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Abstract. Due to A. Lasota, a continuous function f from a real compact interval I into itself is called generically chaotic if the set of all points $[x,y]$, for which $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$, is residual in $I \times I$. In the paper the generically chaotic functions are characterized in terms of behaviour of subintervals of I under iterates of f and also in terms of topological transitivity. Using this characterization some other properties of generically chaotic functions are proved.

Keywords: Generic chaos, dense chaos, topological transitivity, topological entropy

Classification: 58F13, 54H20, 26A18

1. Introduction and main results.

In this paper a function will always be a function belonging to the space $C^0(I,I)$ of all continuous maps of a real compact interval I into itself, endowed with the topology of uniform convergence. For a function f and $\varepsilon > 0$ define the following planar sets:

$$\begin{aligned} C_1(f) &= \{[x,y] \in I^2 : \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0\}, \\ C_2(f) &= \{[x,y] \in I^2 : \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0\}, \\ C_2(f,\varepsilon) &= \{[x,y] \in I^2 : \limsup_{n\to\infty} |f^n(x) - f^n(y)| > \varepsilon\}, \\ C(f) &= C_1(f) \cap C_2(f), \\ C(f,\varepsilon) &= C_1(f) \cap C_2(f,\varepsilon). \end{aligned}$$

A function f is called chaotic in the sense of Li and Yorke (see [**7**]) if there is an uncountable set S such that $C(f) \supset S \times S \setminus \{[x,x] : x \in I\}$. Clearly, S cannot contain more than one asymptotically periodic point, so our definition is equivalent to that in which S must be disjoint with the set of all asymptotically periodic points. Several conditions characterizing chaos in the sense of Li and Yorke are given in [**5**].

Recently A. Lasota (cf. [**11**]) proposed another definition of chaos, so-called generic chaos. Being inspired by his definition we introduce some other kinds of chaos.

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Definition 1.1. A function $f \in C^0(I, I)$ is called

(i) (see [11]) generically chaotic if the set $C(f)$ is residual in I^2,
(ii) generically ε-chaotic if the set $C(f, \varepsilon)$ is residual in I^2,
(iii) densely chaotic if the set $C(f)$ is dense in I^2,
(iv) densely ε-chaotic if the set $C(f, \varepsilon)$ is dense in I^2.

Fix an integer $r \geqslant 2$ and a sequence $0 = a_0 < a_1 < \cdots < a_r = 1$. Let $f : [0,1] \to [0,1]$ be a map such that

(i) $\varphi_i = f \mid [a_{i-1}, a_i]$ is continuous and is differentiable in $]a_{i-1}, a_i[$ for $i = 1, \ldots, r$,
(ii) $\varphi_i([a_{i-1}, a_i]) = [0,1]$ for $i = 1, \ldots, r$,
(iii) there exists a $q > 1$ such that $\inf |\varphi_i'| \geqslant q$ in $]a_{i-1}, a_i[$ for $i = 1, \ldots, r$.

J. Piórek [11] proved that such a function f is generically chaotic.

The main result of this paper is the characterization of generically chaotic functions given in Theorem 1.2. But first the notation used: Throughout the paper an interval will always be a nondegenerate interval lying in I. It will not necessarily be compact. If J is an interval then $\operatorname{diam} J$ is its length. If $A, B \subset I$ then $\operatorname{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. We write $\operatorname{dist}(A, b)$ instead of $\operatorname{dist}(A, \{b\})$. A compact interval J will be called an invariant transitive interval of f if it is f-invariant and the restriction of f to the interval J is topologically transitive. For any set $A \subset I$, $\operatorname{int} A$ is the interior of A and $\operatorname{Orb}(f, A) = \bigcup_{n=0}^{\infty} f^n(A)$.

Theorem 1.2. *Let $f \in C^0(I, I)$. The following conditions are equivalent:*

(a) *f is generically chaotic,*
(b) *for some $\varepsilon > 0$, f is generically ε-chaotic,*
(c) *for some $\varepsilon > 0$, f is densely ε-chaotic,*
(d) *$C_1(f)$ is dense in I^2 and $C_2(f)$ is a second Baire category set in any interval $J^2 \subset I^2$,*
(e) *$C_1(f)$ is dense in I^2 and for some $\varepsilon > 0$, $C_2(f, \varepsilon)$ is dense in I^2,*
(f) *the following two conditions are fulfilled simultaneously:*
(f-1) *for every two intervals J_1, J_2, $\liminf_{n \to \infty} \operatorname{dist}(f^n(J_1), f^n(J_2)) = 0$,*
(f-2) *there exists an $a > 0$ such that for every interval J, $\limsup_{n \to \infty} \operatorname{diam} f^n(J) > a$,*
(g) *the following two conditions are fulfilled simultaneously:*
(g-1) *there exists a fixed point x_0 of f such that for every interval J, $\lim_{n \to \infty} \operatorname{dist}(f^n(J), x_0) = 0$,*
(g-2) *there exists a $b > 0$ such that for every interval J, $\liminf_{n \to \infty} \operatorname{diam} f^n(J) > b$,*
(h) *the following two conditions are fulfilled simultaneously:*
(h-1) *f has a unique invariant transitive interval or two invariant transitive intervals having one point in common,*
(h-2) *for every interval J there is an invariant transitive interval T of f such that $\operatorname{Orb}(f, J) \cap \operatorname{int} T \neq \emptyset$.*

Moreover, the equivalences (b)$\Longleftrightarrow$ (c)$\Longleftrightarrow$(e)$\Longleftrightarrow$(f) *hold with the same* ε *and with* $a = \varepsilon$ *in* (f-2).

It is easy to see that in Theorem 1.2 we may without loss of generality assume that an interval always means a compact interval.

The condition (h-2) is equivalent to the condition that the set of all $x \in I$ with $\mathrm{Orb}(f, \{x\}) \cap \mathrm{int}\, T = \emptyset$ for every invariant transitive interval T of f, is closed and nowhere dense in I.

Any generically chaotic function is densely chaotic but not conversely (see Example 3.6 below).

As a consequence of (h)$\Longrightarrow$(a) we get that topologically transitive functions are generically chaotic. The converse statement is not true (see Example 3.1 and, for a stronger result, Example 3.2).

Let $\varepsilon > 0$. A function f is said to be Lyapunov ε-unstable at a point x if for every neighbourhood U of x, there is $y \in U$ and $n \geqslant 0$ with $|f^n(y) - f^n(x)| > \varepsilon$. From (a)$\Longrightarrow$(f-2) it follows that if f is generically chaotic then there exists an $\varepsilon > 0$ such that f is Lyapunov ε-unstable at every point $x \in I$ (ε does not depend on x). Consequently, any generically chaotic function has sensitive dependence on initial conditions (in the sense of [**4**]) and is topologically sensitive (in the sense of [**3**]). From (a)$\Longrightarrow$(g-2) and [**12**, pp. 83–84] we get the following stronger result: if f is generically chaotic, then there exists $\eta > 0$ such that for each $x \in I$ and for each typical point y chosen in I, we have $|f^n(y) - f^n(x)| > \eta$ for infinitely many $n \geqslant 0$.

A function having positive topological entropy (and consequently a function chaotic in the sense of Li and Yorke) need not be generically chaotic, since it may be constant or identical in some interval. However, the converse statement is true. In fact, it is not difficult to show that every function satisfying the condition (g) has a horseshoe. Using the condition (h) we prove the following stronger result.

Theorem 1.3. *In the space* $C^0(I,I)$ *for every* $0 < \varepsilon < \operatorname{diam} I$ *the number* $(1/2)\log 2$ *is the minimum of the topological entropies of all generically (or, equivalently, densely)* ε*-chaotic functions (and hence also of all generically chaotic functions).*

The position of generically chaotic functions in the Šarkovskii ordering (see [**13**]) is specified by the following

Theorem 1.4. *Let* $f \in C^0(I,I)$ *be generically chaotic. Then* f *has a periodic orbit of period* 2.3 *and may or may not have periodic orbits of odd periods greater than* 1.

In connection with Theorem 1.3 and Theorem 1.4 we would like to emphasize that neither of the conditions (1) $h(f) \geqslant (1/2)\log 2$ and (2) f has a periodic orbit of period 2.3, implies the other.

It is well known (see [**6**] and [**1**]) that the set of all functions which are chaotic in the sense of Li and Yorke is residual in the space $C^0(I,I)$. On the other hand, we have

Theorem 1.5. *The set of all generically chaotic functions is dense in itself but is nowhere dense in* $C^0(I,I)$. *The same is true for densely chaotic functions and also for generically (or, equivalently, densely)* ε*-chaotic functions.*

2. Definitions, notations and known results.

We recall some definitions not given in Section 1. The iterates of $f \in C^0(I,I)$ are defined inductively by f^0 =identity map and $f^{n+1} = f^n \circ f$, $n \geqslant 0$. A point $x \in I$ is periodic if $f^n(x) = x$ for some $n > 0$. The least such n is called the period of x. A point of period one is called a fixed point. A point x is said to be an asymptotically periodic point of f if there is a periodic point p of f with $\lim_{n\to\infty} |f^n(x) - f^n(p)| = 0$. The orbit of $x \in I$ under f is $\mathrm{orb}(f,x) = \{f^n(x) : n \geqslant 0\}$. Thus $\mathrm{orb}(f,x) = \mathrm{Orb}(f,\{x\})$ since for $A \subset I$ we put $\mathrm{Orb}(f,A) = \bigcup_{n=0}^{\infty} f^n(A)$. If no misunderstanding can arise we suppress f and shortly write $\mathrm{orb}(x)$ and $\mathrm{Orb}(A)$. If x is periodic with period n then its orbit is also called periodic with period n. A function f is said to be topologically transitive if there is a point whose orbit under f is dense in I. Such points are called transitive. A set $A \subset I$ is f-invariant if $f(A) \subset A$. $\overline{A}$ denotes the closure of A, $f \mid A$ is the restriction of f to the set A. Functions $f, g \in C^0(I,I)$ are topologically conjugate if there is a homeomorphism $h : I \to I$ such that $f = h \circ g \circ h^{-1}$. In the sequel we will often write $\mathrm{LSD}(f,J)$ or $\mathrm{LID}(f,J)$ instead of $\limsup\limits_{n\to\infty} \mathrm{diam}\, f^n(J)$ or $\liminf\limits_{n\to\infty} \mathrm{diam}\, f^n(J)$, respectively.

We denote the topological entropy of f by $h(f)$. If f is piecewise monotone then $h(f) = \lim_{n\to\infty}(1/n)\log N_n$, where N_n is the number of monotone pieces of f^n (see [**8**], [**9**]). Recall that the following conditions are equivalent (see e.g. [**14**] for references):

(i) $h(f) > 0$,

(ii) f has a periodic orbit of period not a power of 2,

(iii) f has a horseshoe, i.e. there are closed intervals $J, K \subset I$ having at most one point in common, and positive integers m, n such that $J \cup K \subset f^m(J) \cap f^n(K)$.

If f has positive topological entropy then f is chaotic in the sense of Li and Yorke but not conversely.

The set of all fixed points of f will be denoted by $\mathrm{Fix}(f)$ and the set of all $x \in \mathrm{Fix}(f)$ with $\lim_{n\to\infty} \mathrm{dist}(f^n(J), x) = 0$ by $\mathrm{Fix}(f,J)$.

Let $\min I = a_0 < a_1 < \cdots < a_n = \max I$ and $b_i \in I$, $i = 0,1,\dots,n$. Then by $\langle (a_0,b_0),\dots,(a_n,b_n)\rangle$ will be denoted the piecewise linear function which sends a_i to b_i, $i = 0,1\dots,n$ and is linear on each interval $[a_i, a_{i+1}]$, $i = 0,1\dots,n-1$.

3. Examples.

We want to present some simple examples to illustrate Theorem 1.2. Here I will be the interval $[0,1]$. If f is one of the functions considered by J. Piórek (see Section 1) then for every interval J there is an n with $f^n(J) = [0,1]$. Hence by Theorem 1.2, every such function is generically chaotic and is even topologically transitive and generically ε-chaotic with any $\varepsilon < 1$ (since (f)$\Longrightarrow$(b) holds with $a = \varepsilon$). In particular, the standard tent map is such a function.

Example 3.1. $f = \langle(0,2/3),(1/3,1/3),(1/2,2/3),(2/3,1/3),(1,1/2)\rangle$ is generically chaotic. In fact, it is generically ε-chaotic with any $\varepsilon < 1/3$, since for every interval J there is an n with $f^n(J) \supset [1/3,2/3]$. The interval $[1/3,2/3]$ is the unique invariant transitive interval of f. Slightly modifying this example, it is easy to find generically chaotic functions arbitrarily close to a constant function.

Example 3.2. $f = \langle(0,0),(1/5,4/5),(2/5,1/5),(3/5,4/5),(4/5,1/5),(1,1)\rangle$ is generically ε-chaotic with any $\varepsilon < 3/5$. The interval $[1/5,4/5]$ is the unique invariant transitive interval of f. One can easily find generically chaotic functions arbitrarily close to the identity map. The function f is not topologically transitive and though it is generically chaotic and onto, it is not possible to find a finite number of points $x_1,\dots,x_n$ such that $\bigcup_{i=1}^n \overline{\mathrm{orb}(x_i)} = [0,1]$.

Example 3.3. $f = \langle(0,1/3),(1/6,0),(2/3,1),(1,1/3)\rangle$ is generically ε-chaotic with any $\varepsilon < 1/3$. $[0,1/3]$ and $[1/3,1]$ are invariant transitive intervals of f.

Example 3.4. $f = \langle(0,0),(1/6,5/12),(1/4,11/12),(5/12,5/12),(1,1/8)\rangle$ is generically ε-chaotic with any $\varepsilon < 1/2$. $[1/6,11/12]$ is the unique invariant transitive interval of f. Here when investigating behaviour of intervals it is useful to consider the second iterate of $f \mid [1/6,11/12]$.

Example 3.5. $f = \langle(0,2/3),(2/9,1),(4/9,1/3),(5/9,2/3),(2/3,1/3),(1,0)\rangle$ maps $[0,1/3]$ onto $[2/3,1]$ and vice versa. Thus it is not densely chaotic though, as one can show, there are two points x_1,x_2 with $\overline{\mathrm{orb}(x_1)\cup\mathrm{orb}(x_2)} = [0,1]$.

Example 3.6. ([10]) For $n = 0,1,2,\dots$ denote $a_n = 1-1/3^n$, $b_n = 1-1/(4.3^{n-1})$, $c_n = 1-1/(2.3^n)$, $I_n = [a_n,1]$ and put

$$f_n = \langle(a_0,a_0),(b_0,1),(c_0,a_0),\dots,(a_n,a_n),(b_n,1),(c_n,a_n),(a_{n+1},a_{n+1}),(1,1)\rangle.$$

Define the function f as the uniform limit of f_n for $n \to \infty$. Since all the intervals I_n are f-invariant, the function f does not satisfy the condition (f-2) from Theorem 1.2. Thus f is not generically chaotic. We are going to prove that f is densely chaotic. So let J_1, J_2 be intervals. We need to prove that $(J_1 \times J_2) \cap C(f) \neq \emptyset$. But it is not difficult to see that for every interval J, $\mathrm{Orb}(f,J) \ni 1$ and consequently, there are nonnegative integers r,s with $f^s(J_1) \cap f^s(J_2) \supset I_r$. So it suffices to prove that $(I_r \times I_r) \cap C(f) \neq \emptyset$. But this is obvious, since $f \mid I_r$ is chaotic in the sense of Li and Yorke (in fact, $f([a_r,b_r]) = f([b_r,c_r]) = I_r$ and thus f has a horseshoe).

Example 3.7. Let $0 < \varepsilon < 1$. Take $\varepsilon < \delta < 1$ and define

$$f = \langle(0,\delta),(\delta/2,1),(\delta,\delta),(1,0)\rangle$$

Then f is generically ε-chaotic and topologically transitive and a simple computation gives $h(f) = (1/2)\log 2$.

4. Preliminary results.

To prove our main results we need several lemmas. Some of them are stated in stronger versions than necessary, since we hope that they themselves can be interesting.

We omit the proof of the next lemma which follows easily from the uniform continuity of f.

Lemma 4.1. *Let $f \in C^0(I,I)$, $g = f^k$ for some positive integer k, J be an interval and $A, B \subset I$ be nonempty sets. Then*

(i) $\liminf_{n\to\infty} \operatorname{dist}(f^n(A), f^n(B)) = 0$ *if and only if* $\liminf_{n\to\infty} \operatorname{dist}(g^n(A), g^n(B)) = 0$,

(ii) $\lim_{n\to\infty} \operatorname{dist}(f^n(A), f^n(B)) = 0$ *if and only if* $\lim_{n\to\infty} \operatorname{dist}(g^n(A), g^n(B)) = 0$,

(iii) $\mathrm{LID}(f,J) = 0$ *if and only if* $\mathrm{LID}(g,J) = 0$,

(iv) $\mathrm{LSD}(f,J) = 0$ *if and only if* $\mathrm{LSD}(g,J) = 0$,

(v) *f satisfies* (f-2) *if and only if g satisfies* (f-2) *(i.e. the condition* (f-2) *with f replaced by g and $a > 0$ replaced by some $a' > 0$ is fulfilled),*

(vi) *f satisfies* (g-2) *if and only if g satisfies* (g-2) *(i.e. the condition* (g-2) *with f replaced by g and $b > 0$ replaced by some $b' > 0$ is fulfilled. Moreover, here we can put $b' = b$.).*

Lemma 4.2. *Let $f, F \in C^0(I,I)$ be topologically conjugate and let $g = f^k$ for some positive integer k. Then*

(i) *$C_1(f) = C_1(g)$ and $C_2(f) = C_2(g)$,*

(ii) *f is generically or densely chaotic if and only if g is generically or densely chaotic, respectively,*

(iii) *f is generically or densely chaotic if and only if F is generically or densely chaotic, respectively.*

PROOF : (i) is a consequence of Lemma 4.1 (i, ii) and (ii) follows from (i). We prove (iii). Let $f = h \circ F \circ h^{-1}$ be a topological conjugacy. Then $C(f) = \{[h(x), h(y)] : [x,y] \in C(F)\}$ and the implication from the right to the left follows. The converse implication is a consequence of the symmetry of the relation of topological conjugacy. ∎

Lemma 4.3. *Let $f \in C^0(I,I)$. Then the following three conditions are equivalent:*

(i) *$C_1(f)$ is residual in I^2,*

(ii) *$C_1(f)$ is dense in I^2,*

(iii) (f-1) *from Theorem* 1.2.

PROOF : The implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) are obvious. We are going to prove (iii)$\Longrightarrow$(i). So let (f-1) be fulfilled. Since $C_1(f) = \bigcap_{n=1}^{\infty} L(n, 1/n)$ where $L(n,1/n) = \{[x,y] \in I^2 : \inf_{k \geqslant n} |f^k(x) - f^k(y)| < 1/n\}$ are open sets, it suffices to prove that for every n, $L(n,1/n)$ is dense in I^2. So take any positive integer n and intervals J_1, J_2. We prove that $L(n,1/n) \cap (J_1 \times J_2) \neq \emptyset$. From (f-1) it follows that there exists $k \geqslant n$ with $\operatorname{dist}(f^k(J_1), f^k(J_2)) < 1/n$. This implies the existence of points $x \in J_1, y \in J_2$ such that $|f^k(x) - f^k(y)| < 1/n$. Hence $[x,y] \in L(n,1/n)$ and the proof is complete. ∎

Remark 4.4. The equivalence (i)$\Longleftrightarrow$(ii) follows also from the fact that $C_1(f) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{[x,y] \in I^2 : |f^k(x) - f^k(y)| < 1/n\}$ is a G_δ-set. The sets $C_2(f)$ and $C_2(f,\varepsilon)$ are of type $G_{\delta\sigma}$.

Lemma 4.5. *Let $f \in C^0(I,I)$ and let $J_i, i = 1,2,\ldots,n$, be pairwise disjoint intervals such that $f(J_i) \subset J_{i+1}, i = 1,\ldots,n-1$ and $f(J_n) \subset J_1$. Let the condition* (f-1) *from Theorem* 1.2 *be fulfilled. Then $n \leqslant 2$.*

PROOF : Denote $M = \overline{J_1} \cup \cdots \cup \overline{J_n}$.

Case 1. M is not an interval, i.e. M has at least two components. It follows from the assumptions that these components can be denoted by the symbols $M_1,\ldots,M_k$ such that $f(M_i) \subset M_{i+1}, i = 1,\ldots,k-1$ and $f(M_k) \subset M_1$. Since $\operatorname{dist}(M_i, M_j) > 0$ for $i \neq j$, we have $\liminf_{n\to\infty} \operatorname{dist}(f^n(M_1), f^n(M_2)) > 0$, a contradiction with (f-1). Therefore Case 1 is impossible.

Case 2. M is an interval. Let $(Q_1,\ldots,Q_n)$ be such a permutation of the intervals $J_1,\ldots,J_n$ that $\sup Q_i \leqslant \inf Q_j$ for every $i,j \in \{1,\ldots,n\}, i < j$. We are going to prove that $n \leqslant 2$. Assume on the contrary that $n > 2$. Then it cannot be that $f(Q_1) \subset Q_n$ and $f(Q_n) \subset Q_1$ simultaneously. Let, e.g., $f(Q_1) \not\subset Q_n$. Then $f(Q_1) \subset Q_r$ for some $1 < r < n$. Denote $K = Q_2 \cup Q_3 \cup \cdots \cup Q_n$. Then $f(K) \cap Q_1 \neq \emptyset, f(K) \cap Q_n \neq \emptyset, f(K) \cap Q_r = \emptyset$. Since the set $\overline{K} \setminus K$ is finite and the set Q_r is infinite, it is obvious that $f(\overline{K})$ is not an interval. We have a contradiction because $\overline{K}$ is an interval. ∎

Lemma 4.6. *Let J be a bounded real interval and let $f : J \to J$ be a continuous function without fixed points. Let $K \subset J$ be a compact interval. Then* $\lim_{n\to\infty} \operatorname{diam} f^n(K) = 0$.

PROOF : Let $f(x) > x$ for all $x \in J$ (if $f(x) < x$ for all $x \in J$, we proceed analogously). Denote $\sup J = b$. Let F be the continuous extension of f over the interval $J \cup \{b\}$. Then $F(b) = b$ and for every $x \in K, \lim_{n\to\infty} f^n(x) = b$. Let $\varepsilon > 0$ and $x \in K$. Take a positive integer $n(x)$ with $f^{n(x)}(x) > b - \varepsilon$. Since f is continuous and for all $y \in J$ we have $f(y) > y$, there exists an open neighbourhood $U(x)$ of x such that for every $n \geqslant n(x), f^{n(x)}(U(x)) \subset]b-\varepsilon, b[$. Take $x_1,\ldots,x_k$ with $\bigcup_{i=1}^k U(x_i) \supset K$ and put $N = \max\{n(x_i) : i = 1,\ldots,k\}$. Then for every $n \geqslant N, f^n(K) \subset]b-\varepsilon, b[$ and the lemma follows. ∎

Lemma 4.7. *Let $f \in C^0(I,I)$ and J be a compact interval with* $\operatorname{LSD}(f,J) > 0$. *Then* $\operatorname{Orb}(J)$ *contains a periodic point of f. Moreover, if the condition* (f-1) *from Theorem* 1.2 *is fulfilled then* $\operatorname{Orb}(J)$ *contains a periodic point of f with period 1 or 2.*

PROOF : From the assumption $\operatorname{LSD}(f,J) > 0$ it follows the existence of integers $r \geqslant 0$ and $k \geqslant 1$ such that $f^r(J) \cap f^{r+k}(J) \neq \emptyset$. Then the sets $K_i = \bigcup_{j=0}^{\infty} f^{r+i+jk}(J), i = 0,1,\ldots,k-1$, are intervals. Thus the set $\operatorname{Orb}(f^r(J)) = \bigcup_{i=0}^{k-1} K_i$ has at most k components. Since $f(K_i) = K_{i+1}$ for $i = 0,1,\ldots,k-2$ and $f(K_{k-1}) \subset K_0$, it is possible to denote these components by the symbols $J_1, J_2,\ldots,J_n, n \leqslant k$ in such a way that $f(J_i) = J_{i+1}$ for $i = 1,\ldots,n-1$ and $f(J_n) \subset J_1$. (Here notice that by Lemma 4.5 the condition (f-1) would imply $n \leqslant 2$.)

Denote $g = f^n$. Then $g(J_i) \subset J_i, i = 1,\ldots,n$. The compact interval $f^r(J)$ is a subset of J_s for some $s \in \{1,\ldots,n\}$. Since $\operatorname{LSD}(f,J) > 0$ we also have

$\mathrm{LSD}(f, f^r(J)) > 0$ and by Lemma 4.1 (iv), $\mathrm{LSD}(g, f^r(J)) > 0$. Now it follows from Lemma 4.6 that $g = f^n$ must have a fixed point in J_s. This point is a periodic point of f and lies in $\mathrm{Orb}(J)$ since $J_s \subset \mathrm{Orb}(J)$.

The proof of the lemma will be complete if we realize that (f-1) implies $n \leqslant 2$ and that the fixed point of $g = f^n$ where $n = 1$ or 2 is a periodic point of f of period 1 or 2. ∎

Lemma 4.8. *Let $f \in C^0(I, I)$, let $C_1(f)$ be dense in I^2 and let for every interval $A \subset I$, the set $C_2(f)$ be of the second category in A^2. Let x_0 be a fixed point of f. Then there exists $\delta > 0$ such that no interval containing x_0 and having diameter less than δ is f-invariant.*

Proof : Since the closure of an invariant interval is an invariant interval with the same diameter it suffices to prove the claim of our lemma for compact intervals. Assume on the contrary that for every $\delta > 0$ there is a compact interval $J(\delta)$ with the properties: $x_0 \in J(\delta), \operatorname{diam} J(\delta) < \delta, f(J(\delta)) \subset J(\delta)$. Infinitely many of the intervals $J(1/n), n = 1, 2, \ldots$ have the right endpoints greater than x_0 or infinitely many of them have the left endpoints less than x_0. Without loss of generality we may suppose the former possibility. Further take into account that the intersection of two compact invariant intervals is a compact invariant interval. Now it is not difficult to see that there exists a sequence of invariant intervals $J_n, n = 1, 2, \ldots$ of the form $J_n = [x_0 - a_n, x_0 + b_n]$ where $\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 0$ and for every n, $0 < b_{n+1} < b_n, 0 \leqslant a_{n+1} \leqslant a_n$ and $a_{n+1} = a_n$ if and only if $a_n = 0$.

Case 1. For every $n, a_n > 0$. Then for every n we have $J_{n+1} \subset \operatorname{int} J_n$. Let m be a positive integer. By Lemma 4.3 the set $C_1(f)$ is residual in $I \times J_{m+1}$. Thus there exists a set $K_m \subset I$ such that K_m is residual in I and for every $x \in K_m$ there is $y \in J_{m+1}$ with $\liminf\limits_{n\to\infty} |f^n(x) - f^n(y)| = 0$. Since $J_{m+1} \subset \operatorname{int} J_m$ and the interval J_m is invariant, we can see that for every $x \in K_m$ there exists a positive integer $m(x)$ such that $\mathrm{orb}(f^{m(x)}(x)) \subset J_m$. Now consider the set $K = \bigcap_{m=1}^{\infty} K_m$. It is residual in I and it is easy to see that for every $x \in K, \lim_{n\to\infty} f^n(x) = x_0$.

Case 2. For some $n, a_n = 0$. Without loss of generality we may assume that $a_1 = 0$ and consequently $a_n = 0$ for all n. Now we cannot use the inclusions $J_{n+1} \subset \operatorname{int} J_n$ from Case 1. But it suffices to take into account that $f(J_1) \subset J_1$ and analogously as in Case 1 we can prove that the orbits of points from J_1 generically converge to x_0, i.e. that there exists a set $K \subset J_1$, K residual in J_1 such that $\lim_{n\to\infty} f^n(x) = x_0$ for all $x \in K$.

We can see that in either case there are an interval $A \subset I$ and a set $K \subset A$, K residual in A such that for any $x \in K, \lim_{n\to\infty} f^n(x) = x_0$. Consequently, for every $[x, y] \in K^2$ we have $\lim_{n\to\infty} |f^n(x) - f^n(y)| = 0$. Since K^2 is residual in A^2 we get that $C_2(f)$ is a first category set in A^2. This contradiction finishes the proof. ∎

Lemma 4.9. *Let $f \in C_0(I, I)$, x_0 be a fixed point of f and let the condition* (f-1) *from Theorem 1.2 be fulfilled. Let there exist arbitrarily small f-invariant intervals arbitrarily close to the point x_0. Then there exist arbitrarily small f-invariant intervals containing the point x_0.*

PROOF : Let the assumptions be fulfilled. We have a sequence of points x_n and a sequence of invariant intervals $J_n, n = 1, 2, \ldots$, such that $x_n \in J_n, \lim_{n\to\infty} x_n = x_0, \lim_{n\to\infty} \operatorname{diam} J_n = 0$. Without loss of generality we can assume that x_0 is not the right endpoint of I and that $x_n \geqslant x_0$ for all n. If for infinitely many n we have $x_n = x_0$, the lemma is proved. So let us consider the opposite case. We may assume that $x_n > x_0$ for all $n = 1, 2, \ldots$.

Case 1. For some i, $\inf J_i > x_0$. Since $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} \operatorname{diam} J_n = 0$, it is possible to find an interval J_j such that $\operatorname{dist}(J_i, J_j) > 0$. Since the intervals J_i, J_j are f-invariant we have a contradiction with (f-1). Therefore this case is impossible.

Case 2. For all i, $\inf J_i \leqslant x_0$. Then the intervals $\overline{J_i}$ are f-invariant, they contain the point x_0 and $\lim_{n\to\infty} \operatorname{diam}(\overline{J_i}) = 0$. The lemma is proved. ∎

Lemma 4.10. *Let $f \in C^0(I, I)$, x_0 be a fixed point of f, M be an interval containing x_0. Let for every $x \in M$, $\liminf_{n\to\infty} |f^n(x) - x_0| = 0$. Then*

(i) *if $x \in \operatorname{Orb}(M)$ and $x < x_0$ or $x > x_0$ then for every positive integer $k, f^k(x) > x$ or $f^k(x) < x$, respectively,*
(ii) *$M \cup f(M)$ is an f-invariant interval.*

PROOF : (i) If the assumptions of the lemma are fulfilled, $\operatorname{Orb}(M)$ is an interval containing x_0 and for every $x \in \operatorname{Orb}(M)$, $\liminf_{n\to\infty} |f^n(x) - x_0| = 0$. It follows from it that it suffices to prove (i) with $\operatorname{Orb}(M)$ replaced by M. So let $y \in M, y < x_0$ (if $y > x_0$ the proof is similar). Suppose on the contrary that $f^k(y) \leqslant y$ for some positive integer k. Denote $f^k = g$. By Lemma 4.1 (i) we have $\liminf_{n\to\infty} |g^n(y) - x_0| = 0$. Hence $g^n(y) \neq y$ for $n = 1, 2, \ldots$. Thus there exists a positive integer r such that $g^r(y) < g^{r-1}(y) < \cdots < g(y) < y$ and $g^r(y) < g^{r+1}(y)$. Take a fixed point p of g between $g^r(y)$ and $g^{r-1}(y)$ and a point q between y and x_0 with $g^r(q) = p$. Then $q \in M$ and $\liminf_{n\to\infty} |g^n(q) - x_0| > 0$, which is a contradiction (see Lemma 4.1 (i)).

(ii) Clearly, $K = M \cup f(M)$ is an interval. We are going to prove that $f(K) \subset K$. Suppose this is not the case. Then there exists an $a \in f(M)\backslash M$ such that $f(a) \notin K$. Since $a \notin M$, we have $a \geqslant \sup M$ or $a \leqslant \inf M$. Without loss of generality we may assume that $a \geqslant \sup M$. By (i) there is a $b \in M, b < x_0$ with $f(b) = a$. Further, by (i), $f(a) < a$. Since $f(a) \notin K$, we have $f(a) \leqslant \inf M$. Hence $f^2(b) \leqslant b$, which is a contradiction with (i).

∎

Lemma 4.11. *Let $f \in C^0(I, I), x_0$ be a fixed point of f, J be a compact interval. Let for every $x \in J, \liminf_{n\to\infty} |f^n(x) - x_0| = 0$. Then* $\lim_{n\to\infty} \operatorname{diam} f^n(J) = 0$.

PROOF : Let the assumptions be fulfilled and let $\operatorname{LSD}(f, J) > 0$. Then according to Lemma 4.7, $\operatorname{Orb}(J)$ contains a periodic point p of f. Since $\liminf_{n\to\infty} |f^n(p) - x_0| = 0$ it must be $p = x_0$. Hence for some s we have $x_0 \in f^s(J)$. Without loss of generality we may assume that $x_0 \in J$. Denote $K = J \cup f(J)$. By Lemma 4.10, $f(K) \subset K$. To finish the proof it suffices to show that $\lim_{n\to\infty} \operatorname{diam} f^n(K) = 0$. Since $K \supset f(K) \supset f^2(K) \supset \cdots \supset \{x_0\}$, it suffices to prove that $\bigcap_{n=0}^{\infty} f^n(K) = \{x_0\}$. Denote this intersection by A and suppose that $A \neq \{x_0\}$. Then A is a compact interval and it is easy to see that $f(A) = A$. Then (see [14, p. 45]) f has at least two periodic points in the interval A. We have a contradiction since for every $x \in A$, $\liminf_{n\to\infty} |f^n(x) - x_0| = 0$. ∎

Lemma 4.12. *Let $f \in C^0(I, I)$ and J be an interval. Then the following two conditions are equivalent:*

(1) *there are $x, y \in J$ with $\liminf_{n\to\infty} |f^n(x) - f^n(y)| > 0$,*
(2) *there are $x, y \in J$ with $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$.*

Further, the following two conditions are equivalent:

(3) $\mathrm{LID}(f, J) > 0$,
(4) $\mathrm{LSD}(f, J) > 0$,

and either of the conditions (1), (2) *implies either of the conditions* (3), (4). *Moreover, if J is a compact interval then all the conditions* (1)–(4) *are equivalent.*

PROOF : The implications (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) are obvious for arbitrary intervals. Now let J be a compact interval. We are going to prove that then (4) $\Longrightarrow$ (1). So let $\mathrm{LSD}(f, J) > 0$ and for every $x, y \in J$, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$. According to Lemma 4.7 there exist a point $x_0 \in I$, a nonnegative integer s and a positive integer r such that $x_0 \in f^s(J)$ and $f^r(x_0) = x_0$. Denote $f^s(J) = K$ and $f^r = g$. Then K is a compact interval, $g(x_0) = x_0$ and by Lemma 4.1 (i) we have $\liminf_{n\to\infty} |g^n(x) - x_0| = 0$ for every $x \in K$. From Lemma 4.11 we get $\mathrm{LSD}(g, J) = 0$. By Lemma 4.1 (iv), $\mathrm{LSD}(f, J) = 0$. This contradiction finishes the proof that for a compact interval J, all the conditions (1)–(4) are equivalent. To complete the proof of the lemma we need to prove that (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (3) for any interval J. The former implication is a consequence of the facts that it holds for compact intervals and for any $x, y \in J$ there is a compact subinterval of J containing x, y. The latter implication follows from the equalities $\mathrm{LSD}(f, J) = \mathrm{LSD}(f, \overline{J})$ and $\mathrm{LID}(f, J) = \mathrm{LID}(f, \overline{J})$. ∎

Remark 4.13. In several lemmas we have the assumption that some interval is compact. They do not hold without this assumption. For example if we take $I = [0, 1]$ and $f(x) = x^{1/2}$, then (4) $\Longrightarrow$ (1) in Lemma 4.12 does not hold for $J =]0, 1]$.

Remark 4.14. Using Lemma 4.12 one can prove that the following conditions are equivalent (cf. Lemma 4.3):

(i) $C_2(f)$ is dense in I^2,
(ii) for every interval J, $\mathrm{LSD}(f, J) > 0$,
(iii) for every interval J, $\mathrm{LID}(f, J) > 0$.

Example 3.6 shows that if the set $C_2(f)$ is dense in I^2 then it need not be residual in I^2.

Lemma 4.15. *Let $f \in C^0(I, I)$ and let there be an $a > 0$ such that for every interval J, $\mathrm{LSD}(f, J) > a$. Then there are $\varepsilon > 0$ and $b > 0$ such that for every interval J, $\mathrm{LSD}(f, J) > a + \varepsilon$ and $\mathrm{LID}(f, J) > b$.*

PROOF : Let the assumptions be fulfilled and let $J_i, i = 1, 2, \dots, k$ be intervals such that $\bigcup_{i=1}^{k} J_i = I$ and $\operatorname{diam} J_i \leqslant a/2$ for every i. Denote $a' = \min\{\mathrm{LSD}(f, J_i) : i = 1, 2, \dots, k\}$ and $b' = \min\{\mathrm{LID}(f, J_i) : i = 1, 2, \dots, k\}$. Then $a' > a$ and by Lemma 4.12, $b' > 0$. Now take any interval J. Since $\mathrm{LSD}(f, J) > a$ we have $f^n(J) \supset J_j$ for some $j \in \{1, 2, \dots, k\}$ and some nonnegative integer n. Thus $\mathrm{LSD}(f, J) \geqslant \mathrm{LSD}(f, J_j) \geqslant a'$ and $\mathrm{LID}(f, J) \geqslant \mathrm{LID}(f, J_j) \geqslant b'$. The lemma follows. ■

Lemma 4.16. *Let $f \in C^0(I, I)$. Then the following four conditions are equivalent:*

(i) *there exists an $a > 0$ such that for every interval J, $\mathrm{LSD}(f, J) > a$ (i.e. the condition (f-2) from Theorem 1.2),*

(ii) *there exists a $b > 0$ such that for every interval J, $\mathrm{LID}(f, J) > b$ (i.e. the condition (g-2) from Theorem 1.2),*

(iii) *there exists an $a > 0$ such that the set $C_2(f, a)$ is residual in I^2,*

(iv) *there exists an $a > 0$ such that the set $C_2(f, a)$ is dense in I^2.*

The equivalences (i)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) hold with the same $a > 0$. Each of these four conditions implies the condition

(v) *$C_2(f)$ is residual in I^2*

and the condition (v) implies the condition

(vi) *$C_2(f)$ is a second Baire category set in J^2 for every interval J.*

Moreover, if (f-1) from Theorem 1.2 is fulfilled then all the conditions (i)–(vi) are equivalent.

PROOF : (i)$\Longrightarrow$(ii) follows from Lemma 4.15, (ii)$\Longrightarrow$(i) is trivial.

(i)$\Longrightarrow$(iii) Let (i) be fulfilled. By Lemma 4.15 there is an $\varepsilon > 0$ such that for every interval J, $\mathrm{LSD}(f, J) > a + \varepsilon$. Since $C_2(f, a) \supset \bigcap_{n=1}^{\infty} C_2(f, n, a + \varepsilon)$ where $C_2(f, n, c) = \{[x, y] \in I^2 : \sup_{k \geqslant n} |f^k(x) - f^k(y)| > c\}$, it suffices to prove that the sets $C_2(f, n, a + \varepsilon)$ are residual for every n. But these sets are open, so it suffices to prove that they are dense in I^2. Let n be a positive integer and J_1, J_2 be intervals. We are going to prove that there are $x \in J_1, y \in J_2$ with $[x, y] \in C_2(f, n, a + \varepsilon)$.

Case 1. For some r, $f^r(J_1) \subset f^r(J_2)$. Since $\mathrm{LSD}(f, J_1) > a + \varepsilon$ we can take $k \geqslant \max\{r, n\}$ with $\operatorname{diam} f^k(J_1) > a + \varepsilon$. Since $f^k(J_1) \subset f^k(J_2)$ there are $x \in J_1, y \in J_2$ with $|f^k(x) - f^k(y)| > a + \varepsilon$ whence $[x, y] \in C_2(f, n, a + \varepsilon)$.

Case 2. For every r, $f^r(J_1) \setminus f^r(J_2) \neq 0$. Now take $k \geqslant n$ with $\operatorname{diam} f^k(J_2) > a + \varepsilon$. Since $f^k(J_1)$ is not a subset of $f^k(J_2)$, there are $x \in J_1, y \in J_2$ with $|f^k(x) - f^k(y)| > a + \varepsilon$ and again $[x, y] \in C_2(f, n, a + \varepsilon)$.

(iii)$\Longrightarrow$(iv) This is trivial.

(iv)$\Longrightarrow$(i) Let (iv) be fulfilled. Take any interval J. Since $C_2(f, a)$ is dense there are $x, y \in J$ such that $\limsup_{n \to \infty} |f^n(x) - f^n(y)| > a$. It follows from it that $\mathrm{LSD}(f, J) > a$, so we have (i).

(iii)$\Longrightarrow$(v)$\Longrightarrow$(vi) These implications are trivial.

((vi) & (f-1))$\Longrightarrow$(i) Now let (f-1) from Theorem 1.2 and (vi) be fulfilled. We are going to prove (i). With respect to Lemma 4.1 (v) it suffices to prove that

$g = f^2$ satisfies (f-2), i.e. that there exists an $a > 0$ such that for every interval J, $\mathrm{LSD}(g, J) > a$.

Moreover, here it suffices to consider only the intervals containing fixed points of g. In fact, take any interval J. Then (vi) gives $\mathrm{LSD}(f, J) > 0$. Hence $\mathrm{LSD}(f, \overline{J}) > 0$ and by Lemma 4.7, $\mathrm{Orb}(f, \overline{J})$ contains a periodic point of f with period 1 or 2. Thus for some s, $g^s(\overline{J})$ contains a fixed point of g. Now it suffices to take into account that $\mathrm{LSD}(g, J) = \mathrm{LSD}(g, \overline{J}) = \mathrm{LSD}(g, g^s(\overline{J}))$.

So (i) will be proved if we show that $\inf\{\mathrm{LSD}(g, J) : J$ is an interval containing a fixed point of $g\} > 0$. Assume on the contrary that there exist a sequence $x_i, i = 1, 2, \ldots$ of fixed points of g and a sequence $K_i, i = 1, 2, \ldots$ of intervals such that for every i, $x_i \in K_i$ and $\lim_{i\to\infty} \mathrm{LSD}(g, K_i) = 0$. Without loss of generality we can assume that there exists $\lim_{i\to\infty} x_i = p$. Clearly, p is a fixed point of g. Denote $\mathrm{LSD}(g, K_i) = \varepsilon_i$. For every i there is a k_i such that for all $k \geqslant k_i$, $\operatorname{diam} g^k(K_i) \leqslant 2\varepsilon_i$. Denote $J_i = \bigcup_{k=k_i}^{\infty} g^k(K_i)$. Then $x_i \in J_i, \operatorname{diam} J_i \leqslant 4\varepsilon_i$ and $g(J_i) \subset J_i$. Further, the sets J_i are intervals. In fact, by Lemma 4.2 (i) we have $C_2(f) = C_2(g)$ and now (vi) and Lemma 4.12 imply that for every i and k, the set $g^k(K_i)$ is not a singleton.

We have shown that arbitrarily close to the fixed point p of g there are arbitrarily small g-invariant intervals J_i. Further, by Lemma 4.1 (i), g satisfies (f-1). Now it follows from Lemma 4.9 that there exist arbitrarily small g-invariant intervals containing the point p.

On the other hand, by Lemma 4.2 (i), $C_1(f) = C_1(g)$ and $C_2(f) = C_2(g)$. Now using (vi), Lemma 4.3 and Lemma 4.8, we get that there is a $\delta > 0$ such that no interval containing p and having diameter less than δ, is g-invariant. So we have a contradiction and the proof of the lemma is complete. ∎

Lemma 4.17. *Let $f \in C^0(I, I)$, J be an interval and $\mathcal{S}$ be a family of intervals. Then*

(i) *there are $a, b \in \mathrm{Fix}(f)$, not necessarily $a \leqslant b$, with $\mathrm{Fix}(f, J) = \{x \in \mathrm{Fix}(f) : a \leqslant x \leqslant b\}$;*
(ii) *in (i), $\mathrm{Fix}(f, J) \setminus \{a, b\} \subset \mathrm{Orb}(f, J)$;*
(iii) *if $\mathrm{Fix}(f, J_1) \cap \mathrm{Fix}(f, J_2) \neq \emptyset$ for every $J_1, J_2 \in \mathcal{S}$, then $\bigcap_{J\in\mathcal{S}} \mathrm{Fix}(f, J) \neq \emptyset$.*

Proof : The statements (i) and (ii) are trivial. To prove (iii) suppose that $\mathrm{Fix}(f, J_1) \cap \mathrm{Fix}(f, J_2) \neq \emptyset$ for every $J_1, J_2 \in \mathcal{S}$. Using (i) one can prove easily that each finite subfamily of $\{\mathrm{Fix}(f, J) : J \in \mathcal{S}\}$ has nonempty intersection. Since $\mathrm{Fix}(f, J), J \in \mathcal{S}$ are closed sets in the compact space I, we have $\bigcap_{J\in\mathcal{S}} \mathrm{Fix}(f, J) \neq \emptyset$. ∎

Lemma 4.18. *Let $f \in C^0(I, I), x_0, y_0 \in \mathrm{Fix}(f)$, J be an interval containing y_0. Let $\liminf_{n\to\infty} \mathrm{dist}(f^n(J), x_0) = 0$. Then $x_0 \in \mathrm{Fix}(f, J)$.*

Proof : If $x_0 = y_0$, the lemma is trivial. So we may without loss of generality assume that $x_0 < y_0$. Clearly, the set $\mathrm{Orb}(J)$ is an f-invariant interval and $\inf \mathrm{Orb}(J) \leqslant x_0$. If $\inf f^k(J) \leqslant x_0$ for some nonnegative integer k, the lemma holds. So we may assume that $\inf \mathrm{Orb}(J) = x_0 < \inf f^k(J)$ for every $k = 0, 1, 2, \ldots$. Now we distinguish two cases.

Case 1. For every $\varepsilon > 0$ there is an $x \in]x_0, x_0 + \varepsilon[$ with $f(x) \leqslant x$. Then it is easy to see that $x_0 \in \mathrm{Fix}(f, J)$ and the proof is finished.

Case 2. There is an $\varepsilon > 0$ such that for every $x \in]x_0, x_0+\varepsilon[$, $f(x) > x$. We are going to show that this case is impossible. Denote $s = \sup \mathrm{Orb}(J)$. Since $x_0 \notin \mathrm{Orb}(J)$, we have $f(t) > x_0$ for every $t \in]x_0, s[$. There are two possibilities.

Subcase 2A. $f(s) > x_0$. Denote $m = \min f([x_0 + \varepsilon, s])$. Then $\mathrm{Orb}(J) \subset [\min\{\inf J, m\}, s]$ and we have a contradiction, since $\inf \mathrm{Orb}(J) = x_0 < \min\{\inf J, m\}$.

Subcase 2B. $f(s) = x_0$. Then for every nonnegative integer k, $\sup f^k(J) < s$. In fact, in the opposite case we would have a contradiction with the fact that for every k, $x_0 < \inf f^k(J)$. Now denote $M = \max f([x_0, s])$. Since $M \in f(]x_0, s[)$ we have $M \in \mathrm{Orb}(J)$. Then $M < s$ and $\max\{\sup J, M\} < s = \sup \mathrm{Orb}(J)$. If we realize that $\mathrm{Orb}(J) \subset]x_0, \max\{\sup J, M\}]$ we get a contradiction. ∎

Lemma 4.19. *Let $f \in C^0(I, I)$, $a, b \in \mathrm{Fix}(f)$, J_1, J_2 be intervals with $a \in J_1, b \in J_2$. Let $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$. Then $\mathrm{Fix}(f, J_1) \cap \mathrm{Fix}(f, J_2) \neq \emptyset$.*

PROOF : If $a = b$ the lemma is trivial. So let $a \neq b$, say $a < b$. For $i = 1, 2$ denote $F_i = \{x \in \mathrm{Fix}(f) : \liminf_{n\to\infty} \mathrm{dist}(f^n(J_i), x) = 0\}$. By Lemma 4.18, $F_i = \mathrm{Fix}(f, J_i), i = 1, 2$. Denote $c_1 = \max F_1$ and $c_2 = \min F_2$. To prove the lemma it suffices to show that $c_1 \geqslant c_2$. So suppose on the contrary that $c_1 < c_2$. There are two possibilities.

Case 1. $\mathrm{Fix}(f) \cap]c_1, c_2[= \emptyset$. Let for every $x \in]c_1, c_2[$, $f(x) > x$ (if $f(x) < x$ for every x between c_1 and c_2 we proceed similarly). If for some $n, \sup f^n(J_1) > c_1$, then $c_2 \in F_1$ and we have a contradiction. If for every $n, \sup f^n(J_1) \leqslant c_1$, then the assumption that $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$ implies that $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_2), c_1) = 0$. Hence $c_1 \in F_2$ and we have a contradiction again.

Case 2. There exists a fixed point p of f such that $\max F_1 = c_1 < p < c_2 = \min F_2$. From the definition of F_1 and F_2 we have $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_i), p) > 0$ for $i = 1, 2$. But then $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) > 0$, a contradiction. The proof of the lemma is complete. ∎

Lemma 4.20. *Let $f \in C^0(I, I)$, J_1, J_2 be intervals with $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$ and $\mathrm{LSD}(f, J_i) > 0, i = 1, 2$. Then there exists a periodic point x_0 of f such that $\lim_{n\to\infty} \mathrm{dist}(f^n(J_i), f^n(x_0)) = 0$ for $i = 1, 2$ and, consequently, $\lim_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$. Moreover, if the condition (f-1) from Theorem 1.2 is fulfilled, then x_0 can be chosen such that it is a fixed point of f, i.e. $\mathrm{Fix}(f, J_1) \cap \mathrm{Fix}(f, J_2) \neq \emptyset$.*

PROOF : Since $\mathrm{dist}(A, B) = \mathrm{dist}(\overline{A}, \overline{B})$, $\mathrm{diam}\, A = \mathrm{diam}\, \overline{A}$ and $f(\overline{A}) = \overline{f(A)}$ for every sets A, B, we can assume the intervals J_1, J_2 to be compact. According to Lemma 4.7, $\mathrm{Orb}(f, J_i)$ contains a periodic point of f of period $p_i, i = 1, 2$. Let $g = f^p$, where p is the least common multiple of p_1 and p_2. Then $\mathrm{Orb}(g, J_1)$ and $\mathrm{Orb}(g, J_2)$ contain fixed points of g. Without loss of generality we can assume that J_1 and J_2 contain them. So let $a \in J_1, b \in J_2$ be fixed points of g. Since by

Lemma 4.1 (i) $\liminf_{n\to\infty} \operatorname{dist}(g^n(J_1), g^n(J_2)) = 0$, Lemma 4.19 implies the existence of a point $x_0 \in \operatorname{Fix}(g, J_1) \cap \operatorname{Fix}(g, J_2)$. The point x_0 is a periodic point of f and by Lemma 4.1 (ii) we have $\lim_{n\to\infty} \operatorname{dist}(f^n(J_i), f^n(x_0)) = 0$ for $i = 1, 2$. Thus the proof of the first part of the lemma is finished.

Now let, moreover, the condition (f-1) from Theorem 1.2 be fulfilled. Then by Lemma 4.7, we can assume that $p_i \in \{1, 2\}$ for $i = 1, 2$ and consequently, p is 1 or 2. Thus we have $\lim_{n\to\infty} \operatorname{dist}(f^n(J_i), f^n(x_0)) = 0$, $i = 1, 2$, for some periodic point x_0 of f of period 1 or 2. To finish the proof we need to show that if the period of x_0 is 2, then there is a fixed point y_0 of f such that $y_0 \in \operatorname{Fix}(f, J_1) \cap \operatorname{Fix}(f, J_2)$. So suppose that the period of x_0 is 2. Let y_0 be a fixed point of f lying between x_0 and $f(x_0)$. We are going to prove that $y_0 \in \operatorname{Fix}(f, J_1)$. This is clear if for some n, $f^n(J_1) \ni y_0$. So suppose that $y_0 \notin \operatorname{Orb}(f, J_1)$. Then it is not difficult to see that the fixed point $a \in J_1$ of $g = f^2$ mentioned above is a periodic point of f of period 2 and y_0 lies between a and $f(a)$. Since by the condition (f-1) from Theorem 1.2 $\liminf_{n\to\infty} \operatorname{dist}(f^n(J_1), f^n(f(J_1))) = 0$, we have $\liminf_{n\to\infty} \operatorname{dist}(g^n(J_1), y_0) = 0$. Then from Lemma 4.18 we get $y_0 \in \operatorname{Fix}(g, J_1)$ and by Lemma 4.1 (ii), $y_0 \in \operatorname{Fix}(f, J_1)$. Similarly one can show that $y_0 \in \operatorname{Fix}(f, J_2)$. The proof of the lemma is complete. ∎

Lemma 4.21. *Let $f \in C^0(I, I)$, A, B be intervals and let for every subinterval A_0 of A and every subinterval B_0 of B,* $\operatorname{Orb}(f, A_0) \cap B_0 \neq \emptyset$. *Then there is a point $x \in A$ with $\overline{\operatorname{orb}(f, x)} \supset B$.*

PROOF : Let $B_1, B_2, \ldots$ be the sequence of all open intervals with rational endpoints lying in B. Then for every i, the set $A_i = A \cap \bigcup_{k=0}^{\infty} f^{-k}(B_i)$ is an open dense set in A. Hence $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$. So there is a point $x \in A$ with the desired property. ∎

Lemma 4.22. *Let $f \in C^0(I, I)$ be topologically transitive. Then the condition* (f) *from Theorem* 1.2 *is fulfilled.*

PROOF : We are going to prove the condition (f-1) by showing that whenever J_1, J_2 are intervals and ε is positive, we have $\operatorname{dist}(f^n(J_1), f^n(J_2)) < \varepsilon$ for some nonnegative integer n. To prove it, take a transitive point $x \in J_1$ and an m with $f^m(x) \in J_2$. Denote $f^m(x)$ by y. Let x_0 be a fixed point of f. Take some $0 < \delta \leqslant \varepsilon/2$ with $f^m(]x_0 - \delta, x_0 + \delta[) \subset]x_0 - \varepsilon/2, x_0 + \varepsilon/2[$ and a k with $f^k(y) \in]x_0 - \delta, x_0 + \delta[$. Then for $n = m + k$ we have $|f^n(x) - f^n(y)| < \varepsilon$ and thus $\operatorname{dist}(f^n(J_1), f^n(J_2)) < \varepsilon$.

It remains to prove (f-2). Since f is transitive, it has a periodic orbit of period four (in the opposite case every trajectory would have at most two limit points (see [**14**, p. 66]), which is a contradiction with the topological transitivity of f). Then f has a periodic orbit $\{a, b\}$ of period two lying in $\operatorname{int} I$. Because of the transitivity of f the interval with endpoints a, b cannot be f-invariant. It follows from it that there is a point z lying between a and b with $f(z) \in \{a, b\}$. Then there is a $d > 0$ such that none of the points $f^k(z), k = 1, 2, \ldots$ belongs to $]z - d, z + d[$. Now take any interval J, a transitive point $x \in J$ and a positive integer m with $f^m(x) \in J$. Denote $f^m(x)$ by y. Let $\varepsilon > 0$. Take some $0 < \delta \leqslant \varepsilon/2$ with $f^m(]z - \delta, z + \delta[) \subset]f^m(z) - \varepsilon/2, f^m(z) + \varepsilon/2[$ and a k with $f^k(y) \in]z - \delta, z + \delta[$. Then for $n = m + k$ we have $|f^n(x) - f^n(y)| > d - \varepsilon$ and thus $\operatorname{diam} f^n(J) > d - \varepsilon$. We have proved that for every interval J and every $\varepsilon > 0$ there is an n with

$\operatorname{diam} f^n(J) > d - \varepsilon$. Hence for every interval J, $\mathrm{LSD}(f, J) \geqslant d$ and the proof is finished. ∎

Lemma 4.23. *Let $f \in C^0(I, I)$ be topologically transitive and x_0 be a fixed point of f. Then for every interval J, $\lim_{n\to\infty} \operatorname{dist}(f^n(J), x_0) = 0$.*

PROOF : Without loss of generality we may assume that J is a compact interval. By Lemma 4.22 the assumptions of Lemma 4.7 are fulfilled. Hence, $\mathrm{Orb}(f, J)$ contains a periodic point y_0 of f of period 1 or 2. We may assume that $y_0 \in J$. Denote $g = f^2$. Since J contains a transitive point of f, we have $\liminf_{n\to\infty} \operatorname{dist}(f^n(J), x_0) = 0$ and by Lemma 4.1 (i), $\liminf_{n\to\infty} \operatorname{dist}(g^n(J), x_0) = 0$. Since $x_0, y_0 \in \mathrm{Fix}(g)$, Lemma 4.18 gives that $x_0 \in \mathrm{Fix}(g, J)$. From Lemma 4.1 (ii) we obtain $x_0 \in \mathrm{Fix}(f, J)$ and the proof is finished. ∎

5. Proofs of main results.

PROOF of Theorem 1.2: We prove (a)⟹(b)⟹(c)⟹(d)⟹(e)⟹(f)⟹(g) ⟹(h) ⟹(f)⟹(a).

(a)⟹(b) This follows from Lemma 4.3 ((i)⟹(iii)) and Lemma 4.16 (((v)& (f-1))⟹(iii)).

(b)⟹(c) This is trivial.

(c)⟹(d) See Lemma 4.16 ((iv)⟹(vi)).

(d)⟹(e) This follows from Lemma 4.3 ((ii)⟹(iii)) and Lemma 4.16 (((vi)& (f-1))⟹(iv)).

(e)⟹(f) See Lemma 4.3 ((ii)⟹(iii)) and Lemma 4.16 ((iv)⟹(i)).

(f)⟹(g) The implication (f-2)⟹(g-2) is proved in Lemma 4.16 ((i)⟹(ii)). Further, if (f-1) and (f-2) are fulfilled then by Lemma 4.20, $\mathrm{Fix}(f, J_1) \cap \mathrm{Fix}(f, J_2) \neq \emptyset$ for any two intervals J_1, J_2. But then by Lemma 4.17 (iii) we get (g-1).

(g)⟹(h) We will say that an interval A f-covers an interval B if for some non-negative integer n, $f^n(A) \supset B$. Let (g) be fulfilled. Denote $L = [x_0 - b/2, x_0 - b/4]$ and $R = [x_0 + b/4, x_0 + b/2]$. Then at least one of the intervals L and R is a subset of I and every subinterval of I f-covers L or R. Let, for example, $L \subset I$. Either every subinterval of L f-covers L or some subinterval L_1 of L does not f-cover L. But then also $R \subset I$ and every subinterval of L_1 f-covers R. So in either case, by Lemma 4.21, there is a point y whose orbit is dense in some compact interval K. Since $\overline{\mathrm{orb}(f, y)} \supset K$ we have $\overline{\mathrm{orb}(f, y)} \supset \mathrm{Orb}(f, K)$. Clearly, (g) implies (f). Therefore the function f and the compact interval K satisfy all the assumptions of Lemma 4.7. Thus either there is an r such that $f^r(K)$ contains a fixed point of f or there is no such r but then for some s, $f^s(K)$ contains a periodic point of f of period 2. In the former case it is easy to see that $\mathrm{Orb}(f, f^r(K))$ is an invariant transitive interval of f. In the latter case denote by z a fixed point of f lying between $f^s(K)$ and $f^{s+1}(K)$. Then z lies between the intervals $\mathrm{Orb}(f^2, f^s(K))$ and $\mathrm{Orb}(f^2, f^{s+1}(K))$. Since from (g-1) we have $\lim_{n\to\infty} \operatorname{dist}(f^n(f^s(K)), f^n(f^{s+1}(K))) = 0$, the set $\overline{\mathrm{Orb}(f, f^s(K))}$ is an interval. Clearly, it is an invariant transitive interval of f.

So we have proved that f has at least one invariant transitive interval. On the other hand, it is easy to see that two invariant transitive intervals cannot have more than one point in common and in view of (g-1) their distance cannot be positive. Hence f cannot have more than two invariant transitive intervals and if it has two such intervals, their intersection is a singleton which is obviously a fixed point of f. The proof of (h-1) is finished.

It remains to prove (h-2). Denote by U the union of all invariant transitive intervals of f and suppose on the contrary that for some interval J, $\mathrm{Orb}(f,J) \cap \operatorname{int} U = \emptyset$. Since the interval U is f-invariant and by (g-1) we have $x_0 \in \mathrm{Fix}(f,U) \cap \mathrm{Fix}(f,J)$, the point x_0 must be an endpoint of U, say the left one. Then $x_0 - \min I > b$. Denote $[x_0 - b, x_0[$ by M. Since $\mathrm{Orb}(f,J) \supset M$ we have $\mathrm{Orb}(f,M) \cap \operatorname{int} U = \emptyset$. Hence $\mathrm{Orb}(f,M)$ is an interval whose right endpoint is x_0. For every subinterval S of M we have $\mathrm{Orb}(f,S) \cap \operatorname{int} U = \emptyset, x_0 \in \mathrm{Fix}(f,S)$ and $\mathrm{LSD}(f,S) > b$. This implies $\mathrm{Orb}(f,S) = \mathrm{Orb}(f,M)$. By Lemma 4.21, $\overline{\mathrm{Orb}(f,M)}$ is an invariant transitive interval of f. This is a contradiction with the assumption that U is the union of all invariant transitive intervals of f.

(h)$\Longrightarrow$(f) By Lemma 4.22, to every invariant transitive interval T of f it is possible to assign an $a(T) > 0$ such that for every interval $A \subset T$, $\mathrm{LSD}(f,A) > a(T)$. Since f has one or two invariant transitive intervals, there exists the minimum $a > 0$ of such $a(T)'$s. Using (h-2) we get that for every interval $J \subset I, \mathrm{LSD}(f,J) > a$. Thus we proved (f-2).

It remains to prove (f-1). Take any two intervals J_1, J_2. We are going to prove that $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$. Since for every interval J, $\mathrm{Orb}(f,J)$ meets the interior of some invariant transitive interval, we may without loss of generality assume that $J_1 \subset T_1$ and $J_2 \subset T_2$ for some invariant transitive intervals T_1 and T_2 of f. If $T_1 = T_2$ we get $\liminf_{n\to\infty} \mathrm{dist}(f^n(J_1), f^n(J_2)) = 0$ from Lemma 4.22. If T_1 and T_2 are different, we get it from Lemma 4.23, if we realize that T_1 and T_2 have a common fixed point of f.

(f)$\Longrightarrow$(a) This follows from Lemma 4.3 ((iii)$\Longrightarrow$(i)) and Lemma 4.16 ((i)$\Longrightarrow$(v)).

Finally, from the fact that in Lemma 4.16 the implications (iv)$\Longrightarrow$(i) and (i)$\Longrightarrow$(iii) hold with the same a, we get that (e)$\Longrightarrow$(f) and (f)$\Longrightarrow$(b) hold with $a = \varepsilon$. Since (b)$\Longrightarrow$(c) and (c)$\Longrightarrow$(e) hold with the same ε trivially, the proof of the theorem is complete. ∎

PROOF of Theorem 1.3: Let f be generically ε-chaotic. By Theorem 1.2 ((a)$\Longrightarrow$(h)) there is an interval J such that $f \mid J$ is topologically transitive. Now $h(f) \geqslant (1/2)\log 2$ follows from the facts that every topologically transitive function has topological entropy greater than or equal to $(1/2)\log 2$ (see [**3**]) and that the entropy of a function cannot be less than the entropy of its restriction. On the other hand, for any $0 < \varepsilon < \operatorname{diam} I$ there is a generically ε-chaotic function $f \in C^0(I,I)$ with $h(f) = (1/2)\log 2$. In Example 3.7 this is shown for $I = [0,1]$. The general case is similar. ∎

PROOF of Theorem 1.4: Let f be generically chaotic. By Theorem 1.2 ((a)$\Longrightarrow$(h)) there is an interval J such that $g = f \mid J$ is topologically transitive. Then by [**2**, pp. 10 and 12], g^2 has a periodic orbit of period 3. Thus f has a periodic orbit of period 2.3. Examples 3.1 and 3.7 show that generically chaotic functions may or may not have periodic orbits of odd periods greater than 1. ∎

PROOF of Theorem 1.5: We leave some technical details to the reader. As far as nowhere density is concerned, it suffices to prove it for densely chaotic functions. So let $B(f,\varepsilon)$ be an open ball in $C^0(I,I)$. Since f has at least one fixed point x_0, it is possible to define a function $g \in B(f,\varepsilon)$ such that for some x_1 and x_2 very close to x_0 and for some small $\alpha > 0$ the intervals $J_i = [x_i - \alpha, x_i + \alpha]$, $i = 1,2$ are disjoint and $g(J_i) = \{x_i\}, i = 1,2$. Then there is a $\delta > 0$ such that simultaneously $B(g,\delta) \subset B(f,\varepsilon)$ and for every $h \in B(g,\delta), h(J_i) \subset J_i, i = 1,2$. Then $\liminf_{n\to\infty} \operatorname{dist}(h^n(J_1), h^n(J_2)) > 0$ and thus $C(h) \cap (J_1 \times J_2) = \emptyset$. So $B(g,\delta)$ contains no densely chaotic function.

Further, from Lemma 4.2 (iii) and from the fact that arbitrarily close to any function there are functions topologically conjugate with it, we get that generically chaotic functions and also densely chaotic functions are dense in itself. The same result can be obtained for ε-chaos as follows. Take a ball $B(f,\delta)$ where f is generically ε-chaotic. From Theorem 1.2 ((b)$\Longleftrightarrow$(f) with $a = \varepsilon$) and from Lemma 4.15 it follows that f is generically ε_1-chaotic for some $\varepsilon_1 > \varepsilon$. Then it is easy to see that for a homeomorphism h sufficiently close to the identity, the function $F = h \circ f \circ h^{-1}$ belongs to $B(f,\delta)$ and satisfies the condition (f) with $a = \varepsilon$. This completes the proof. ∎

6. Open problems.

We finish our paper with the following problems:

(1) Are the conditions (f-1) and (g-1) from Theorem 1.2 equivalent?

(2) Is it true that if $J \subset I$ is a compact interval, $x_0 \in I$ and $\liminf_{n\to\infty} |f^n(x) - f^n(x_0)| = 0$ for every $x \in J$, then $\lim_{n\to\infty} \operatorname{diam} f^n(J) = 0$? (Cf. Lemma 4.11.)

(3) Under what conditions does dense chaos imply generic chaos?

(4) Characterize densely chaotic functions.

(5) It turns out that densely chaotic functions have positive topological entropy. Find $\inf\{h(f) : f \text{ is densely chaotic}\}$.

(6) Prove an analogue of Theorem 1.4 for densely chaotic functions.

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