condition iv) is in general necessary. More precisely, given a special representation of G, it is still NP–hard to compute $\omega(G)$. However, we do not know whether this is also true in the case of geometric representations, i.e. whether computing $\omega(G)$ for SEG graphs is more difficult than for k–DIR graphs. Particularly, this leaves us with the following open question

Problem. *Determine the computational complexity of the CLIQUE problem restricted to intersection graphs of straight line segments in the plane.*

4.6. Middendorf and Pfeiffer [MP] proved the NP–completeness of CLIQUE for special class of string graphs, having a representation which every curve is either a straight line segment, or consists of two straight segments parallel with axis all being of the same type (say $\ulcorner$, i.e. the upper left corner of some isothetic rectangle). This is presently as far as one can go in solving the above problem.

4.7. Note that the answer to the above problem may vary depending on whether the size of the input is measured as the size of the intersection graph or as the size of the complete description of a SEG representation (cf. section 3).

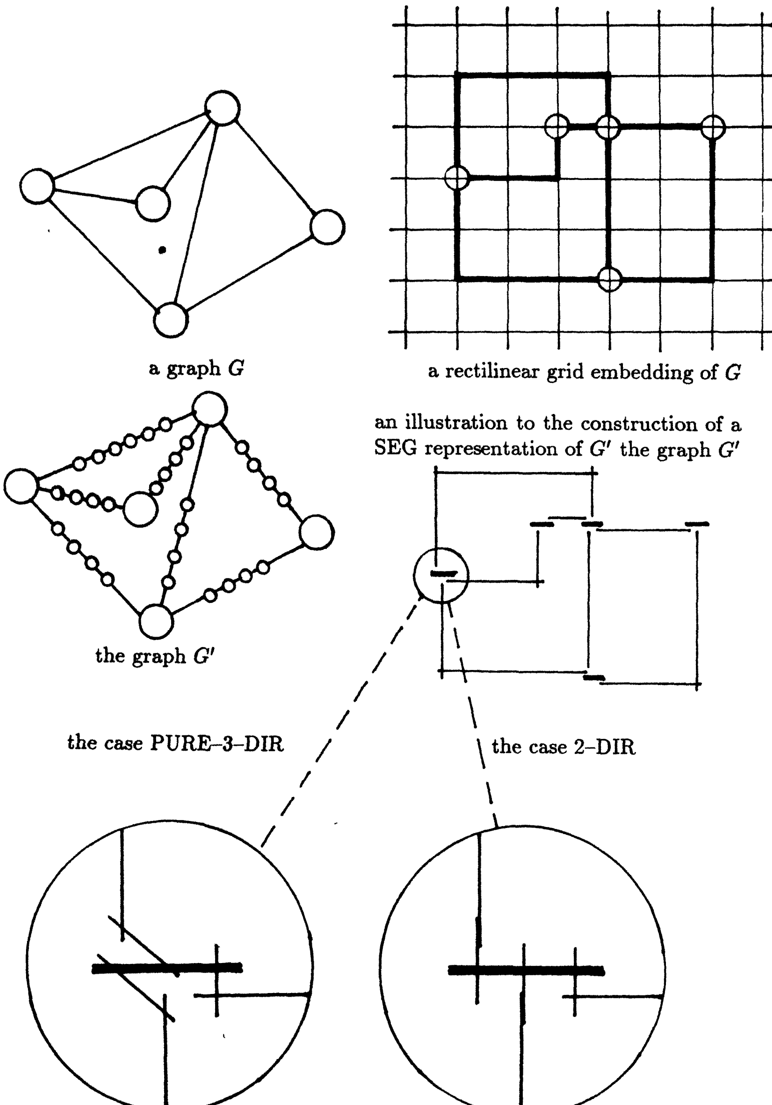
a graph G
a rectilinear grid embedding of G
the graph G'
an illustration to the construction of a SEG representation of G' the graph G'
the case PURE–3–DIR
the case 2–DIR

Figure 1

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