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Existence of unstable sets for invariant sets in compact semiflows. Applications in order–preserving semiflows

PETER POLÁČIK

Abstract. A compact local semiflow on a metric space is considered. Conditions are presented which guarantee that for a given compact invariant set K there exists a negative orbit whose α–limit set is contained in K. By checking these conditions, it is proved that such a negative orbit exists if K is an unstable isolated equilibrium in a gradient–like semiflow or a compact invariant set, not containing equilibria, in a strongly order preserving semiflow.

Keywords: Semiflow, invariant set, limit set, order–preserving

Classification: 58F25, 35B40, 34C35

1. Introduction.

Consider a (local) semiflow $S(t)$, $t \geq 0$, on a metric space(X, d). Let $K \subset X$ be a compact set, invariant under the semiflow $S(t)$. In this paper we show some conditions guaranteeing that there exists a nontrivial unstable set of K. More precisely, these conditions imply that there exists a relatively compact negative orbit whose α–limit set is contained in K.

Our investigation was inspired by a result of Matano. In [M1], he has proved that in a certain class of semiflows, an unstable equilibrium always has a nontrivial unstable set. Specifically, he assumes that X is an ordered Banach space and the semiflow preserves the ordering in the following strong sense: For any $t > 0$ and any two different related points $x \leq y$ in the domain of $S(t)$, the relation $S(t)\tilde{x} \leq S(t)\tilde{y}$ is satisfied for all $\tilde{x}$, $\tilde{y}$ sufficiently close to x, y, respectively. Under certain compactness conditions, he has proved existence of a negative orbit, which lies above a given unstable equilibrium and approaches this equilibrium as $t \to -\infty$. This result has appeared useful in various applications. In particular, it can be used to establish an orbit connection between related equilibria (see [M1], [M3], where also further applications can be found).

Our approach to the question of existence of nontrivial unstable sets equilibria, and more general compact invariant sets, is different from that of Matano. We do not restrict our consideration to a special class of semiflows from the very beginning. Instead, we start by identifying a general property of a given invariant set, which ensures existence of a nontrivial unstable set. Then, by checking this property, we establish existence of such sets in various particular cases. Thus, for strongly order–preserving semiflows we prove that any compact invariant set, which does not contain a stable equilibrium, has a nontrivial unstable set. This extends the result of Matano mentioned above. As an application, we prove existence of an orbit connection from a compact invariant set to an equilibrium.

Another consequence of our general theorem asserts that any unstable isolated compact invariant set has a nontrivial unstable set. In particular, any isolated unstable equilibrium in a gradient-like semiflow has such a set.

We now prepare formulation of the main theorem by recalling some definitions.

By a *negative orbit* we mean a function $v\colon (-\infty, 0] \longrightarrow X$ such that $S(t)v(s) = v(t+s)$ for all $t \geq 0 \geq s$ with $t + s \leq 0$. If $v(0) = x$, we say that v is a negative orbit of x. For a negative orbit $v(\,.\,)$, the α-limit set $\alpha(v(\,.\,))$ of $v(\,.\,)$ is the set of all limit points of $v(t)$ as $t \to -\infty$. Of course, $\alpha(v(\,.\,))$ is nonempty, provided $v(\,.\,)$ (more precisely its image) is relatively compact.

A set $K \subset X$ is called *positively invariant* if for any $t \geq 0$, K is in the domain $D(S(t))$ of $S(t)$ and $S(t)K \subset K$. K is called *negatively invariant* if any element of K has a negative orbit which takes values in K. If $K \neq \emptyset$ is both negatively and positively invariant, it is called *invariant*.

We use the notation $\mathrm{cl}(U)$ or $\overline{U}$ for the closure of a set U and ∂U for the boundary of U. By $\mathrm{dist}(x, B)$ we denote the distance from a point x to a set A: $\mathrm{dist}(x, A) = \inf\{d(x,y)\colon y \in A\}$.

We assume the following *compactness* of the semiflow.

(C) For any bounded set $G \subset X$ there exists a $\delta > 0$ such that $G \subset D(S(\delta))$ and $S(\delta)G$ is relatively compact.

The main result of this paper reads as follows.

Theorem 1. *Let (C) hold. Let $K \subset X$ be a compact positively invariant set. Assume that there exist a neighbourhood U of K and a set $M \subset \overline{U}$ with the following properties:*

(i) *M is closed.*
(ii) *$K \subset M$.*
(iii) *K is not open in M (i.e., there exists a sequence $y_n \in M \setminus K$ such that $\mathrm{dist}(y_n, K) \to 0$ as $n \to \infty$).*
(iv) *For any $y \in M \setminus K$ there exists a constant $\tau > 0$ such that $S(t)y \in M$ for all $t \in [0, \tau]$ and $S(\tau)y \in \partial U$.*

Then there exists a negative orbit $v(\,.\,)$ with relatively compact image such that $v(t) \in M \setminus K$ for all $t \geq 0$ and $\alpha(v(\,.\,)) \subset K$.

Freely speaking, the hypotheses of Theorem 1 require that there exist a closed set $M \supset K$, which is locally positively invariant (i.e., any positive orbit starting in M remains in M as long as it stays close to K), and in which K is totally unstable. By total instability we mean that there is a neighbourhood of K in M from which every positive orbit eventually escapes, unless it lies in K. We return to the discussion of these hypotheses in Section 3.

The idea of the proof of this theorem is quite simple. Consider the ω-limit set $\omega(M)$ of the set M (see Section 2 for the definition). At this point assume that the orbit

$$\bigcup_{t\geq 0} S(t)M$$

of M is relatively compact. By well known property [H1], [H2], this ensures that $\omega(M)$ is nonempty , compact and invariant. Hence through any point y in $\omega(M)$ there exists a relatively compact negative orbit. The instability property (iv) guarantees that if such a y is chosen sufficiently close to K, then its negative orbit (at least one of them) satisfies the conclusion of Theorem 1.

Theorem 1, together with some immediate consequences, is proved in Section 3. In Section 2 we have collected definitions and basic properties of semiflows used throughout the paper. In that section we also introduce the concept of a relative ω-limit set of a set, which allows to skip the unnatural assumption of compactness of the orbit of M, as assumed in the above outline.

A major part of the paper (Section 4) is devoted to strongly order preserving semiflows. We derive from Theorem 1 the existence theorem of Matano and prove its extension indicated above. Then we prove that for any minimal compact invariant set there exists an entire orbit connecting this invariant set to an equilibrium.

We do not show concrete examples of strongly order preserving semiflows in this paper. We refer an interested reader to the papers [Hi4–6], [M1–3], [MP], [P1–3] for examples in parabolic equations and to [Hi1–4], [Mi], [S1–2], [ST] for examples in ordinary and delay differential equations.

The paper is finished with some remarks concerning extensions of our results to discrete dynamical systems (Section 5).

2. Invariance of limit sets.

In this preliminary section we introduce further definitions and state some properties of the limit sets which we need below.

Throughout the paper, (X,d) is a metric space and $S(t)$ is a local semiflow (for brevity only semiflow) on X. The latter means that for any $t\geq 0$, $S(t)$ is a mapping of an open subset $D(S(t))\subset X$ into X such that the following conditions are satisfied.

(a) $D(S(0))=X$ and $S(0)$ is the identity on X.
(b) The set $D(S):=\{(t,x)\colon t\geq 0,\ x\in D(S(t))\}$ is open in $[0,+\infty)\times X$ and the mapping $(x,t)\longmapsto S(t)x$ is continuous on $D(S)$.
(c) For any $t,\ t'\geq 0$ we have $D(S(t+t'))=S(t')^{-1}D(S(t))$ and $S(t+t')x=S(t)S(t')x$ for any $x\in D(S(t+t'))$.

For $x\in X$ we denote by τ_x the escape time of x:

$$\tau_x=\sup\{t\geq 0\colon x\in D(S(t))\}$$

The positive orbit (or the trajectory) of x is the set

$$O^+(x)=\{S(t)x\colon 0\leq t<\tau_x\}$$

The positive orbit $O^+(H)$ of a set $H \subset X$ is the union of the positive orbits of all elements of H.

The ω–limit set of a point x with a global trajectory (i.e., $\tau_x = +\infty$) is defined by

$$\omega(x) = \bigcap_{s>0} \mathrm{cl}\left(\bigcup_{t>s} S(t)x\right)$$

Replacing in this definition x by H, where H is a set of points with global trajectories, we obtain the definition of the ω–limit set of H.

It is well known (see e.g. [H1], [H2]) that the ω–limit set of a point with a relatively compact trajectory is nonempty, compact, connected and invariant. The same properties has the α–limit set of a relatively compact negative orbit (as defined in the introduction). Moreover, relatively compact trajectories have the following property:

$$\mathrm{dist}(S(t)x, \omega(x)) \to 0 \text{ as } t \to \infty .$$

The statements concerning the ω–limit sets hold true if the positive orbit of a connected set H (instead of a point) is considered. We now state a modified version of the latter property. For this we need the following definitions.

Let $H \subset G$ be two subsets of X. We define the *positive orbit of H relative to G* by

$$O_G^+(H) := \{S(t)x \colon x \in H,\, 0 \le t < \tau_x \text{ and } S(s)x \in G \text{ for all } s \in [0,t]\} .$$

Clearly, if G is open, then $O_G^+(H)$ is just the positive orbit of H for the restricted semiflow $S(t)\,|_G$, $t \ge 0$.

If $H \subset G$, and G is closed, we define the *ω–limit set of H relative to G* by

$$\omega_G(H) := \{y \in X \colon \text{there are two sequences } x_n \in H \text{ and } t_n \to +\infty \text{ such that } S(s)x_n \in G \text{ for all } s \in [0,t_n] \text{ and } S(t_n)x_n \to y\} .$$

Clearly, $\omega_G(H) \subset G$ and

$$\omega_G(H) = \bigcap_{s>0} \mathrm{cl}\left(\bigcup_{t>s} S(t)H_t\right) , \tag{2.1}$$

where

$$H_t := \{x \in H \colon S(s)x \in G \text{ for all } s \in [0,t]\} .$$

Using (2.1), an obvious modification of standard arguments (see e.g. [H1], [H2]) proves the following property:

Lemma 2.1. *Let $G \subset X$ be closed and let $H \subset G$. If $O_G^+(H)$ is relatively compact, then $\omega_G(H)$ is compact and negatively invariant. Moreover,*

$$\mathrm{dist}(S(t)H_t, \omega_G(H)) \to 0 \textit{ as } t \to +\infty .$$

Note that $\omega_G(H)$ may be empty.

Remark 2.2. From (2.1) it follows that $\omega_G(H) = \omega_G(S(t)H_t)$ for any $t > 0$. Therefore the conclusion of Lemma 2.1 remains valid if instead of compactness of $O_G^+(H)$ one assumes that $O_G^+(S(t)H_t)$ is relatively compact for some $t > 0$. The latter holds provided there is a $\delta > 0$ such that $G \subset D(S(\delta))$ and $S(\delta)G$ is relatively compact. This is easily seen from the inclusions

$$O_G^+(S(\delta)H_\delta) = \bigcup_{t\geq\delta} S(t)H_t = S(\delta)\bigcup_{t\geq\delta} S(t-\delta)H_t \subset S(\delta)G\,.$$

3. Existence of unstable sets.

In this section we prove Theorem 1 and a few related results. Before doing so, we make some remarks concerning the hypotheses of this theorem. In the introduction we have rephrased them as requiring that K is totally unstable in some locally positively invariant closed set M. As we will verify in a moment, if closedness of M is not required, then such a set M can be found for any unstable positively invariant set K. Still there need not exist a nontrivial unstable set of K. So closedness of M cannot be omitted. However, it can be replaced by another assumption (see Theorem 1' below).

Let K be a compact positively invariant set. We show that if K is unstable, then a neighbourhood U and a set M can be found, which have all the properties required in Theorem 1, possibly except for (i) (closedness of M). Recall that K is stable in case that for any neighbourhood V of K there exists a neighbourhood W such that for any $x \in W$ we have $\tau_x = +\infty$ and $O^+(x) \subset V$. So if K is unstable (i.e. not stable), then there exists a neighbourhood U of K and a sequence $x_n \in U$ such that $\operatorname{dist}(x_n, K) \to 0$ and for any n we have either $\tau_{x_n} < +\infty$ or $O^+(x_n)$ is not contained in U. By compactness of K, neighbourhood U can be chosen bounded. Hence, if we assume the compactness condition (C), then any trajectory starting in U is either contained in U (and then it is global by (C)) or else it hits ∂U. It is then easy to see that if we put

$$M := K \cup \bigcup_{n=1}^{\infty} O_{\overline{U}}^+(x_n)\,, \tag{3.1}$$

then $M \subset \overline{U}$ and all the conditions (ii)–(iv) of Theorem 1 are satisfied. (Recall that $O_{\overline{U}}^+(x)$ denotes the positive orbit of x relative to $\overline{U}$). We now borrow a simple example from [M1] which shows that without additional assumptions (like (i)) there need not exist a nontrivial unstable set of K. Consider the system

$$\dot{x} = y, \quad \dot{y} = 0\,. \tag{3.2}$$

The origin (as well as any other point on the x–axis) is an unstable equilibrium of this system and there is no orbit approaching the origin in the backward direction.

This example is rather special in that 0 is not an isolated equilibrium. As we shall see, the assumption that there is no compact invariant set in $\overline{M} \setminus K$ can replace (i) in Theorem 1.

Theorem 1'. *Let the hypotheses of Theorem 1 be satisfied with (i) replaced by the following:*

(i') *If $\tilde{K} \subset M$ is compact and invariant then $\tilde{K} \subset K$.*

Then there exists a negative orbit $v(\,.\,)$ with relatively compact image such that $v(t) \in \overline{M} \setminus K$ for all $t \leq 0$ and $\alpha(v(\,.\,)) \subset K$.

PROOF of Theorems 1, 1': Assume that a neighbourhood U of K and a set $M \subset \overline{U}$ satisfy (ii)–(iv) and (i) or (i'). Choose a bounded neighbourhood V of K such that $\overline{V} \subset U$ and let $H := M \cap V$. Consider the relative positive orbit $O^+_{\overline{V}}(H)$. By (C) and Lemma 2.1, $O^+_{\overline{V}}(H)$ has a compact negatively invariant limit set $\omega_{\overline{V}}(H)$ (cf. Remark 2.2). Since $O^+_{\overline{V}}(H) \subset M$, by (iv) we have

$$\omega_{\overline{V}}(H) \subset \overline{M} \cap \overline{V}. \tag{3.3}$$

We now proceed in two steps. First we prove that there exists an $x \in \omega_{\overline{V}}(H) \setminus K$. Then we verify that the negative orbit of x (which exists by invariance) has its α–limit set in K.

Consider the sequence y_n as in (iii). By compactness of K, passing to a subsequence we may assume that $y_n \to y \in K$. By positive invariance of K, the positive orbit of y is global and lies in $K \subset V$. Using this and continuity of the semiflow, we obtain that the positive orbit of y_n stays in V for an arbitrarily long prescribed time if n is large enough (formally, for any $T > 0$ there is an n_0 such that $S(t)y_n \in V$ for all $0 \leq t \leq T$ and $n > n_0$). This, in conjunction with (iv), implies that there is a sequence $t_n \to +\infty$ such that $S(t_n)y_n \in V$ for $0 \leq t \leq t_n$ and $S(t_n)y_n \in \partial V$. Since the points $S(t_n)y_n = S(\delta)S(t_n - \delta)y_n$ lie in a relatively compact set $S(\delta)V$, we can find a convergent subsequence of this sequence. Clearly the limit is an element of $\omega_{\overline{V}}(H) \cap \partial V \subset \omega_{\overline{V}}(H) \setminus K$.

Now take any $x \in \omega_{\overline{V}}(H) \setminus K$. By compactness and negative invariance of $\omega_{\overline{V}}(H)$, x has a relatively compact negative orbit $v(\,.\,)$ such that $v(t) \in \omega_{\overline{V}}(H)$ for all $t \leq 0$. By (3.3), $v(t) \in \overline{M}$ for all $t \leq 0$. Since $x = v(0) \notin K$, we have $v(t) \notin K$ for $t < 0$ (otherwise $x \in K$, by positive invariance). It remains to prove that $\alpha(v(\,.\,)) \subset K$. This is trivial if (i') is assumed (for $\alpha(v(\,.\,)) \subset \overline{M}$ is a compact invariant set). This proves Theorem 1'. Assume (i). By invariance, for any $z \in \alpha(v(\,.\,))$ we have $O^+(z) \subset \alpha(v(\,.\,)) \subset M \cap \overline{V} \subset M \cap U$. Thus $z \notin K$ would clearly contradict (iv). This proves $\alpha(v(\,.\,)) \subset K$ and completes the proof. ∎

We now state two consequences. First recall that a compact invariant set K is called an ***isolated compact invariant set*** if it is a maximal compact invariant set in some neighbourhood of K.

Corollary 3.1. *Let (C) hold. Let K be an isolated compact invariant set. If K is unstable, then there exists a negative orbit $v(\,.\,)$ with relatively compact image disjoint from K such that $\alpha(v(\,.\,)) \subset K$.*

PROOF : As was remarked above, there exist a neighbourhood U of K and a set $M \subset \overline{U}$ satisfying (ii)–(iv) of Theorem 1 (see (3.2)). Shrinking U, if necessary, we may assume that any compact invariant set $\widetilde{K} \subset \overline{U}$ is contained in K. It follows that assumption (i') of Theorem 1' is satisfied. The conclusion now follows from Theorem 1'. ∎

The next simple consequence of Theorem 1' establishes an unstable set of an unstable isolated equilibrium in a gradient-like semiflow, i.e., in a semiflow with a Lyapunov function. The latter means that there exists a continuous function $V: X \longrightarrow \mathbf{R}$ such that for any $x \in X$ the function $t \longmapsto V(S(t)x)$ is strictly decreasing, unless x is an equilibrium. It is well-known that if $S(t)$ is gradient-like, then the α-limit set of any compact negative orbit consists of equilibria. This property is what we actually need in the next proposition.

Proposition 3.3. *Let (C) hold. Assume that the α-limit set of any relatively compact negative orbit consists of equilibria. Then for any unstable isolated equilibrium e, there exists a relatively compact negative orbit $v(\,.\,)$ such that $v(t) \neq e$ for all $t \leq 0$ and $\alpha(v(\,.\,)) = \{e\}$.*

PROOF : If $\{e\}$ is an isolated compact invariant set, then the conclusion follows from Corollary 3.2. If, on the contrary, $\{e\}$ is not an isolated compact invariant set, then there exist a neighbourhood U of e, not containing other equilibria, and a compact invariant set $\widetilde{K}$, $\{e\} \neq \widetilde{K} \subset U$. So there exists a negative orbit $v(\,.\,)$ with relatively compact image contained in $\widetilde{K} \setminus e$. Since, by assumption, $\alpha(v(\,.\,))$ consists of equilibria, the choice of U implies $\alpha(v(\,.\,)) = \{e\}$. This proves Proposition 3.3. ∎

We finish this section with one more remark. Return to the system 3.2. It is a special case of the plane coöperative system, i.e., system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

where f and g are C^1 with the nonnegative partial derivatives f_y, g_x. In such a system, the α-limit set of any bounded negative orbit is an equilibrium [**Hi1**]. Thus, by Proposition 3.3, the only obstacle, which prevents an unstable equilibrium from having a nontrivial unstable set, is this equilibrium not being an isolated equilibrium (like in (3.2)).

4. Order-preserving semiflows.

In this section we apply Theorem 1 in strongly order-preserving semiflows. We reprove, using this theorem, Matano's result establishing a nontrivial unstable set for a semiunstable equilibrium. Then we proceed with an investigation of compact invariant sets not containing an equilibrium. In strongly order-preserving semiflows such sets are automatically unstable (by a result of Hirsch), and we prove existence of unstable sets for them. As a consequence of this result, we then prove existence of a connecting orbit from a compact invariant set to an equilibrium.

From now on we assume that X is an ordered metric space. This means that there is a partial ordering $\leq$ on X, which is compatible with the topology: $x \leq y$ whenever x, y are the limits of sequences $x_n \leq y_n$. In other words, the relation $\leq$ is closed in $X \times X$.

For two points we write $x < y$ if $x \le y$ and $x \ne y$. For two sets A, $B \subset X$ we write $A \le B$ ($A < B$) if $x \le y$ ($x < y$) whenever $x \in A$, $y \in B$. The reserved inequality signs are used in the usual way.

Given a set $A \subset X$ we denote

$$X^+(A) := \{y \in X : y \ge x \text{ for some } x \in A\}.$$

In particular, for $x \in X$, $X^+(x) = \{y \in X : y \ge x\}$.

A semiflow $S(t)$ is said to be *order-preserving* (or *monotone*) if $x \le y$ implies $S(t)x \le S(t)y$ for all $0 \le t < \min\{\tau_x, \tau_y\}$. Following [M3], we call $S(t)$ *strongly order-preserving* if it is order-preserving and for any $x < y$ there exist a $t_0 > 0$ and neighbourhoods U, V of x, y, respectively, such that

$$U \cup V \subset D(S(t_0)) \text{ and } S(t_0)U < S(t_0)V .$$

In the whole section we assume that $S(t)$ is an order-preserving semiflow.

Let E denote the set of all equilibria of $S(t)$. Note that if $e \in E$ then, by monotonicity, the set $X^+(e)$ contains positive orbits of all its elements. Thus $S(t)$ restricts to a semiflow on $X^+(e)$ (taken with the induced metric). An equilibrium e is said to be unstable from above if it is unstable for the restricted semiflow on $X^+(e)$.

For a convenient application of Theorem 1 the following lemma is useful. (Note that a similar property is implicitly contained in the proof of Theorem 5 in [M1].)

Lemma 4.1. *Assume that $S(t)$ is strongly order-preserving and that (C) holds. Let e be an equilibrium unstable from above. Then there exists a neighbourhood U of e with the following property: For any $x \in X^+(e) \cap U$, $x \ne e$, there exists a $t \in [0, \tau_x)$ such that $S(t)x \notin U$.*

In the proof of this lemma and later in this section we shall use the following lemma which follows immediately from the strong order-preservence.

Lemma 4.2. *Assume that $S(t)$ is strongly order-preserving. Let $N \subset X$ be a compact set and let $z \in X$ satisfy $N < z$. Then there exist a $t_1 > 0$ and neighbourhoods U and V of z and N, respectively, such that*

$$S(t_1)U < S(t_1)V .$$

An analogous statement holds for the reserved inequality sign.

PROOF of Lemma 4.1: By instability of e, there exist an $\varepsilon_0 > 0$ and sequences $x_n \in X^+(e)$, $t_n \in (0, +\infty)$ such that

$$x_n \to x ,$$

$$d(S(t)x_n, e) < \varepsilon_0 \text{ for } t \in [0, t_n),$$

$$d(S(t_n)x_n, e) = \varepsilon_0 .$$

(Here we have used (C), which guarantees that the distance ε_0 is actually achieved.) Continuity of the semiflow implies $t_N \to +\infty$. From this and (C) (where we take $G = B(e, \varepsilon_0)$ – the ε_0–ball with center e) we easily obtain that the set $N := \{S(t_n)x_n : n = 1, 2, \dots\}$ is relatively compact (just observe that $S(t_n))x_n = S(\delta)S(t_n - \delta)x_n \in S(\delta)G$). By Lemma 4.2, there exist a t_1 and a neighbourhood U of e such that

$$S(t_1)U < S(t_1)N . \tag{4.1}$$

Of course, U can be chosen bounded. We claim that U has the property required in the conclusion. Suppose not. Then there exists an $x \in U$, $x > e$, such that $O^+(x) \subset U$. Since U is bounded, by (C) we have $\tau_x = +\infty$. Now, using the strong order–preservence and the facts that $x > e$ and $x_n \to e$, we find a $t_2 > 0$ and an integer n_0 such that

$$S(t_2)x > S(t_2)x_n \text{ for } n \geq n_0 .$$

Consequently

$$S(t)x > S(t)x_n \text{ for all } t > t_2 \text{ and } n \geq n_0 .$$

In particular,

$$S(t_1)S(t_n)x > S(t_n + t_1)x > S(t_n + t_1)x_n = S(t_1)S(t_n)x_n$$

if $n \geq n_0$ is sufficiently large (such that $t_n + t_1 \geq t_2$). This is a contradiction to (4.1) because $S(t_n))x \in U$ and $S(t_n)x_n \in N$. Thus U indeed has the required property. Lemma 4.1 is proved. ∎

As any easy consequence of Lemma 4.1 and Theorem 1 we now obtain existence of an unstable set of an unstable equilibrium.

Corollary 4.3 [Matano]. *Let the assumptions of Lemma 4.1 hold. Then there exists a negative orbit $v(\,.\,)$ with relatively compact image such that $v(t) > e$ for all $t < 0$ and $v(t) \to e$ as $t \to -\infty$.*

PROOF : Let U be as in Lemma 4.1. The assertion follows immediately from Theorem 1, where we take $K = \{e\}$ and $M = X^+(e) \cap \overline{U}$. ∎

It is worth mentioning that another possible choice of M is

$$M = K \cup \bigcup_n O^{\pm}_{\overline{U}}(x_n) ,$$

where $x_n > e$ is any sequence approaching e (cf. (3.1)). The resulting negative orbit $v(\,.\,)$ then takes values in $\overline{M} \subset X^+(e)$. If all these x_n are subsolutions (i.e., $O^+(x_n) \geq x_n$) then $v(t)$ is monotone in t. To see this, one first observes that the set of subsolutions is closed and contains the positive orbits of all its points. So if the x_n are subsolutions, then $\overline{M} \setminus K$ consists of subsolutions which implies monotonicity of $v(t) \in \overline{M} \setminus K$.

A sufficient condition for existence of a sequence of subsolutions as above is given in **[M1]** (see the end of the proof of the Theorem 5). It requires that X is a Banach lattice and any order bounded set B (i.e. a set satisfying $a \le B \le b$ for some a, $b \in X$) is bounded. Thus under this condition the negative orbit from the conclusion of Corollary 4.3 can be chosen monotone. (This is the second part of Theorem 5 of **[M1]**.)

Now we focus our attention to compact invariant sets. Assume that K is a compact invariant set, which is minimal in the sense that K has only trivial compact invariant subsets. Note that, by Zorn's lemma, any compact invariant set contains a minimal compact invariant set (see e.g. **[PM**, Lemma 2.2]). We want to find a nontrivial unstable set of K. Of course this is not possible if K is stable. While the latter may happen in an order-preserving semiflow (in **[Hi5]** is an example of stable periodic orbit), in strongly order-preserving semiflows all such sets are unstable. This is a consequence of the following property:

Proposition 4.4 (Limit set dichotomy). *Let $S(t)$ be strongly order-preserving. If x, $y \in X$ have relatively compact positive orbits and $x > y$ then either $\omega(x) > \omega(y)$ or else $\omega(x) = \omega(y) \subset E$.*

The limit set dichotomy has been proved by Hirsch under a slightly more restrictive assumptions on the semiflow **[Hi6]**. Modifying his arguments Smith and Thieme **[ST2]** have proved this property for strongly order-preserving semiflows, as defined above. (Proposition 4.4 is also stated without proof in **[M3]**.)

Now we can prove

Theorem 4.5. *Assume that $S(t)$ is strongly order-preserving and that (C) holds. Let K be a minimal compact invariant set which is not open in $X^+(K)$ and which is not an equilibrium. Then there exists a negative orbit $v(\,.\,)$ with relatively compact image such that $v(t) \in X^+(K) \setminus K$ for all $t \le 0$ and $\alpha(v(\,.\,)) = K$.*

Proof : First observe that, by compactness of K, the set $X^+(K)$ is closed. Further, by monotonicity and invariance of K, we have $O^+(x) \subset X^+(K)$ for any $x \in X^+(K)$.

We now claim that there exists a neighbourhood U of K such that for any $y \in (U \cap X^+(K)) \setminus K$ there exists a $t \in [0, \tau_y)$ such that $S(t)y \notin U$. Once this is established, one easily verifies that for this neighbourhood and for the set $M := \overline{U} \cap X^+(K))$ all the hypotheses of Theorem 1 are satisfied (note that (iii) holds by assumption). So the conclusion of Theorem 4.5 follows from Theorem 1, provided we prove the claim.

In searching for such a neighbourhood U, we start with the observation that there are no two points x, $z \in K$ with $x < z$. Indeed, if two such points existed, then, by compactness, invariance and minimality of K, we would have $\omega(x) = K = \omega(z)$.Hence, by Proposition 4.4, $K \subset E$ –a contradiction. It follows that the closed set $W := \{g \in X : g \ge K\}$ is disjoint from K. We can thus find a bounded neighbourhood $U \subset X \setminus W$ of K such that there is no $y \in \overline{U}$ with $y \ge K$. We finish the proof by showing that U has the desired property. Suppose not. Then there exist two points $y \in U$ and $x \in K$ such that $y > x$ and $O^+(y) \subset U$. By (C) and boundedness of U, $O^+(y)$ is relatively compact. Now, since $\omega(x) \cap E = K \cap E = \emptyset$, by Proposition 4.4, we have $\omega(y) > \omega(x)$. But this is impossible by the choice of U (we clearly have $\omega(y) \subset \overline{U}$). This contradiction proves that U indeed is the sought neighbourhood. The proof is complete. ■

Remarks 4.6.

(a) From Theorem 4.1 it immediately follows that any nontrivial closed orbit K has a nontrivial unstable set in $X^+(K)$ (if it is not open in $X^+(K)$).

(b) Previous results establish an unstable set above a given invariant set. Analogous assumptions and arguments can be used to prove existence of invariant sets lying below given invariant sets.

In [**M1**], Matano has proved, under stronger compactness conditions, that between two related equilibria $e_1 \le e_2$ there exists a connecting orbit, provided there is no other equilibrium in the order interval $[e_1, e_2] := \{x \in X : e_1 \le x \le e_2\}$. Our next theorem is an extension of this result. Before we formulate the theorem we need some definitions.

A function $w\colon (-\infty, +\infty) \longrightarrow X$ is called an entire orbit if $S(t)w(s) = w(t+s)$ for any $t \ge 0$, $s \in \mathbf{R}$.

Let K_1, K_2 be two minimal compact invariant sets. An entire orbit $w(\,.\,)$ is called a connecting orbit from K_1 to K_2 if $w(\,.\,)$ has relatively compact image and $\alpha(w(\,.\,)) = K_1$ and $\alpha(w(0)) = K_2$.

In the remaining part of this section we assume that for any $x \in X$ there are two sequences x_n, y_n both approaching x such that $x_n < x < y_n$. Note that this implies that no set K is open in $X^+(K)$, neither it is open in $X^-(K) := \{y \in X : y \le x \text{ for some } x \in K\}$.

Now we are in position to prove the next theorem. Note that its formulation implicitly contains the obvious fact that if $K_1 < K_2$ are two invariant sets then the set $X^+(K_1) \cap X^-(K_2)$ contains the positive orbits of all its elements.

Theorem 4.7. *Let $S(t)$ be strongly order-preserving and let (C) hold. Assume that $K_1 < K_2$ are two minimal compact invariant sets such that all positive orbits in $D := X^+(K_1) \cap X^-(K_2)$ are relatively compact and there is no equilibrium e satisfying $K_1 < e < K_2$. Further assume that K_1 is not an equilibrium.*

Then $K_2 = \{e\}$ for some equilibrium e and there is a connecting orbit from K_1 to e. The image of this connecting orbit lies in D. An analogous statement holds if K_2 is not an equilibrium.

In the proof we use the following convergence criterion. Its proof can be found in [**Hi6**], [**M1**], [**ST2**].

Proposition 4.8. *Let $x \in X$ have relatively compact positive orbit and let $S(T)x > x$ for some $T > 0$. Then $S(t)x$ approaches an equilibrium as $t \to +\infty$.*

Proof of Theorem 4.7: Assume that K_1 is not an equilibrium (if K_2 is not an equilibrium, the proof is analogous). By Theorem 4.5, there is a negative orbit

$v(t) \in X^+(K_1)$ such that

$$\alpha(v(\,.\,)) = K_1 \tag{4.3}$$

To prove the conclusion we show that $v(\,.\,)$ extends to an entire orbit which lies in D and connects K_1 to an equilibrium e. Then we prove that $e = K_2$. First we prove that $v(t) \in X^-(K_2)$ for all $t \leq 0$. For this we choose a $z \in K_1$ and find a sequence $t_n \to -\infty$ such that $v(t_n) \to z$. Using Lemma 4.2 with $N = K_2 > z$, we obtain that there is a sequence $t_n' \to -\infty$ such that $v(t_n') < K_2$ for all n. This, in conjunction with invariance of K_2 clearly implies $v(t) \in X^-(K_2)$ for all $t \leq 0$. Setting $v(t) := S(t)v(0)$ for $t > 0$, we thus obtain an entire orbit with values in D, which is relatively compact by assumption. Using Lemma 4.2 again (this time with $z = v(0)$ and $N = K_1$), we find two numbers $t_2 < t_1$ such that $v(t_2) < v(t_1)$. Proposition 44.8 now implies that $v(t)$ approaches, as $t \to +\infty$, an equilibrium $\{e\}$. Recalling (4.3) we see that $v(t)$ is a connecting orbit from K_1 to e.

It remains to prove that $K_2 = \{e\}$. By closedness of D (which follows from compactness of K_1, K_2), we have $e \in D$, i.e., $z_1 \leq e \leq z_2$ for some $z_1 \in K_1$, $z_2 \in K_2$. By minimality of K_1, K_2 and Proposition 4.4, we have $K_1 = \omega(z_1) \leq e \leq \omega(z_2) = K_2$. Since $K_1 \cap E = \emptyset$ and e cannot satisfy $K_1 < e < K_2$ (by assumption), we must have $K_2 = \{e\}$. This completes the proof. ∎

Remark 4.9. By (C), the hypothesis that all trajectories in $D := X^+(K_1) \cap X^-(K_2)$ be relatively compact is satisfied if D is bounded. The latter is the case if X is a subset of an ordered Banach space possessing a normal positive cone (e.g. $C(\Omega)$, $L^p(\Omega)$).

5. Discrete dynamical systems.

We finish the paper with some remarks concerning extensions of our results to discrete dynamical systems.

Consider a compact continuous mapping T on a metric space X. The discrete dynamical system defined by T is the set of iterations T^n, $n = 1, 2, \dots$ The concepts of positive and negative orbits, and ω resp. α–limit sets are defined in a natural way. It is well known [**H1**], [**H2**] that the limit sets of points and sets have similar invariance properties as in the continuous case. It is therefore quite easy to extend some of our results to discrete dynamical systems. We now state a discrete version of Theorem 1. Its proof is straightforward "production" of discrete analogs of the arguments of the proof of Theorem 1 and is therefore omitted.

Theorem 5.1. *Let $K \subset X$ be compact and positively invariant (i.e. $TK \subset K$). Assume that there exist a neighbourhood U of K and a set $M \subset \overline{U}$ satisfying the properties (i)–(iii) of Theorem 1 and*

(iv) *For any $y \in M \setminus K$ there exists an m such that $T^n y \in M$ for $n = 0, 1, \dots, m-1$ and $T^m y \notin U$.*

Then there exists a negative orbit (v_k) such that $v_k \in M \setminus K$ for all $k = 0, -1, -2, \dots$, and $\alpha(v_k) \subset K$.

Applying this theorem analogously as in the continuous case, one can prove existence of a nontrivial unstable set of an unstable fixed point of T if T is strongly order–preserving (which is defined analogously as in the continuous case). Though it seems possible (and not difficult) to prove this discrete version of Matano's existence theorem by "discretizing" his original proof, in our approach the discretization is more convenient.

We do not know whether a discrete version of Theorem 4.7 holds. The reason is that it is not known, up to our knowledge, whether a discrete analog of the limit set dichotomy is valid.

For recent results in discrete strongly order–preserving dynamical systems and their applications in nonautonomous periodic parabolic differential equations see [**AHM**], [**He**].

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