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On the λ-property of Orlicz space L_M

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Abstract. In this paper, we show that each Orlicz space L_M with the Orlicz norm has the λ-property and give a criterion of that L_M has the uniform λ-property.

Keywords: Orlicz space, λ-property, uniform λ-property

Classification: 46E30

Notation.

Let X be a Banach space, $B(X)$ the closed unit ball, $U(X)$ the open unit ball and $S(X)$ the unit sphere. A point e of a convex subset A of X is an extreme point of A if $x, y \in A$ and $e = \frac{1}{2}x + \frac{1}{2}y$ imply $e = x = y$. The set of the extreme points of A is denoted by $\operatorname{ext}(A)$. A point $x \in B(X)$ is said a λ-point if there exist $e \in \operatorname{ext}(B(X)), y \in B(X)$ and $\lambda \in (0,1]$ such that $x = \lambda e + (1-\lambda)y$. In this case, the triple (e, y, λ) is said to be amenable to x. X is called to have the λ-property if each $x \in B(X)$ is a λ-point. If X has the λ-property and satisfies

$$\inf\{\lambda(x) : x \in B(X)\} > 0,$$

where $\lambda(x) = \sup\{\lambda : (e, y, \lambda) \text{ is amenable to } x\}$, X is called to have the uniform λ-property (see [**1**]).

Let $M : R \to R^+$ satisfy the following conditions:

a) $M(u)$ is even, convex and continuous;
b) $M(0) = 0$ and $M(u) > 0$ for $u \neq 0$;
c) $\lim_{u\to 0} M(u)/u = 0$, $\lim_{u\to\infty} M(u)/u = \infty$,

and G be a bounded closed set of n-dimensional Euclidean space E^n. The Orlicz space L_M is the family of all real Lebesgue measurable functions $x(t)$, defined on G, for which $\varrho_M(kx) = \int_G M(kx(t))\,dt < \infty$ for some $k > 0$. L_M with the Orlicz norm

$$\|x\| = \sup\{\int_G x(t)y(t)\,dt : \varrho_N(y) \leq 1\}$$

is a Banach space, where $N(v)$ is the conjugate function of $M(u)$.

We denote the set of all points on which $M(u)$ is not strictly convex by D, i.e., for $v \in D$ there exist a, b such that $a < v < b$ and $M(u)$ is affine on (a, b). It is clear that $D = \bigcup_i (a_i, b_i)$, where (a_i, b_i) are non-overlapping intervals. We also define k_x^* and k_x^{**} by

$$k_x^* = \inf\{k > 0 : \int_G N(p(kx(t)))\,dt \geq 1\}$$

and

$$k_x^{**} = \sup\{k > 0 : \int_G N(p(kx(t)))\, dt \le 1\},$$

respectively. By [2] or Theorem 1.27 in [3],

$$\|x\| = \frac{1}{k}(1 + \int_G M(kx(t))\, dt, \quad x \neq 0,$$

iff $k \in [k_x^*, k_x^{**}]$.

In [4], we obtain that each Orlicz space L_M with the Luxemburg norm ($\|x\|' = \inf\{k > 0 : \varrho_M(x/k) \le 1\}$) has the λ-property and it has the uniform λ-property iff $M(u)$ is strictly convex. In this paper, we shall see that the condition "L_M with the Orlicz norm has the uniform λ-property" is different from "L_M with the Luxemburg norm has the uniform λ-property", and the proving methods are completely different.

Main results.

Lemma 1. *If $x \in U(L_M)$, x is a λ-point.*

PROOF : Since $\text{ext}(B(L_M)) \neq \emptyset$ by [5], taking $e \in \text{ext}(B(L_M))$, we have for any $x \in U(L_M))$

$$\begin{aligned} x &= x + (1 - \|x\|)(\frac{1}{2}e - \frac{1}{2}e) \\ &= \frac{1}{2}(1 - \|x\|)e + \frac{1}{2}(1 + \|x\|)(x - \frac{1}{2}(1 - \|x\|)e)/\frac{1}{2}(1 + \|x\|) \\ &= \frac{1}{2}(1 - \|x\|)e + \frac{1}{2}(1 + \|x\|)y, \end{aligned}$$

where $y = 2(x - \frac{1}{2}(1 - \|x\|)e)/(1 + \|x\|)$ and $y \in B(L_M)$. This shows that x is a λ-point. ■

Theorem 1. *L_M has the λ-property.*

PROOF : By Lemma 1, we only need to prove that for any $x \in S(L_M)$, x is a λ-point. By [5] or Theorem 2.3 in [3], $x \in S(L_M)$ is an extreme point of $B(L_M)$ iff for all $k \in [k_x^*, k_x^{**}], m\{t \in G : kx(t) \in D\} = 0$. Hence for $x \in S(L_M)$ but $x \in \text{ext}(B(L_M))$, there exists $k_x \in [k_x^*, k_x^{**}]$ such that $m\{t \in G : k_x x(t) \in D\} > 0$. Define

$$G_i = \{t \in G : k_x x(t) \in (a_i, b_i)\}, \quad i = 1, 2 \ldots,$$

then $m \bigcup_i G_i > 0$. Without loss of generality, we may assume $x(t) \ge 0$. Let

$$E_i' = \{t \in G_i : k_x x(t) \le \frac{1}{4}b_i + \frac{3}{4}a_i\},$$
$$E_i'' = \{t \in G_i : k_x x(t) \ge \frac{1}{4}a_i + \frac{3}{4}b_i\},$$

G_i', G_i'' be partitions of G_i with $G_i' \supset E_i', G_i'' \supset E_i'', i = 1, 2, \dots$, and

$$k_e = 1 + \sum_i (M(a_i) mG_i' + M(b_i) mG_i'') + \int_{G \setminus \bigcup_i G_i} M(k_x x(t))\, dt,$$

then $k_e < \infty$. Indeed, if $\sum_i M(b_i) mG_i < \infty$, it is clear. Otherwise, we set $c_i = \|x\chi_{G_i}\|_\infty, c_i' = \min\{b_i, 4c_i - a_i\}$ and

$$E_i = \{t \in G_i : k_x x(t) \geq \frac{3}{4} a_i + \frac{1}{4} c_i\}, \quad i = 1, 2, \dots .$$

Obviously, $mG_i > 0$ implies $mE_i > 0$. As $M(u)$ is linear on (a_i, b_i) and

$$\begin{aligned}
&\sum_i M(\frac{3}{4} a_i + \frac{1}{4} c_i) mE_i \\
=&\sum_i \{(M(b_i) - M(a_i))(\frac{3}{4} a_i + \frac{1}{4} c_i - a_i)/(b_i - a_i) + M(a_i)\} mE_i \\
=&\sum_i \{(M(b_i) - M(a_i))(c_i - a_i)/4(b_i - a_i) + M(a_i)\} mE_i \\
\leq&\sum_i \int_{G_i} M(k_x x(t))\, dt \leq \int_G M(k_x x(t))\, dt < \infty,
\end{aligned}$$

we have

$$\begin{aligned}
\sum_i M(c_i') mE_i \leq &\sum_i \{4(M(b_i) - M(a_i))(c_i - a_i)/(b_i - a_i) + M(a_i)\} mE_i \\
=&16 \sum_i M(\frac{3}{4} a_i + \frac{1}{4} c_i) mE_i - 15 \sum_i M(a_i) mE_i < \infty .
\end{aligned}$$

Remarking $mE_i \geq mG_i''$, as $c_i \leq b_i$, and $mG_i'' = 0$ whenever $b_i > c_i', i = 1, 2, \dots$, we obtain

$$k_e \leq 1 + \rho_M(k_x x) + \sum_i M(b_i) mG_i'' \leq 1 + \rho_M(k_x x) + \sum_i M(c_i') mE_i < \infty .$$

Set

$$\lambda = \min\{\frac{1}{4}, k_e/4k_x\}, \quad 1/k_x = \lambda/k_e + (1 - \lambda)/k_y,$$

$$e(t) = \frac{1}{k_e} (\sum_i (a_i \chi_{G_i'} + b_i \chi_{G_i''}) + k_x x(t) \chi_{G \setminus \bigcup_i G_i})$$

and $x = \lambda e + (1 - \lambda) y$. For $t \in G \setminus \bigcup_i G_i$, $k_x x(t) = k_e e(t)$ and

$$\begin{aligned}
&k_x x(t)/k_y = k_x x(t)(\frac{1}{k_x} - \frac{\lambda}{k_e})/(1 - \lambda) \\
=&(k_e - \lambda k_x) x(t)/k_e(1 - \lambda) = (x(t) - \lambda k_x x(t)/k_e)/(1 - \lambda) \\
=&(x(t) - \lambda e(t))/(1 - \lambda) = y(t),
\end{aligned}$$

so $k_x x(t) = k_e e(t) = k_y y(t)$. Since $\lambda k_x/k_e \le \frac{1}{4}$ and $(1-\lambda)k_x/k_y \ge \frac{3}{4}$, for $t \in G_i'$,

$$\begin{aligned} a_i \le & k_x x(t) = \lambda k_x a_i/k_e + (1-\lambda)k_x k_y y(t)/k_y \\ \le & \frac{1}{4}a_i + \frac{3}{4}b_i \le \lambda k_x a_i/k_e + (1-\lambda)k_x b_i/k_y . \end{aligned}$$

From $a_i \le \lambda k_x a_i/k_e + (1-\lambda)k_x k_y y(t)/k_y$, we have $k_y y(t) \ge a_i$ and from

$$\lambda k_x a_i/k_e + (1-\lambda)k_x k_y y(t)/k_y \le \lambda k_x a_i/k_e + (1-\lambda)k_x b_i/k_y ,$$

$k_y y(t) \le b_i$. Similarly, for $t \in G_i''$,

$$\begin{aligned} b_i \ge & k_x x(t) = \lambda k_x b_i/k_e + (1-\lambda)k_x k_y y(t)/k_y \\ \ge & \frac{1}{4}b_i + \frac{3}{4}a_i \ge \lambda k_x b_i/k_e + (1-\lambda)k_x a_i/k_y \end{aligned}$$

and $a_i \le k_y y(t) \le b_i$. Hence we have $k_y y(t) \in [a_i, b_i]$ for $t \in G_i$, $i = 1, 2, \ldots$. This shows that

$$\begin{aligned} 1 = & \|x\| = \frac{1}{k_x}(1 + \int_G M(k_x x(t))\,dt) \\ = & \frac{(1-\lambda)k_e + \lambda k_y}{k_e k_y}(1 + \int_G M(\frac{k_e k_y}{(1-\lambda)k_e + \lambda k_y}(\lambda e(t) + (1-\lambda)y(t)))\,dt) \\ = & \frac{(1-\lambda)k_e + \lambda k_y}{k_e k_y}(1 + \frac{\lambda k_y}{(1-\lambda)k_e + \lambda k_y}\int_G M(k_e e(t))\,dt \\ & \qquad + \frac{(1-\lambda)k_e}{(1-\lambda)k_e + \lambda k_y}\int_G M(k_y y(t))\,dt) \\ = & \frac{\lambda}{k_e}(1 + \varrho_M(k_e e)) + \frac{(1-\lambda)}{k_y}(1 + \varrho_M(k_y y)) = \lambda + \frac{(1-\lambda)}{k_y}(1 + \varrho_M(k_y y)) \end{aligned}$$

and $\|y\| \le \frac{1}{k_y}(1 + \varrho_M(k_y y)) = 1$ by Theorem 10.5 in [**6**].

Now, if we have $e \in \text{ext}(B(L_M))$, then x is a λ-point. To prove $e \in \text{ext}(B(L_M))$, it is enough to show that $k_e = k_e^* = k_e^{**}$ by Theorem 2.3 in [**3**].

Let $k_x = k_x^* = k_x^{**}$. For any $k > k_e$, we fix $k' \in (k_e, k)$ and define $k'' = k'k_e/(\frac{1}{4}k_e + \frac{3}{4}k')$, $k_0 = \min\{k_x k/k', k_x k''/k_e\}$. If $k''(\frac{1}{4}a_i + \frac{3}{4}b_i)/k_e \ge b_i$, then

$$\begin{aligned} & k'(\frac{1}{4}a_i + \frac{3}{4}b_i)/(\frac{1}{4}k_e + \frac{3}{4}k') \ge b_i, \\ & k'(\frac{1}{4}a_i + \frac{3}{4}b_i) \ge \frac{3}{4}k'b_i + \frac{1}{4}k_e b_i, \\ & \frac{1}{4}k'a_i \ge \frac{1}{4}k_e b_i, \text{ and } k'a_i/k_e \ge b_i . \end{aligned}$$

For $t \in G_i', i = 1, 2, \ldots$, if $k''(\frac{1}{4}a_i + \frac{3}{4}b_i)/k_e \ge b_i$, then

$$p(ke(t)) = p(k'ka_i/k_e k') \ge p(kb_i/k') \ge p(kk_x x(t)/k') \ge p(k_0 x(t))$$

and if $k''(\frac{1}{4}a_i + \frac{3}{4}b_i)/k_e < b_i$, then

$$\begin{aligned} p(ke(t)) \geq & p(k_e e(t)) = p(a_i) = p(k''(\frac{1}{4}a_i + \frac{3}{4}b_i)/k_e) \\ \geq & p(k'' k_x x(t)/k_e) \geq p(k_0 x(t)), \end{aligned}$$

as $p(u)$ is right-continuous and $k''(\frac{1}{4}a_i + \frac{3}{4}b_i)/k_e \geq a_i$. Noticing that for $t \in G_i'', i = 1, 2 \dots$,

$$p(ke(t)) = p(kb_i/k_e) \geq p(kk_x x(t)/k_e) \geq p(kk_x x(t)/k') \geq p(k_0 x(t)),$$

$k''/k_e > 1, k/k' > 1$, and $k_0 > k_x^{**}$, we obtain

$$\int_G N(p(ke(t)))\, dt \geq \int_G N(p(k_0 x(t)))\, dt > 1.$$

This yields $k_e \geq k_e^{**}$. Similarly, we have $k_e \leq k_e^*$. So $k_e = k_e^* = k_e^{**}$.

Now, let $k_x^* < k_x^{**}$. For any $s', s'' \in (k_x^*, k_x^{**}), s' < s'', N(p(s'x(t))) \leq N(p(s''x(t)))$ and

$$1 = \int_G N(p(s'x(t)))\, dt \leq \int_G N(p(s''x(t)))\, dt = 1.$$

Hence $N(p(s'x(t))) = N(p(s''x(t)))$ a.e.. As $N(v)$ is convex and $N(v) > 0$ for $v \neq 0$, $p(s'x(t)) = p(s''x(t))$ a.e.. We assume, for simplicity, $p(s'x(t)) = p(s''x(t))$ for all $t \in G$. This implies that for any $s \in (k_x^*, k_x^{**})$ and $t \in G$ with $x(t) \neq 0$, there exist a, b such that $a < sx(t) < b$ and $p(u)$ is constant in (a, b), i.e. $sx(t) \in D$. Remarking $p(u_i) \neq p(u_j)$ for $u_i \in (a_i, b_i), u_j \in (a_j, b_j), i \neq j$, we have

$$\{t \in G : s'x(t) \in (a_i, b_i)\} = \{t \in G : s''x(t) \in (a_i, b_i)\}$$

for any $s', s'' \in (k_x^*, k_x^{**})$ and $k_x^* x(t) \geq a_i, k_x^{**} x(t) \leq b_i$, whenever for some $k \in (k_x^*, k_x^{**})$ with $kx(t) \in (a_i, b_i), i = 1, 2, \dots$. Let

$$N' = \{i : m\{t \in G : kx(t) \in (a_i, b_i)\} > 0, \quad k \in (k_x^*, k_x^{**})\}.$$

Obviously, N' is not empty. If there exist $k_1 \in (k_x^*, k_x^{**})$ and $j \in N'$ such that

$$\begin{aligned} & m\{t \in G : a_j \leq k_1 x(t) \leq a_j/4 + 3b_j/4\} > 0, \\ & m\{t \in G : 3a_j/4 + b_j/4 \leq k_1 x(t) \leq b_j\} > 0, \end{aligned}$$

then taking k_1 instead of k_x, we can choose G_j', G_j'' with $mG_j' > 0, mG_j'' > 0$. For $k > k_e$, without loss of generality, we may assume $k_0 < k_x^{**}$. Hence for $t \in G_j''$

$$p(ke(t)) = p(kb_j/k_e) > p(a_j) = p(k_0 x(t))$$

and for $t \in G \setminus G_j''$, $p(ke(t)) \geq p(k_0 x(t))$. Therefore

$$\int_G N(p(ke(t)))\,dt \geq \int_{G\setminus G_j''} N(p(k_0 x(t)))\,dt + \int_{G_j''} N(p(ke(t)))\,dt =$$
$$= \int_G N(p(k_0 x(t)))\,dt + \int_{G_j''} N(p(ke(t)))\,dt - \int_{G_j''} N(p(k_0 x(t)))\,dt > 1$$

i.e. $k_e \geq k_e^{**}$. Similarly, we can get $k_e \leq k_e^*$. So $k_e = k_e^* = k_e^{**}$.

Otherwise, for all $i \in N'$ either

$$m\{t \in G : a_i \leq k_x^* x(t) < \frac{1}{4}a_i + \frac{3}{4}b_i\} = 0$$

or

$$m\{t \in G : \frac{3}{4}a_i + \frac{1}{4}b_i < k_x^{**} x(t) \leq b_i\} = 0.$$

If there exist $i', i'' \in N'$ such that

$$m\{t \in G : \frac{3}{4}a_{i'} + \frac{1}{4}b_{i'} < k_x^{**} x(t) \leq b_{i'}\} = 0,$$
$$m\{t \in G : a_{i''} \leq k_x^* x(t) < \frac{1}{4}a_{i''} + \frac{3}{4}b_{i''}\} = 0,$$

we may assume $mG_{i'}' > 0, mG_{i''}'' > 0$, or else take $k \in (k_x^*, k_x^{**})$ instead of k_x. As above, we have $k_e = k_e^* = k_e^{**}$.

If for all $i \in N'$

$$m\{t \in G : \frac{3}{4}a_i + \frac{1}{4}b_i < k_x^{**} x(t) \leq b_i\} = 0,$$

then

$$m\{t \in G : k_x^{**} x(t) \in (a_i, b_i)\} > 0, \quad i \in N'.$$

Let $k_x = k_x^{**}$, then $mG_i' > 0, mG_i'' = 0$ for $i \in N'$. In the same way as above we have $k_e = k_e^* = k_e^{**}$. If for all $i \in N'$

$$m\{t \in G : a_i \leq k_x^* x(t) < \frac{1}{4}a_i + \frac{3}{4}b_i\} = 0,$$

let $k_x = k_x^*$, the result is the same. ∎

Lemma 2. *If $D \neq \emptyset$ and $K = \sup\{b_i/a_i : b_i > 1\} < \infty$, then $M(b_i)/N(p(a_i)) \leq 2(K-1)$ provided $M(b_i)/M(a_i) \geq 2K$.*

PROOF : Let $d_i = M(b_i)/M(a_i) \geq 2K$, then

$$d_i = ((b_i - a_i)p(a_i) + M(a_i))/M(a_i)$$

by Theorem 1.1 in [6]. Hence

$$(d_i - 1)M(a_i) = (b_i - a_i)p(a_i)$$

and

$$M(b_i)/(b_i - a_i)p(a_i) = d_i/(d_i - 1).$$

Using the equality in Young inequality, we have

$$(b_i - a_i)p(a_i)/(N(p(a_i)) + M(a_i)) = (b_i - a_i)p(a_i)/a_i p(a_i) \leq K - 1$$

and

$$N(p(a_i)) \geq (b_i - a_i)p(a_i)\left(\frac{1}{K-1} - \frac{1}{d_i - 1}\right).$$

This means

$$M(b_i)/N(p(a_i)) \leq d_i(K-1)(d_i-1)/(d_i-K)(d_i-1)$$
$$\leq d_i(K-1)/(d_i - \frac{1}{2}d_i) \leq 2(K-1).$$

■

Theorem 2. *L_M has the uniform λ-property iff*

$$\sup\{b_i/a_i : b_i > 1\} < \infty.$$

PROOF : If $K = \sup\{b_i/a_i : b_i > 1\} < \infty$, let $N'' = \{i : b_i > 1\}, K' = M(1)mG + 4K + 1$ and $\lambda = 1/4K'$. For $x \in S(L_M)\backslash \text{ext}(B(L_M))$, we define $k_x, G_i, i = 1, 2, \ldots$, and k_e as in Theorem 1. Denote

$$C = 1 + \int_{G\backslash\bigcup_i G_i} M(k_e e(t))\,dt = 1 + \int_{G\backslash\bigcup_i G_i} M(k_x x(t))\,dt.$$

Using Lemma 2 and

$$\sum_i N(p(a_i))mG_i \leq \int_G N(p(x(t)))\,dt \leq 1,$$

by Lemma 9.1 in [6], we have

$$\begin{aligned}
k_x/k_e &= (1+\int_G M(k_x x(t))\,dt)/(1+\int_G M(k_e e(t))\,dt)\\
&\le (\sum_i M(b_i)mG_i + C)/(\sum_i M(a_i)mG_i + C)\\
&\le \frac{M(1)mG + 2K\sum_{i\in N''} M(a_i)mG_i + 2(K-1)\sum_{i\in N''} N(p(a_i))mG_i + C}{\sum_i M(a_i)mG_i + C}\\
&\le M(1)mG + 2K + 2(K-1) + 1 \le K'\,.
\end{aligned}$$

Hence $\lambda k_x/k_e \le \lambda K' \le \frac{1}{4}$. Setting e and $x = \lambda e + (1-\lambda)y$ as in Theorem 1, we may prove that (e, y, λ) is amenable to x in the same way as in Theorem 1. This implies $\lambda(x) \ge 1/4K'$ for $x \in S(L_M)$. By [1], for $x \in B(L_M)$

$$\lambda(x) \ge \frac{1}{2}(1+\|x\|)\lambda(x/\|x\|) \ge 1/8K', \; x \ne 0,$$

and $\lambda(\Theta) = \frac{1}{2}$. Thus, we obtain that L_M has the uniform λ-property.

Let L_M have the uniform λ-property, then

$$\inf\{\lambda(x) : x \in B(L_M)\} = \lambda_0 > 0\,.$$

If $\sup\{b_n/a_n : b_n > 1\} = \infty$, without loss of generality, we may assume $b_n/a_n > n^3$, $n = 1, 2, \ldots$, and $N(p(a_1))mG > 1$. Fix the disjoint sets $F', F'' \subset G$ satisfying $mF' = mF''$ and $N(p(a_1))mF' = \frac{1}{4}$. For $n > 3$, taking $G_n \subset G \setminus F' \cup F''$ such that $N(p(a_n))mG_n = \frac{1}{2}$ and a partition of the same measure $\{E_{n_i}\}_1^n$ of G_n, we define

$$\begin{aligned}
u_{n_i} &= (1 - 1/i\ln n)a_n + b_n/i\ln n \quad 1 \le i \le n,\\
k_n &= 1 + \sum_1^n M(u_{n_i})mE_{n_i} + M(a_1)mF' + m(b_1)mF''
\end{aligned}$$

and

$$x_n = \frac{1}{k_n}(\sum_i u_{n_i}\chi_{E_{n_i}} + a_1\chi_{F'} + b_1\chi_{F''})\,.$$

For $k < k_n$, $kx_n(t) < b_n, t \in G_n$; $kx_n(t) < b_1, t \in F''$, and $kx_n(t) < a_1, t \in F'$ imply $p(kx_n(t)) \le p(a_n), t \in G_n$; $p(kx_n(t)) \le p(a_1), t \in F''$ and $p(kx_n(t)) < p(a_1), t \in F'$. Hence

$$\begin{aligned}
&\int_G N(p(kx_n(t)))\,dt\\
&= \int_{G_n} N(p(kx_n(t)))\,dt + \int_{F''} N(p(kx_n(t)))\,dt + \int_{F'} N(p(kx_n(t)))\,dt\\
&< N(p(a_n))mG_n + N(p(a_1))mF'' + N(p(a_1))mF' = 1.
\end{aligned}$$

For $k > k_n$, as $kx_n(t) > b_1, p(kx_n(t)) > p(a_1), t \in F''$,

$$\int_G N(p(kx_n(t)))\,dt > N(p(a_n))mG_n + N(p(a_1))mF' + N(p(a_1))mF'' = 1.$$

Thus $k_n = k^*_{x_n} = k^{**}_{x_n}$. By Theorem 1.27 in [3],

$$\|x_n\| = \frac{1}{k_n}(1 + \varrho_M(k_n x_n)) = 1.$$

By Theorem 1, x_n is a λ-point, $n = 3, 4, \dots$. Let (e_n, y_n, λ_n) be amenable to x,

$$\|e_n\| = (1 + \varrho_M(k_{e_n} e_n))/k_{e_n},$$
$$\|y_n\| = (1 + \varrho_M(k_{y_n} y_n))/k_{y_n}$$

and

$$k_n' = k_{e_n} k_{y_n} / (\lambda_n k_{y_n} + (1 - \lambda_n) k_{e_n}),$$

then

$$\begin{aligned} &\|x_n\| = \lambda_n \|e_n\| + (1 - \lambda_n)\|y_n\| \\ =&\frac{\lambda_n}{k_{e_n}}(1 + \varrho_M(k_{e_n} e_n)) + \frac{(1 - \lambda_n)}{k_{e_n}}(1 + \varrho_M(k_{y_n} y_n)) \\ =&\frac{1}{k_n'}(1 + \frac{\lambda_n k_n'}{k_{e_n}} \int_G M(k_{e_n} e_n(t))\,dt + \frac{(1 - \lambda_n)k_n'}{k_{y_n}} \int_G M(k_{y_n} y_n(t))\,dt) \\ \geq&\frac{1}{k_n'}(1 + \int_G M(k_n' x_n(t))\,dt) \geq \|x_n\|. \end{aligned}$$

By Theorem 1.27 in [3], $k_n' \in [k_n^*, k_n^{**}]$, hence $k_n' = k_n$. Considering $t \in G_n$, $k_n x_n(t) \in (a_n, b_n)$, we have $k_{e_n} e_n(t), k_{y_n} y_n(t) \in [a_n, b_n]$ for $t \in G_n$ and $k_n x_n(t) = k_{e_n} e_n(t) = k_{y_n} y_n(t)$ for $t \in F' \cup F''$. By Theorem 2.3 in [3], for $t \in G_n$, either $k_{e_n} e_n(t) = a_n$ or $k_{e_n} e_n(t) = b_n$. Since

$$M(b_n) = \int_0^{b_n} p(u)\,du = M(a_n) + \int_{a_n}^{b_n} p(u)\,du \geq (b_n - a_n)p(a_n)$$

and

$$N(p(a_n)) = a_n p(a_n) - M(a_n) \leq a_n p(a_n)$$

by Young inequality, we have

$$M(b_n)/N(p(a_n)) \geq (b_n - a_n)p(a_n)/a_n p(a_n) \geq n^3 - 1.$$

Thus

$$M(b_n)mG_n \geq (n^3 - 1)N(p(a_n))mG_n = \frac{1}{2}(n^3 - 1).$$

Let $E_n' = \{t \in G_n : k_{e_n} e_n(t) = b_n\}$. If $mE_n' = 0$, then

$$\begin{aligned}\lambda_n k_n/k_{e_n} &= \frac{\lambda_n(\sum_1^n M(u_{n_i})mE_{n_i} + C')}{M(a_n)mG_n + C'} \\ &\geq \frac{\lambda_n \sum_1^n((1-1/i\ \ln n)M(a_n) + M(b_n)/i\ \ln n)mE_{n_i}}{M(a_n)mG_n + C'} \\ &\geq \frac{\lambda_n M(b_n)mG_n/n\ \ln n}{M(a_n)mG_n + C'} = \frac{\lambda_n/n\ \ln n}{M(a_n)/M(b_n) + C'/M(b_n)mG_n} \\ &\geq \frac{\lambda_n/n\ \ln n}{M(a_n)/M(n^3 a_n) + 2C'/(n^3-1)} \\ &\geq \frac{\lambda_n/n\ \ln n}{M(a_n)/n^3 M(a_n) + 4C'/n^3} = \lambda_n n^2/(4C'+1)\ln n,\end{aligned}$$

where $C' = M(a_1)mF' + M(b_1)mF'' + 1$. Remarking $\lambda_n k_n/k_{e_n} \leq 1$, as $1/k_n = \lambda_n/k_{e_n} + (1-\lambda_n)/k_{y_n}$, and $\lambda_n \geq \lambda_0$, we have

$$\lambda_0 n^2/(4C'+1)\ln n \leq 1.$$

The contradiction for large n implies that there exists n' such that for $n > n'$, $mE_n' > 0$. Let

$$i(n) = \max\{i : m(E_n' \cap E_{n_i}) > 0, \quad 1 \leq i \leq n\}.$$

For $t \in E_n' \cap E_{n_{i(n)}} \subset G_n$,

$$\begin{aligned}&(1 - 1/i(n)\ \ln n)a_n + b_n/i(n)\ \ln n \\ =&k_n x_n(t) = \frac{\lambda_n k_n}{k_{e_n}} k_{e_n} e_n(t) + \frac{(1-\lambda_n)k_n}{k_{y_n}} k_{y_n} y_n(t) \\ =&\frac{\lambda_n k_n}{k_{e_n}} b_n + \frac{(1-\lambda_n)k_n}{k_{y_n}} k_{y_n} y_n(t) \geq \frac{\lambda_n k_n}{k_{e_n}} + \frac{(1-\lambda_n)k_n}{k_{y_n}} a_n,\end{aligned}$$

as $k_{y_n} y_n(t) \in [a_n, b_n]$. Hence $\lambda_n k_n/k_{e_n} \leq 1/i(n)\ \ln n$.

On the other hand, as $\sum_1^n 1/i < \ln n$,

$$\begin{aligned}\lambda_n k_n/k_{e_n} &= \frac{\lambda_n(\sum_1^n M(u_{n_i})mE_{n_i} + C')}{\sum_1^n M(a_n)mE_{n_i} \setminus E_n' + \sum_1^n M(b_n)mE_{n_i} \cap E_n' + C'} \\ &\geq \frac{\lambda_n \sum_1^n M(b_n)mE_{n_i}/i\ \iota_n n}{\sum_{i>i(n)} M(a_n)mE_{n_i} + \sum_{i\leq i(n)} M(b_n)mE_{n_i} + C'} \\ &\geq \frac{(\lambda_n M(b_n)mG_n \sum_1^n 1/i)/\ \ln n}{(M(a_n)mG_n + i(n)M(b_n)mG_n/n + C')n} \\ &\geq \frac{\lambda_n M(b_n)mG_n}{nM(a_n)mG_n + i(n)M(b_n)mG_n + nC'} \\ &\geq \frac{\lambda_n}{i/n^2 + i(n) + nC'/(n^3-1)} \geq \lambda_0/3i(n)\end{aligned}$$

for large n. Take $n > n'$ satisfying $nC'/(n^3 - 1) < 1$ and $\lambda_0 > 3/\ln n$. Then

$$1/i(n)\ln n \geq \lambda_n k_n/k_{e_n} \geq \lambda_0/3i(n) > 1/i(n)\ln n.$$

The contradiction shows that L_M does not have the uniform λ-property. ∎

Notes.

1. Theorem 1.27 in [**3**] had been used in Chen Shutao's paper "Some rotundities of Orlicz spaces with Orlicz norm" (Bull. Acad. Polon. Sci. **34** (1986), No. 9–10, 585–596).

2. $\text{ext}(B(L_M)) \neq \emptyset$. In fact, $L_M = E^*_{(M)}$, where

$$E_{(M)} = \{u \in L_M : \varrho_M(ku) < \infty \text{ for all } k > 0\}$$

with norm $\|\cdot\|_{(M)}$ (see § 14.5 and Theorem 14.2 in [**6**]). By Krein–Milman theorem,

$$B(L_M) = \overline{\text{co}}^{w^*}\,\text{ext}(B(L_M)).$$

Therefore, $\text{ext}(B(L_M)) \neq \emptyset$.

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