Josef Kolomý

On accretive multivalued mappings in Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 4, 701--710

Persistent URL: http://dml.cz/dmlcz/106905

Comment.Math.Univ.Carolinae 31,4 (1990)701–710

On accretive multivalued mappings in Banach spaces

JOSEF KOLOMÝ

Dedicated to the memory of Zdeněk Frolík

Abstract. Let X be a uniformly Fréchet smooth Banach space such that X^* is Fréchet smooth, $A : X \to 2^X$ a maximal accretive mapping with $W = \operatorname{int} D(A) \neq \emptyset$. Then the set $C(A)$ of all points of W, where A is singlevalued and norm to norm upper semicontinuous is completely characterized. A different result for monotone operators was proved by Fabian [9]. The asymptotic behavior of resolvents of accretive mappings is considered with the solvability of operator equations.

Keywords: Multivalued mappings, accretive operators, resolvents, operator equation, solvability

Classification: 47H06, 47H15, 47H17, 54C60

Introduction.

The theory of monotone and accretive mappings, intensively studied in the last period, has fruitful applications in the theory of nonlinear partial, ordinary differential and integral equations.

The purpose of this note is two-fold. We prove that in a uniformly Fréchet smooth Banach space X having the Fréchet smooth dual X^*, the set of all points of W where a maximal accretive mapping $A : X \to 2^X$ with $W = \operatorname{int} D(A) \neq \emptyset$ is singlevalued and norm to norm upper semicontinuous, is strictly equal to a dense G_δ subset of W of all points of W where the function of the minimum modulus of the operator A (the Kenderov function), is continuous. The proof of this result relies among others on the adaptation of the Kenderov [18] method and the modification of the Fitzpatrick [10] lemma for monotone operators. Recall that Kenderov proved the following important result: If X is a Banach space which admits an equivalent norm such that its dual norm on X^* is rotund and $T : X \to 2^{X^*}$ is monotone with $D_0 = \operatorname{int} D(T) \neq \emptyset$, then there exists a dense G_δ subset $C(f)$ of D_0 such that T is singlevalued at the points of $C(f)$, where $C(f)$ is a set of all points of D_0, where the function f of the minimum modulus of the mapping T is continuous.

The second result of this note concerns the asymptotic behavior of resolvents of accretive mappings acting in Banach spaces with the suitable geometric structure. The second result is related to the results of Reich [25], [26], [28] and Morosanu [24]. Recall that the various properties of monotone and accretive mappings were studied, for instance, by Calvert, Fitzpatrick and Solomon [4], Fabian [8], [9], Kato [17], Kido [19] and Veselý [31] (see also references in these papers), while the further properties of asymptotic behavior of resolvents of accretive operators were investigated by Gobbo [12], Reich [27], Takahashi and Ueda [30]. For the basic properties of accretive mappings and their applications, we refer to Barbu [1], Ciorănescu [5] and Kato [17].

Definitions and notation.

Let X be a real normed linear space, X^* its dual space, $\langle\, , \rangle$ the pairing between X and X^*, $B_r(u)$ the closed ball centered about $u \in X$ and with the radius $r > 0$, $S_r(u)$ its sphere. For the given set $M \subset X$, int M(int$_a$ M) denotes the interior of M (the algebraic interior of M). According to [17], we define $|G|$ for a set $G \subset X$, as follows:

$$|G| = \begin{cases} \inf\{\|u\| : u \in G\}, & \text{if } G \neq \emptyset \\ +\infty & \text{if } G = \emptyset . \end{cases}$$

By R, R_+, we denote the set of all real and nonnegative numbers, respectively. We use the standard notions for rotund (i.e. strictly convex) and uniformly rotund normed linear spaces and Gâteaux and Fréchet derivatives of functionals. By symbols $\sigma(X, X^*)$ and $\sigma(X^*, X)$ we mean the weak and the weak* topology on X and X^*, respectively. Recall that X is said to be: (i) smooth (Fréchet smooth), if the norm of X is Gâteaux (Fréchet) differentiable on $S_1(0)$; (ii) uniformly Fréchet smooth, if the norm of X is uniformly Fréchet differentiable on $S_1(0)$; (ii) an (H)-space, if for each $(u_n) \subset X, u_n \to u$ weakly, $u \in X, \|u_n\| \to \|u\|$, we have $u_n \to u$ in the norm of X; (iv) a dual Banach space, if there is a Banach space Z such that $X = Z^*$ (in the sense of topology and the norm). We shall say that X satisfies the Opial condition (see for instance [13]), if for each $u_n \subset X, u_n \to u_0$ weakly in $X, u_0 \in X$, there is $\liminf_{n\to\infty} \|u_n - u\| > \liminf_{n\to\infty} \|u_n - u_0\|$ for all $u \neq u_0, u \in X$. Let E, G be topological spaces, $A : E \to 2^G$ (where 2^G denotes a system of all subsets of G) a multivalued mapping, $D(A) = \{u \in E : A(u) \neq \emptyset\}$ its domain, $G(A) = \{(u, v) \in E \times G : u \in D(A), v \in A(u)\}$ its graph in the space $E \times G$. A mapping $A : E \to 2^G$ is said to be: (i) upper semicontinuous at $u_0 \in D(A)$, if for each open subset W of G such that $A(u_0) \subset W$, there exists an open neighborhood V of u_0 such that $A(u) \subset W$ for each $v \in V \cap D(A)$; (ii) sequentially closed at $u_o \in D(A)$, if $(u_n) \subset D(A), u_n \to u_0, v_n \in A(u_n)$ and $v_n \to v_0$ imply that $v_0 \in A(u_0)$. Let X be a real normed linear space. A duality mapping $J : X \to 2^{X^*}$ is defined by $J(u) = \{u^* \in X^*, \langle u^*, u\rangle = \|u\|^2, \|u^*\| = \|u\|\}$ for each $u \in X$. Recall that $J(u)$ is a nonempty convex weakly* compact subset of X^* for each $u \in X$ and that X is smooth (Fréchet smooth), if and only if J is singlevalued (continuous) on X (see [6]). Recall that a mapping $A : X \to 2^X$ is said to be: (i) accretive if $I + \lambda A$, where I is an identity mapping in X, is expansive for each $\lambda > 0$, i.e. if for each $u, v \in D(A)$ and each $x \in A(u), y \in A(v)$ there is $\|(u-v)+\lambda(x-y)\| \geq \|u-v\|$ for each $\lambda > 0$ (equivalently, if for each $u, v \in D(A)$ and each $x \in A(u), y \in A(v)$, there exists a point $x^* \in J(u - v)$ such that $\langle x - y, x^*\rangle \geq 0$); (ii) maximal accretive, if A is accretive and if $(u, x) \in X \times X$ is a given element such that for each $v \in D(A)$ and $y \in A(v)$ there exists a point $x^* \in J(u - v)$ such that $\langle x - y, x^*\rangle \geq 0$, then $u \in D(A)$ and $x \in A(u)$; (iii) m-accretive, if A is accretive and the range $R(I + \lambda A)$ of $I + \lambda A$ is equal to X for some $\lambda > 0$. If $A : X \to 2^X$ is an accretive mapping with $D(A) \subseteq X$, then the so-called resolvent $J_\lambda = (I + \lambda A)^{-1}$ of A exists for each $\lambda > 0$ and is singlevalued with the domain $D(J_\lambda) = R(I + \lambda A)$

and the range $R(J_\lambda) = D(A)$. The Yoshida approximations (see [**17**]) A_λ of A are defined by $A_\lambda = \lambda^{-1}(I - J_\lambda)$ for each $\lambda > 0$. A mapping $A : X \to 2^X$ is said to be locally bounded at $u_0 \in D(A)$, if there exists a neighborhood U of u_0 such that $A(U) = \{A(u) : u \in U \cap D(A)\}$ is bounded in X. Notice (see [**2**]) that each accretive mapping $A : X \to 2^X$ can be extended (by the Zorn lemma) to a maximal accretive mapping.

Results. Let X, Y be normed linear spaces, Y a dual Banach space (i.e. $Y = Z^*$ for some Banach space Z), $A : X \to 2^Y$ a mapping with $D(A) \subseteq X$. We shall say that A has a property (P) at $u_0 \in D(A)$, if the following condition is satisfied: If $(u_\alpha) \in D(A)$ is a net, $u_\alpha \to u_0$ in the norm of X, $v_\alpha \in A(u_\alpha)$ is such that $\|v_\alpha\| \le C$ for some constant $C > 0$, then there exists a subnet (v_{α_i}) of (v_α) with the weak* limit point v_0 such that $v_0 \in A(u_0)$.

Clearly, if A is closed at u_0 from the norm topology of X to the weak* topology of Y, then A has the property (P) at u_0.

Lemma 1. *Let X, Y be normed linear spaces, Y a dual Banach space, $A : X \to 2^Y$ a mapping with $D(A) \subseteq X$. Suppose that A is locally bounded and possesses the property (P) at u_0. Then A is norm to weak* upper semicontinuous at u_0.*

PROOF : Standard. Assume that A is not norm to weak* upper semicontinuous at u_0. Then there exists a weak* open set V in Y such that $A(u_0) \subset V$ and that for each open neighborhood U of u_0 there is a point $u \in U$ such that $A(u) \cap (X \setminus V) \neq \emptyset$. For every $n \ge 1$ there exists $u_n \in \operatorname{int} B_{1/n}(u_0)$ such that $A(u_n) \cap (X \setminus V) \neq \emptyset$. Then $u_n \to u_0$ and we choose $v_n \in A(u_n)$ such that $v_n \in X \setminus V$. As A is locally bounded at u_0, v_n is bounded and therefore there exists a subnet (v_{n_α}) of (v_n) and a point $v_0 \in Y$ such that $v_{n_\alpha} \to v_0$ weakly* in Y. Since A satisfies the condition (P) at u_0, $v_0 \in A(u_0) \subset V$. On the other hand, $(v_n) \subset X \setminus V$ and $X \setminus V$ is weakly* closed. Hence $v_0 \in X \setminus V$, a contradiction. ∎

Note that a direct proof of a similar result was given in [**22**].

Lemma 2 [31]. *Let X be a Banach space, $A : X \to 2^X$ an accretive mapping with $W = \operatorname{int} D(A) \neq \emptyset$. Then A is locally bounded on a dense open subset of W.*

Lemma 3. *Let X be a reflexive Banach space, $A : X \to 2^X$ an accretive mapping with $W = \operatorname{int} D(A) \neq \emptyset$. If A has the property (P) on W (in particular, A is norm to weak closed at the points of W), then there exists an open dense subset Q of W such that A is norm to weak upper semicontinuous at the points of Q.*

PROOF : It depends on Lemma 1 and Lemma 2. ∎

Let X be a Banach space, $A : X \to 2^X$ a mapping with $D(A) \subseteq X$. Define the Kenderov function (of the minimum modulus of A) $f : X \to R_+ \cup \{+\infty\}$ by $f(u) = \inf\{\|x\| : x \in A(u)\}, u \in X$.

Lemma 4. *Let X be a dual Banach space, $A : X \to 2^X$ a mapping such that $W = \operatorname{int} D(A) \neq \emptyset$ and that $A(u)$ is bounded for each $u \in W$. If the graph $G(A)$ of A is closed in $(X, \|\cdot\|) \times (X, \sigma(Z^*, Z))$, then f is lower semicontinuous on X and* *finite on W. Moreover, there exists a dense G_δ subset $C(f)$ of W such that f is continuous at the points of $C(f)$.*

PROOF : Let $a \in R$ be arbitrary, we show that $M = \{u \in X : f(u) \leq a\}$ is closed. Suppose that $u \in \overline{M}$ and let $(u_n) \subset M$ be such that $u_n \to u$. For each n there exists $x_n \in A(u_n)$ such that $f(u_n) \leq \|x_n\| < f(u_n) + 1/n \leq a + 1/n$. Hence there exists a point $x \in X$ and a subnet (x_{n_α}) of (x_n) such that $x^*_{n_\alpha} \to x$ in the $\sigma(Z^*, Z)$-topology of X. Since $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(Z^*, Z))$, we see that $(u, x) \in G(A)$ and $\|x\| \leq \liminf_\alpha \|x_{n_\alpha}\| \leq a$. As $x \in A(u)$, we conclude that $f(u) \leq \|x\| \leq a$ and therefore $u \in M$. Moreover, $A(u)$ is $\sigma(Z^*, Z)$-compact for each $u \in W$ and since the norm in X is $\sigma(Z^*, Z)$-lower semicontinuous, we conclude that $f(u) = \min\{\|x\| : x \in A(u)\}$ for each $u \in W$ and $f : W \to R_+$. The rest is well-known. ∎

Lemma 5. *Let X be a reflexive smooth Banach space, $A : X \to 2^X$ an accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$ and that $A(u)$ is a bounded set for each $u \in W$. If $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$, then for each (fixed) $u \in C(f)$ there is $f(u) = \|x\|$ for each $x \in A(u)$.*

PROOF : See [**21**, Lemma 5]. ∎

Lemma 6. *Let X be a reflexive smooth and rotund Banach space, $A : X \to 2^X$ an accretive mapping with $W = \operatorname{int} D(A) \neq \emptyset$. Suppose that $A(u)$ is convex and bounded for each $u \in W$. If $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$, then A is singlevalued at the points of the dense G_δ subset $C(f)$ of W.*

PROOF : It is based on Lemmas 4, 5, and the fact that a convex set in the rotund space has at most one point with the minimum norm. ∎

Proposition 1. *Let X be a reflexive smooth rotund Banach space, $A : X \to 2^X$ an accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Suppose that $A(u)$ is bounded convex for each $u \in W$. If $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$, then there exists a dense G_δ subset W_0 of W such that A is single valued and norm to weak upper semicontinuous on W_0.*

PROOF : According to Lemma 6, A is single valued at the points of the dense G_δ subset $C(f)$ of W. Since $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$, A is norm to weak closed at the points of W. In view of Lemma 3, we conclude that A is norm to weak upper semicontinuous on a dense open subset Q of W. Now it is sufficient to set $W_0 = Q \cap C(f)$, which proves our result. ∎

Let X be a normed linear space, then each accretive mapping $A : X \to 2^X$ can be extended by the Zorn lemma to the maximal accretive mapping $\tilde{A}$ (see [**2**]). Recall that if $A : X \to 2^X$ is maximal accretive, then the following assertions are valid (compare [**5**] and [**17**]):

(i) If X is smooth, then $A(u)$ is convex and closed for each $u \in D(A)$;
(ii) if X is Fréchet smooth, then the graph $G(A)$ of A is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$.

S. Fitzpatrick [10] proved the following result: Let X be a Banach space, $T : X \to 2^{X^*}$ a maximal monotone mapping with $D_0 = \operatorname{int} D(T) \neq \emptyset$. Suppose that $u_0 \in D_0$ is a point of continuity of the Kenderov function $\varphi : D_0 \to R_+$ defined by $\varphi(u) = \min\{\|u^*\| : u^* \in T(u)\}, u \in D_0$. If $u_n \in D_0, u_n \to u_0, u_n^* \in T(u_n), u_0^* \in T(u_0)$, then $\|u_n^*\| \to \|u_0^*\|$.

We prove the following lemma

Lemma 7. *Let X be a uniformly Fréchet smooth Banach space such that X^* is Fréchet smooth, $A : X \to 2^X$ an accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$ and that $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$. If $u_0 \in C(f), u_n \in W, u_n \to u_0, x_n \in A(u_n), x_0 \in A(u_0)$, then $\|x_n\| \to \|x_0\|$, where $C(f)$ is the set of all points of W, where the function f is continuous.*

Proof : Since X^* is Fréchet smooth, X is a reflexive rotund (H)-space. As X^* is uniformly rotund, we conclude that A is locally bounded at each point of W ([11]). In particular, $A(u)$ is bounded for each x of W. First of all, we show that $\liminf_{n\to\infty} \|x_n\| > \|x_0\|$. Suppose that $\liminf_{n\to\infty} \|x_n\| < \|x_0\|$. Without loss of generality, one can assume that $\|x_n\| < \|x_0\| - \alpha$ for some $\alpha > 0$ and infinitely many indexes n. Then there exists a subsequence of (x_n), say (x_n), such that $x_n \to \overline{x}$ weakly in X. Since $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*)), \overline{x} \in A(u_0)$ and $\|\overline{x}\| \leq \liminf_{n\to\infty} \|x_n\|$. According to Lemma 5, we have that $\|\overline{x}\| = \|x_0\|$ and hence $\|x_0\| \leq \liminf_{n\to\infty} \|x_n\| \leq \|x_0\| - \alpha$, a contradiction.

According to Lemma 4, the set $C(f)$ of all points of W, where f is continuous, is a dense G_δ subset of W. Assume that $\limsup_{n\to\infty} \|x_n\| > \|x_0\|$. Without loss of generality, one can assume that $\|x_n\| > \|x_0\| + \alpha$ for infinitely many indexes n and some $\alpha > 0$. Let $z_n^* \in X^*$ be such that $\|z_n^*\| = 1$ and $\langle z_n^*, x_n \rangle = \|x_n\|$. As X is reflexive and smooth, then the duality mapping J is singlevalued and surjective. Hence there exists $z_n \in X$ such that $z_n^* = J(z_n)$ and $\|z_n\| = 1$. Now choose $v_n \in \operatorname{int} B_{1/n}(z_n)$ such that $u_n + n^{-1} v_n \in C(f)$ for sufficiently large n. Note that this choice is possible, since $u_n + \operatorname{int} n^{-1} B_{1/n}(z_n)$ is open and $C(f)$ is dense in W. Choose $\overline{y}_n \in A(u_n + n^{-1} v_n)$. According to Lemma 5, we have that $\|\overline{y}_n\| = f(u_n + n^{-1} v_n) \to f(u_0) = \|x_0\|$ as $n \to \infty$. Since A is accretive, we have that $0 \leq \langle \overline{y}_n - x_n, J(u_n + n^{-1} v_n - u_n) \rangle = n^{-1} \langle \overline{y}_n - x_n, J(v_n) \rangle$. Hence

$$
\begin{aligned}
(1) \quad \langle \overline{y}_n, J(v_n) \rangle \geq \langle x_n, J(v_n) \rangle = \langle x_n, J(z_n) \rangle + \langle x_n, J(v_n) - J(z_n) \rangle \geq \\
\geq \langle x_n, z_n^* \rangle - \|x_n\| \, \|J(v_n) - J(z_n)\|.
\end{aligned}
$$

Since A is locally bounded on W, we get that $\|x_n\| \leq k$ for each n and some $k > 0$. But the uniform Fréchet smoothness implies that a duality mapping J is uniformly continuous on bounded subsets of X (see [6]). As $v_n \in \operatorname{int} B_{1/n}(z_n), \|z_n\| = 1$, for a given $\alpha/(2k)$ there exists an integer n_0 such that $\|J(v_n) - J(z_n)\| < \alpha 2^{-1} k^{-1}$ for each $n \geq n_0$. From (1) and from the inequality $\|x_n\| > \|x_0\| + \alpha$ which is valid for infinitely many indexes, we conclude that $\langle \overline{y}_n, J(v_n) \rangle > \|x_0\| + \frac{\alpha}{2}$ for each $n \geq n_0$. On the other hand, $\langle \overline{y}_n, J(v_n) \rangle \leq \|\overline{y}_n\| \, \|v_n\| \leq \|\overline{y}_n\| (\|z_n\| + n^{-1}) = \|\overline{y}_n\| (1 + n^{-1})$.

Hence

$$\|x_0\| + \frac{\alpha}{2} \le \limsup_{n\to\infty} \frac{\langle \overline{y}_n, J(v_n)\rangle}{1+n^{-1}} \le \limsup_{n\to\infty} \|\overline{y}_n\|,$$

a contradiction, since $\lim_{n\to\infty} \|\overline{y}_n\| = \|x_0\|$, which proves the assertion. ∎

Let X be a reflexive Banach space, $A : X \to 2^X$ a mapping such that $\mathrm{int}_a D(A) \neq \emptyset$. We shall say that A is (i) hemiclosed (i.e. closed from the line segments of X into the weak topology of X) at $u_0 \in \mathrm{int}_a D(A)$ if for each $u \in X$ and each null sequence of positive numbers (t_n) and $x_n \in A(u_n)$, where $u_n = u_0 + t_n u, u_n \in D(A)$ for sufficiently large $n \ge n_0, \|x_n\| \le C$ for some $C > 0$, there exists a subsequence (x_{n_k}) of (x_n) with the weak limit point x_0 such that $x_0 \in A(u_0)$; (ii) hemicontinuous (see [5]) at $u_0 \in \mathrm{int}_a D(A)$, if for each sequence (t_n), where $t_n > 0, t_n \to 0$, and each $u \in X$ and $x_n \in A(u_n)$ with $u_n = u_0 + t_n u, u_n \in D(A)$ for a sufficiently large n, we have that $x_n \to x_0$ weakly in X and $x_0 \in A(u_0)$.

Lemma 8. *Let X be a reflexive Banach space, $A : X \to 2^X$ a mapping with $\mathrm{int}\, D(A) \neq \emptyset$. Then: (i) If A is singlevalued and hemiclosed and locally bounded at $u_0 \in \mathrm{int}\, D(A)$, then A is hemicontinuous at u_0; (ii) If X is smooth rotund, $A : X \to 2^X$ is an accretive locally bounded mapping on $\mathrm{int}\, D(A)$ such that $A(u)$ is convex for each $u \in \mathrm{int}\, D(A)$ and $G(A)$ is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$, then A is singlevalued at the points of the dense G_δ subset $C(f) \subset \mathrm{int}\, D(A)$ and A is norm to weak upper semicontinuous on $\mathrm{int}\, D(A)$.*

Proof : (i) Let (t_n) be a sequence of positive numbers, $t_n \to 0, u_n = u_0 + t_n z$, where $z \in X$ is arbitrary, $u_0 \in \mathrm{int}\, D(A)$. Then $u_n \in D(A)$ for sufficiently large $n \ge n_0$. Let $x_n \in A(u_n), n \ge n_0$. Since $u_n \to u_o$ and A is locally bounded at u_0, we have that (x_n) is bounded and hence there exists a subsequence (x_{n_k}) of (x_n) and $x_0 \in X$ such that $x_{n_k} \to x_0$ weakly in X. According to our hypotheses, A is singlevalued and hemiclosed at u_0. Therefore $x_0 = A(u_0)$ and the whole sequence (x_n) converges to $A(u_0)$, and thus A is hemicontinuous at u_0.

(ii) According to Lemma 6, A is singlevalued on a dense G_δ subset $C(f)$ of $\mathrm{int}\, D(A)$. By Lemma 1, A is norm to weak upper semicontinuous on $\mathrm{int}\, D(A)$. ∎

Theorem 1. *Let X be a uniformly Fréchet smooth Banach space such that X^* is Fréchet smooth, $A : X \to 2^X$ a maximal accretive mapping with $W = \mathrm{int}\, D(A) \neq \emptyset$.*

Then the set $C(A)$ of all points of W, where A is singlevalued and norm to norm upper semicontinuous, is equal to the dense G_δ subset $C(f)$ of all points of W, where the function f is continuous.

Proof : Since X is smooth, $A(u)$ is convex and closed for each $u \in W$. Moreover, A is locally bounded at each point of W, in particular, $A(u)$ is bounded for each $u \in W$ (see [11]). As X is Fréchet smooth and A is maximal accretive, we have that the graph $G(A)$ of A is closed in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$. According to Lemma 4, the function f (defined in Lemma 4, where $W \subseteq X$) is lower semicontinuous and finite on W. Hence there exists a dense G_δ subset $C(f)$ of W such that minimum modulus function f of A is continuous at the points of $C(f)$. Let $u_0 \in C(f)$ be arbitrary, we show that $u_0 \in C(A)$. According to Lemma 8, we have that $A(u_0)$ is a singleton and A is norm to weak upper semicontinuous at u_0, i.e. whenever $u_n \to u_0, u_n \in W$, and $x_n \in A(u_0)$, then $x_n \to x_0 \equiv A(u_0)$ weakly in X. By Lemma 5 (with $W \subseteq X$), we conclude that $\|x_n\| \to \|x_0\|$. As X^* is Fréchet smooth, X is an (H)-space and therefore $x_n \to x_0$ in the norm of X. Hence $u_0 \in C(A)$ and we have proved that $C(f) \subseteq C(A)$. Suppose now that $u_0 \in C(A)$. Since f is lower semicontinuous on X, it is sufficient to prove that f is upper semicontinuous at u_0. Assume that $(u_n) \subset W, u_n \to u_0, x_n \in A(u_n)$. By our assumption, for a given $\varepsilon > 0$ there exists an integer n_0 such that $\|x_n - A(u_0)\| \le \varepsilon$ for each $n \ge n_0$. According to Lemma 5, we have that $f(u_0) = \|A(u_0)\|$ and $f(u_n) \le \|x_n\|$. Therefore $f(u_n) \le \|x_n\| \le \|A(u_0)\| + \varepsilon = f(u_0) + \varepsilon$ for each $n \ge n_0$, which gives that $C(A) \subseteq C(f)$. Hence $C(A) = C(f)$ and Theorem 1 is proved. ∎

Remark 1. Let $X = L_\Phi(G)$ be an Orlicz space provided with the Orlicz norm, where $G \subset R^n, \operatorname{mes} G < +\infty, R^n$ denotes an n-dimensional Euclidean space and $\operatorname{mes} G$ is the Lebesgue measure of G. If N-function Φ is strictly convex on $[0, +\infty)$ and Φ and its dual function Φ^* satisfy the Δ_2-condition for the large arguments and, moreover, if Φ^* is uniformly convex on $[0, +\infty)$ in the sense that for a given $\varepsilon \in (0,1)$ there exists a constant $p(\varepsilon) \in (0,1)$ such that

$$\Phi^*\left(\frac{u+\varepsilon u}{2}\right) \le \frac{1-p(\varepsilon)}{2}\left(\Phi^*(u) + \Phi^*(\varepsilon u)\right)$$

for each $u \in [0, +\infty)$, then the assumptions of Theorem 1 are satisfied for the Orlicz space $X = L_\Phi(G)$. Recall that $L_\Phi(G)$ provided with an Orlicz norm, is uniformly rotund, if Φ is uniformly convex on $[0, +\infty)$ and satisfies the Δ_2-condition (compare Hudzik **[14]**, **[15]**, Kamińska **[16]** and Shutao **[29]**).

Recall that the continuity properties of accretive mappings were studied in reflexive Fréchet smooth spaces (see **[31]** and **[22]**). M. Fabian **[8]** proved a similar result under the additional assumptions on X and X^* and a given operator A, see also **[9]**.

We prove the second result of this note. First of all we shall use the following two lemmas.

Lemma 9 (**[5]**). *Let X be a normed linear space, $A : X \to 2^X$ an accretive mapping with $D(A) \subseteq X$. Then:*

(i) *$\|J_\lambda(x) - J_\lambda(y)\| \le \|x - y\|$ for each $x, y \in D(A_\lambda)$;*

(ii) *$A_\lambda x \in AJ_\lambda x$ and $|AJ_\lambda(x)| \le \|A_\lambda x\|$ for each $x \in D(A_\lambda)$. If $x \in D(A) \cap D(A_\lambda)$, then $\|A_\lambda x\| \le |Ax|$.*

Lemma 10. *Let X be a Banach space, $K \subset X$ a closed subset, $T : K \to X$ a nonexpansive mapping. Assume that one of the following two conditions is satisfied:*

(i) *X is uniformly rotund, K is convex and bounded (see* **[3]**);

(ii) *X satisfies the Opial condition (see* **[7]**).

Then $I - T$ is sequentially weak to norm closed on K (i.e. if $u_n \to u_0$ weakly in X, $u_0 \in K, u_n \in K$ and $(I - T)u_n \to w_0$, then $(I - T)u_0 = w_0$).

Theorem 2. *Let X be a Banach space, $A : X \to 2^X$ an accretive mapping.*

(i) *If X is reflexive, $A^{-1}(0) \neq \emptyset, A^{-1}$ is strong to weak closed at 0, $J : X \to 2^{X^*}$ is sequentially weak to weak closed at 0 and $\overline{D(A)} \subseteq \bigcap\{D(A_\lambda) : \lambda > 0\}$, then the strong $\lim\limits_{\lambda\to+\infty} J_\lambda x$ exists for each $x \in \overline{D(A)}$ and belongs to $A^{-1}(0)$.*

(ii) *Suppose that $A^{-1}(0)$ is a singleton and (λ_n) is a sequence of positive numbers such that $\lambda_n \to +\infty$ as $n \to \infty$.*
(a) *If X is uniformly rotund and A is m-accretive, then $J_{\lambda_n}x \to x_0$ weakly for each $x \in X$ and $Ax_0 = 0$.*
(b) *If X is reflexive and satisfies the Opial condition, $\overline{D(A)}$ is weakly closed and $\overline{D(A)} \subseteq \bigcap\{D(A_\lambda) : \lambda > 0\}$, then $J_{\lambda_n}x \to x_0$ weakly for each $x \in \overline{D(A)}$ and $Ax_0 = 0$.*

PROOF : (i) Let (λ_n) be a sequence of positive numbers such that $\lambda_n \to +\infty$ as $n \to \infty$. Take $x \in \overline{D(A)}$ and set $x_n = J_{\lambda_n}x$. Then $\lambda_n^{-1}(x - x_n) = \lambda_n^{-1}(x - J_{\lambda_n}x) = A_{\lambda_n}x \in AJ_{\lambda_n}x$. Choose $v \in A^{-1}(0)$. By accretivity of A there exists $x_n^* \in J(v - x_n)$ such that $0 \leq \langle -(x - x_n)\lambda_n^{-1}, x_n^*\rangle$. Therefore $\langle (v - x_n) + (x - v), x_n^*\rangle \leq 0$, which gives that

$$\|v - x_n\|^2 \leq \langle v - x, x_n^*\rangle \leq \|v - x\|\,\|x_n^*\| = \|v - x\|\,\|v - x_n\|.$$

Therefore $\|v - x_n\| \leq \|v - x\|$ and hence (x_n) is bounded. There exists a subsequence of (x_n), say (x_{n_k}), such that $x_{n_k} \to x_0$ weakly in X. Setting $y_n = x - x_n$, we get that $y_n \in \lambda_n A x_n$ and $\lambda_n^{-1}y_n \to 0$ as $n \to \infty$. As $(x_n, \lambda_n^{-1}y_n) \in G(A)$, we have that $(\lambda_n^{-1}y_n, x_n) \in G(A^{-1})$. Since A^{-1} is norm to weak closed at 0, we conclude that $x_0 \in A^{-1}(0)$, i.e. $Ax_0 \ni 0$. Setting $v = x_0$, there exist elements in $J(x_0 - x_n)$, say x_n^*, such that $\|x_0 - x_n\|^2 \leq \langle x_0 - x, x_n^*\rangle$ for each $n \geq 1$. As (x_n) is bounded, $x_n^* \in J(x_0 - x_n)$, we see that (x_n^*) is bounded. By reflexivity of X^*, there exist a subsequence $(x_{n_k}^*)$ of (x_n^*) and a point $x_0^* \in X^*$ such that $x_{n_k}^* \to x_0^*$ weakly in X^*. Since $x_0 - x_{n_k} \to 0$ weakly in X and J is sequentially weak to weak closed at 0, we get that $x_0^* = J(0) = 0$. Hence $x_{n_k}^* \to 0$ weakly in X^* and we conclude that the whole sequence (x_n^*) converges weakly to 0. From the last inequality we obtain that $x_n \to x_0$ in the norm of X, which proves (i).

Now assume (ii). Let (λ_n) be a sequence of positive numbers such that $\lim\limits_{n\to\infty} \lambda_n = +\infty$. Put $x_n = J_{\lambda_n}x$, where $x \in X$ in the case (a) or $x \in \overline{D(A)}$ in the case (b). As in (i), we obtain that (x_n) is bounded and hence there exists a subsequence of (x_n), say (x_{n_k}), such that $x_{n_k} \to x_0$ weakly. There exists $R > 0$ such that $(x_n) \subset B_R(0)$. Moreover, $x_0 \in B_R(0)$ and $x_n \in D(A)$ and $\overline{D(A)}$ is weakly closed, $x_0 \in \overline{D(A)}$ in the case (b). Now we consider the nonexpansive mapping J_λ, where $\lambda = 1$. Put $J^0 = J_1|_{B_R(0)}$ in the case (a), and $J^0 = J_1|_{\overline{D(A)}}$ in the case (b). We have that $\|x_n - J^0x_n\| = \|x_n - J_1x_n\| = \|A_1x_n\| \leq |Ax_n| \leq \lambda_n^{-1}\|y_n\|$ by Lemma 9, since $\lambda_n^{-1}y_n \in Ax_n$ and $y_n = x - x_n$. As $\lambda_n^{-1}\|y_n\| \to 0, x_n - J^0x_n \to 0$ and we apply Lemma 10 to the restrictions $J_1|_{B_R(0)}$ and $J_1|_{\overline{D(A)}}$, respectively. As $x_0 \in B_R(0)$ and $x_0 \in \overline{D(A)} \subseteq \bigcap\{D(A_\lambda) : \lambda > 0\} \subseteq R(I + A) = D(J_1)$ in the case (b), we have that

$(I-J_1)x_0 = 0$ in the cases (a) and (b). Hence $0 = x_0 - J_1x_0 = A_1x_0 = AJ_1x_0 = Ax_0$ and, moreover, the whole sequence (x_n) is convergent to x_0 weakly, which proves the assertion. ■

Recall that Reich [28] proved the following result: Let X be a smooth uniformly rotund Banach space with a duality map that is sequentially weak to weak continuous at 0, $A : X \to 2^X$ an accretive operator such that $\overline{D(A)}$ is convex and $R(I+\lambda A) \supset \overline{D(A)}$ for all $\lambda > 0$. If $0 \in R(A)$, then the strong $\lim_{\lambda\to+\infty} J_\lambda x = Qx$ for each $x \in \overline{D(A)}$, where Q is the unique sunny nonexpansive retraction of $\overline{D(A)}$ onto $A^{-1}(0)$. Compare also Reich [25], [26]. Another result concerning the asymptotic behavior of resolvents of accretive multivalued mappings acting in smooth Banach spaces X having X^* Fréchet smooth, has been proved in [23].

Remark 2.

If X is a uniformly Gâteaux smooth Banach space such that for each sequence (x_n) which is weakly convergent in X to x_0, there is $\liminf_{n\to\infty} \|x_n - x\| \geq \liminf_{n\to\infty} \|x_n - x_0\|$ for all $x \in X$, then the duality mapping J is sequentially weak to weak* continuous at 0 (see Gossez and Lami Dozo [13]). Note [28] that each smooth Orlicz sequence space has a duality mapping which is sequentially weakly continuous at 0.

References

[1] Barbu V., *Semigrupuri de contracţii nelineare in spatii Banach*, Ed. Acad. Rep. Soc. Romania, Bucaresti 1974.

[2] Berens H., Hetzelt L., *On accretive operators on ℓ_n^∞*, Pacific J. Math. **125** (1986), 301–315.

[3] Browder F.E., *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[4] Calvert B.,Fitzpatrick S., Solomon W., *Boundedness of approximations in summation of accretive operators*, Ann. sti Univ. Al. I. Cuza, Iaşi **33** (1987), 229–234.

[5] Ciorănescu I., *Aplicaţii da dualitate in analiza funkţionala neliniara*, Ed. Acad. Rep. Soc. Romania, Bucuresti 1974.

[6] Cudia D.F., *The geometry of Banach spaces: Smoothness*, Trans. Amer. Math. Soc. **110** (1964), 284–314.

[7] Engl H.W., *Weak convergence of asymptotically regular sequences for nonexpansive mappings and connections with certain Chebyshef-centers*, Nonlinear Anal. **1** (1977), 495–501.

[8] Fabian M., *On strong continuity of monotone mappings in reflexive Banach spaces*, Acta Polytechnica **6** (1977), 85–98.

[9] Fabian M., *On singlevaluedness (and strong continuity) of maximal monotone mappings*, Comment. Math. Univ. Carolinae **18** (1977), 19–39.

[10] Fitzpatrick S.P., *Continuity of non-linear monotone operators*, Proc. Amer. Math. Soc. **62** (1977), 111–116.

[11] Fitzpatrick P.M., Hess P., Kato T., *Local boundedness of monotone type operators*, Proc. Japan Acad. Ser. A Math. Sci. **48** (1972), 275–277.

[12] Gobbo L., *On the asymptotic behavior of the inverse of an m-accretive operator in a Banach space*, Rend. Sem. Mat. Univ. Politechn. Torino **42** (1984), 47–64.

[13] Gossez J.P., Lami Dozo E., *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40** (1972), 565–573.

[14] Hudzik H., *Uniform convexity of Musielak–Orlicz spaces with Luxemburg's norm*, Comment. Math. Univ. Carolinae **23** (1983), 21–32.

[15] Hudzik H., *Strict convexity of Musielak–Orlicz spaces with Luxemburg's norm*, Bull. Acad. Pol. Sci. Math. **29** (1981), 137–144.
[16] Kamińska A., *On uniform convexity of Orlicz spaces*, Proc. Koninklijke Nederl. Akad. Wetensch., Amsterdam, ser. A 85.
[17] Kato T., *Accretive operators and nonlinear equations in Banach spaces*, Proc. Symp. Pure Math. **18**, Amer. Math. Soc., Providence, R.I., 1970, 138–161.
[18] Kenderov P.S., *Multivalued monotone operators are almost everywhere single-valued*, Studia Math. **56** (1976), 199–203.
[19] Kido K., *Strong convergence of resolvents of monotone operators in Banach spaces*, Proc. Amer. Math. Soc. **103** (1988), 752–758.
[20] Köthe G., *Topologische lineare Räume I*, Springer–Verlag, Heidelberg 1960.
[21] Kolomý J., *On mapping and invariance of domain theorem and accretive operators*, Boll. U.M.I. **1-B** (1987), 235–246.
[22] Kolomý J., *Maximal monotone and accretive multivalued mappings and structure of Banach spaces*, Function Spaces, Proc. Int. Conf. Poznań, 1986, (ed. J. Musielak), Teubner-Texte zur Mathematik **103** (1988), 170–177.
[23] Kolomý J., *Resolvents and selections of accretive mappings in Banach spaces*, Proc. 18th Winter School, Srní 1990, Acta Univ. Carolinae (to appear).
[24] Morosanu G., *Asymptotic behavior of resolvents for a monotone mapping in a Hilbert space*, Atti Accad. Naz. dei Lincei, Ser. 8, **59** (1976), 565–570.
[25] Reich S., *Approximating zeros of accretive operators*, Proc. Amer. Math. Soc. **51** (1975), 381–384.
[26] Reich S., *Extension problems for accretive sets in Banach spaces*, J. Functional Anal. **26** (1977), 378–395.
[27] Reich S., *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.
[28] Reich S., *Asymptotic behavior of resolvents in Banach spaces*, Atti Accad. Naz. dei Lincei, Ser. 8, **67** (1980), 27–30.
[29] Chen Shutao, *Convexity and smoothness of Orlicz spaces, Geometry of Orlicz spaces I*, Function Spaces, Proc. Int. Conf. Poznań, 1986 (ed. J. Musielak), Teubner-Texte zur Mathematik **103** (1988), 12–19.
[30] Takahasi W., Ueda Y., *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl. **194** (1984), 546–553.
[31] Veselý L., *Some new results on accretive multivalued operators*, Comment. Math. Univ. Carolinae **30** (1989), 45–55.

Mathematical Institute of Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(*Received* September 4, 1990)