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ON STRUCTURAL PROPERTIES OF NORMALITY RELATIONS ON LATTICES

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In the papers [2], [3], [4], [5], [7], [8] and [9] Zelinka, Chajda and Niederle have presented the structure of tolerance relations and compatible tolerance relations on abstract algebras and lattices. A binary relation T on an algebra $\mathfrak{A} = (A, \mathscr{F})$ is a tolerance on $\mathfrak{A}$, if T is a reflexive and symmetric relation on the support A of $\mathfrak{A}$. T is compatible with respect to $\mathfrak{A}$, if T has the substitution property over all operations $f \in \mathscr{F}$ on $\mathfrak{A}$, and in particular, $aTb \Leftrightarrow a \wedge bTa \vee b$ in each lattice L, $a, b \in L$ [3, Thm. 1]. In [1] Beran considered some properties of two kinds of normality relations on lattices; the purpose of this paper is to consider a weakened form of these two relations called normality relation, and to show that some properties proved for compatible tolerance relations on abstract algebras are valid already for normality relations. The base of the considerations here are the ideas of Chajda and Zelinka in [4]. As a general reference we have used the monograph [6] by Szász.

A binary relation N on a lattice L is called a normality relation on L, if it satisfies the following four conditions:

1° aNa for each $a \in L$.
2° $aNb \Rightarrow a \leqq b$, $a, b \in L$.
3° aNb and $cNd \Rightarrow a \wedge cNb \wedge d$, $a, b, c, d \in L$.
4° aNb and $aNc \Rightarrow aNb \vee c$, $a, b, c \in L$.

As $aTb \Leftrightarrow a \wedge bTa \vee b$ for compatible tolerance relations T on lattices, normality relations can be considered as a weakened form of compatible tolerance relations on L. As in the case of tolerances (cf. Chajda, Niederle and Zelinka [5]), one can associate with normality relations on a lattice L a certain covering of L called η-covering of L.

Definition 1. *Let L be a lattice and $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ be a family of convex sublattices M_γ of L with a least element 0_γ, where Γ is a set of indices. $\mathfrak{M}$ is called an η-covering of L, if $\mathfrak{M}$ satisfies the following conditions;*

1° *For each $x \in L$ there is in $\mathfrak{M}$ an element M_γ such that $x = 0_\gamma$.*

2° *For each two elements* M_γ, $M_\beta \in \mathfrak{M}$ *there is in* $\mathfrak{M}$ *an element* M_α *such that* $0_\alpha = 0_\gamma \wedge 0_\beta$ *and* $\{x \wedge y \mid x \in M_\gamma \text{ and } y \in M_\beta\} \subseteq M_\alpha$.

3° *For each two elements* M_γ, $M_\beta \in \mathfrak{M}$, *where* $0_\gamma = 0_\beta$, *there is in* $\mathfrak{M}$ *an element* M_δ *such that* $0_\delta = 0_\gamma = 0_\beta$ *and* $\{x \vee y \mid x \in M_\gamma \text{ and } y \in M_\beta\} \subseteq M_\delta$.

The lemma below shows a connection between η-coverings of L and normality relations on L.

Lemma 1. *Let L be a lattice. Each normality relation N on L generates an η-covering $\mathfrak{M}_N$ of L, and conversely, each η-covering $\mathfrak{M}$ of L determines a normality relation $N_{\mathfrak{M}}$ on L as follows; $aNb \Leftrightarrow$ there is in $\mathfrak{M}$ an element M_γ such that $a = 0_\gamma$ and $b \in M_\gamma$.*

Proof. I: Let $\mathfrak{M}$ be an η-covering of L and we shall show that $\mathfrak{M}$ generates a normality relation $N_{\mathfrak{M}}$ on $L : aN_{\mathfrak{M}}b \Leftrightarrow a = 0_\gamma$ for some $\gamma \in \Gamma$ and $b \in M_\gamma$.

$xN_{\mathfrak{M}}x$ for each $x \in L$, as there exists an index $\gamma \in \Gamma$ for each $x \in L$ such that $x = 0_\gamma$; hence 1° holds for $N_{\mathfrak{M}}$. 2°: As $aN_{\mathfrak{M}}b$ implies $a = 0_\gamma$ and $b \in M_\gamma$ for some $\gamma \in \Gamma$, and, on the other hand, 0_γ is the least element of M_γ, $a \leqq b$. 3°: Let $aN_{\mathfrak{M}}b$ and $cN_{\mathfrak{M}}d$. According to 2° in Definition 1, $a \wedge cN_{\mathfrak{M}}b \wedge d$; the property 4° follows from the point 3° in Definition 1.

II: Let N be a normality relation on L; we shall show that N determines an η-covering $\mathfrak{M}_N$ on L given by the property: $aNb \Leftrightarrow$ there is in $\mathfrak{M}$ an element M_γ such that $a = 0_\gamma$ and $b \in M_\gamma$.

The definition of an element $M_\gamma \in \mathfrak{M}_N$ implies that each M_γ contains at least one element 0_γ. Let $a \in M_\gamma$ and $0_\gamma \leqq x \leqq a$; we show that $x \in M_\gamma$, from which the convexity of M_γ follows in L. $0_\gamma Na$ and xNx imply that $0_\gamma \wedge xNx \wedge a$, where $0_\gamma \wedge x = 0_\gamma$ and $a \wedge x = x$, whence $0_\gamma Nx$, and consequently, $x \in M_\gamma$. M_γ is a sublattice of L, as $0_\gamma Na$ and $0_\gamma Nb$ imply $0_\gamma Na \wedge b$ and $0_\gamma Na \vee b$, whence $a \wedge b$, $a \vee b \in M_\gamma$.

As xNx for each $x \in L$, there is an index $\gamma \in \Gamma$ in $\mathfrak{M}_N$ such that $0_\gamma = x$ is the least element of M_γ for each $x \in L$, and 1° of Definition 1 follows. 2°: Let M_γ, $M_\beta \in \mathfrak{M}_N$. As aNb and cNd imply that $a \wedge cNb \wedge d$, there is in $\mathfrak{M}_N$ a set M_α such that $0_\gamma \wedge 0_\beta = 0_\alpha$ and $\{x \wedge y \mid x \in M_\gamma \text{ and } y \in M_\beta\} \subseteq M_\alpha$; the validity of 3° in Definition 1 can be seen similarly.

Let $\{N_\sigma, \sigma \in \Sigma\}$ be a collection of normality relations on L, where Σ is a set of indices, and $\{\mathfrak{M}_\sigma, \sigma \in \Sigma\}$ the corresponding collection of η-coverings of L. If $\Sigma_x \subseteq \Sigma$ is an index set denoting the subset of all elements M_{γ_σ} belonging to one of the families $\mathfrak{M}_\sigma$ and having a fixed least element. say $x \in L$, then obviously the family $\{ \bigcap_{\sigma \in \Sigma_x} M_{\gamma_\sigma} \mid M_{\gamma_\sigma} \in \mathfrak{M}_\sigma, \Sigma_x \subseteq \Sigma \text{ and } x \in L\}$ is an η-covering of L and generates the greatest normality relation with respect to the set inclusion contained in each of the relations N_σ; this relation is denoted by $\bigwedge_{\sigma \in \Sigma} N_\sigma$. According to this observation and to the corollary of Theorem 17 in [6], we can write the following theorem (cf. Theorem 1 of Chajda and Zelinka in [4]).

Theorem 1. *Let L be a lattice. Then the set $N(L)$ of all normality relations on L is a complete lattice with the least element I ($xIy \Leftrightarrow x = y$) and the greatest element U ($xUy \Leftrightarrow x \leqq y$) with respect to the set inclusion. The meet in $N(L)$ is equal to the set intersection.*

In order to illuminate the properties of $N(L)$, we should at first find a formal way for constructing the join of two normality relations on L. This is done in the following and the basic tool is the concept of a polynomial used by Chajda and Zelinka in the case of the join of tolerance relations [4, Def. 1].

Definition 2. *Let L be a lattice and $X = \{x_1, x_2, \ldots\}$ a countable set of symbols such that $L \cap X = \emptyset = X \cap \{\wedge, \vee\}$. A polynomial of L is defined as follows:*

(*a*) *each element of X is a polynomial of L;*
(*b*) *if p_1 and p_2 are polynomials of L, then $p_1 \wedge p_2$ and $p_1 \vee p_2$ are also polynomials of L;*
(*c*) *there are no other polynomials in L except those defined in* (*a*) *and* (*b*).

Elements of X contained in a polynomial of L are called variables, and if a polynomial contains the variables $x_1, \ldots, x_n$ and no other elements of X, then it is denoted by $p(x_1, \ldots, x_n)$. If each x_i is substituted by an element $a_i \in L$, $i = 1, \ldots, n$, at all places in the polynomial $p(x_1, \ldots, x_n)$, we obtain an element of L denoted by $p(a_1, \ldots, a_n)$.

As the definition above shows, each polynomial $p = p(x_1, \ldots, x_n)$ of L without constants can be decomposed into two different subpolynomials $p_1 = p_1(x_1, \ldots, x_k)$ and $p_2 = p_2(x_{k+1}, \ldots, x_n)$ of p such that $p = p_1 \wedge p_2$ or $p = p_1 \vee p_2$, if $n \geqq 2$. The decomposition of p can be continued as a process until p is decomposed into subpolynomials, each of which contains a single variable only. The subpolynomials given by each step of the decomposition process constitute a rooted tree, and naturally, a polynomial can have several different decomposition trees. We shall say that a *polynomial $p = p(x_1, \ldots, x_n)$* without constants *is regularly joindecomposable with respect* to *a set $A = \{a_1, \ldots, a_n\} \subseteq L$*, if there exists a decomposition tree of p such that each decomposition of p or its every subpolynomial $p_i = p_i(x_{i1}, \ldots, x_{ik})$ ($\{x_{i1}, \ldots, x_{ik}\} \subseteq \{x_1, \ldots, x_n\}$) given by the decomposition process in this tree satisfies, when decomposing p_i with respect to the join operation of L, the demand: if $p_i(x_{i1}, \ldots, x_{ik}) = p_{i1}(x_{i1}, \ldots, x_{id}) \vee p_{i2}(x_{id+1}, \ldots, x_{ik})$, then $p_{i1}(a_{i1}, \ldots, a_{id}) = p_{i2}(a_{id+1}, \ldots, a_{ik})$. Now we are able to prove our construction for the join operation in $N(L)$.

Theorem 2. *Let L be a lattice and let $N_\sigma \in N(L)$ for each σ from some subscript* *set Σ. Let R be a binary relation on L defined as follows: $aRb \Leftrightarrow$ there exist subscripts $\sigma_1, \sigma_2, \ldots, \sigma_m$ of Σ, elements $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ of L, where m is a positive integer, and a polynomial $p(x_1, \ldots, x_m)$ of L without constants such that $a_i N_{\sigma_i} b_i$ for $i = 1, \ldots, m$, $a = p(a_1, \ldots, a_m)$, $b = p(b_1, \ldots, b_m)$ and $p(x_1, \ldots, x_m)$ is regularly joindecomposable with respect to the set $\{a_1, \ldots, a_m\}$. Then R is a normality relation on L and $R = \bigvee_{\sigma \in \Sigma} N_\sigma$.*

Proof. According to the definition of R, $N_\sigma \subseteq R$ for each $\sigma \in \Sigma$; this can be seen by putting $m = 1$, $p(x_1) = x_1$, $a_1 = a$ and $b_1 = b$. The same reason implies the reflexivity of R, too. As $a_i N_{\sigma_i} b_i$ for $i = 1, \ldots, m$, $a_i \leqq b_i$, which implies that in each level of the decomposition $p_j(a_{j1}, \ldots, a_{jk}) \leqq p_j(b_{j1}, \ldots, b_{jk})$, whence $p(a_1, \ldots, a_m) \leqq p(b_1, \ldots, b_m)$, too. Hence aRb implies $a \leqq b$. Let aRb and aRc. Then there are two polynomials $p_1(x_1, \ldots, x_m)$ and $p_2(x_{m+1}, \ldots, x_r)$ and the sets $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_m\}, \{a_{m+1}, \ldots, a_r\}, \{c_{m+1}, \ldots, c_r\}$ such that $p_1(a_1, \ldots, a_m) = = a = p_2(a_{m+1}, \ldots, a_r)$, $b = p_1(b_1, \ldots, b_m)$ and $c = p_2(c_{m+1}, \ldots, c_r)$. On the other hand, $p(x_1, \ldots, x_r) = p_1(x_1, \ldots, x_m) \vee p_2(x_{m+1}, \ldots, x_r)$ is regularly joindecomposable with respect to the set $\{a_1, \ldots, a_m, a_{m+1}, \ldots, a_r\}$ as $p_1(x_1, \ldots, x_m)$ and $p_2(x_{m+1}, \ldots, x_r)$ have this property with respect to the sets $\{a_1, \ldots, a_m\}$ and $\{a_{m+1}, \ldots, a_r\}$, respectively, and $a = p_1(a_1, \ldots, a_m) = p_2(a_{m+1}, \ldots, a_r)$. Moreover, $p(b_1, \ldots, b_m, c_{m+1}, \ldots, c_r) = b \vee c$ and $a_i N_{\sigma_i} d_i$ for each $i = 1, \ldots, r$, where $d_i = b_i$, when $1 \leqq i \leqq m$, and $d_i = c_i$ when $m + 1 \leqq i \leqq r$. Hence $aRb \vee c$. The proof is similar for the case: aRb and cRd imply $a \wedge cRb \wedge d$, and thus R is a normality relation on L.

As $N_\sigma \subseteq R$ for each $\sigma \in \Sigma$, $\bigvee_{\sigma \in \Sigma} N_\sigma \subseteq R$. Let now aRb, $a, b \in L$. Then there exist elements $a_1, \ldots, a_m, b_1, \ldots, b_m$ and a polynomial $p(x_1, \ldots, x_m)$ which is regularly joindecomposable with respect to the set $\{a_1, \ldots, a_m\}$ such that $a_i N_{\sigma_i} b_i$ for $i = = 1, \ldots, m$ and $a = p(a_1, \ldots, a_m)$, $b = p(b_1, \ldots, b_m)$. But then also $a_i (\bigvee_{\sigma \in \Sigma} N_\sigma) b_i$ holds for each $i = 1, \ldots, m$, and as $\bigvee_{\sigma \in \Sigma} N_\sigma \in N(L)$ and $p(x_1, \ldots, x_m)$ is regularly joindecomposable with respect to the set $\{a_1, \ldots, a_m\}$, also $a = p(a_1, \ldots, a_m)$ $(\bigvee_{\sigma \in \Sigma} N_\sigma)$ $p(b_1, \ldots, b_m) = b$ is valid, whence $R \subseteq \bigvee_{\sigma \in \Sigma} N_\sigma$. Consequently, $R = \bigvee_{\sigma \in \Sigma} N_\sigma$, and the theorem follows.

Let N_{ab} denote the least normality relation collapsing a and b, i.e. $aN_{ab}b$. An element c of a complete lattice is called compact, if for each subset $S \subseteq L$ such that $c \leqq \bigvee_{q \in S} q$, there exists a finite subset $S' \subseteq S$ such that $c \leqq \bigvee_{q \in S'} q$. A lattice L is called compactly generated, if each element of L is a join of compact elements of L.

As the join in $N(L)$ is defined so that $a(\bigvee_{\sigma \in \Sigma} N_\sigma)\, b$ reduces to a join of finite number of normality relations N_σ, $a(\bigvee_{i=1}^{m} N_i)\, b$, the normality relations N_{ab} of L are compact elements in $N(L)$. Moreover, the relation $N = \bigvee_{a,b} \{N_{ab} \mid aNb\}$ holds obviously, and so the elements of $N(L)$ are compactly generated. Hence we can write

Theorem 3. *Let L be a lattice. The normality relations N_{ab} on L are compact elements of $N(L)$ for each $a, b \in L$, $a \leqq b$, and $N(L)$ is compactly generated.*

In a specified case the normality relations N_{ab} on L can be defined explicitely; this is done in the following lemma.

Lemma 2. *Let L be a distributive lattice. Then the binary relation R on L, where $rRt \Leftrightarrow$ there is an element $k \in L$ such that $r = k \wedge a$ and $t = k \wedge b$, or $r = t$ (a and b are two fixed elements of L and $a \leqq b$), is a normality relation on L, and moreover, $R = N_{ab}$.*

Proof. At first we shall show that R is a normality relation on L. According to the definition of R, rRt when $r = t$, whence R is reflexive. Further, as $a \leqq b$, $r = a \wedge k \leqq b \wedge k = t$, or $r = t$, and thus 2° holds for R, too.

3°: Let rRt and sRq. Thus there are elements $h, k \in L$ such that $r = a \wedge k$, $t = b \wedge k$, $s = a \wedge h$ and $q = b \wedge h$. By combining these we see that $r \wedge s = a \wedge (h \wedge k)$ and $t \wedge q = b \wedge (h \wedge k)$, whence $r \wedge sRt \wedge q$. If $r = t$ or $s = q$, the proof is obvious.

4°: Let rRt and rRq, which imply the existence of two elements $h, k \in L$ such that $r = a \wedge k = a \wedge h$, $t = b \wedge k$ and $q = b \wedge h$. Now $r = (a \wedge k) \vee (a \wedge h) = a \wedge (k \vee h)$ and $t \vee q = (b \wedge k) \vee (b \wedge h) = b \wedge (k \vee h)$, whence $rRt \vee q$, too. The proof is obvious, if $r = t$ or $r = q$.

Accordingly, R is a normality relation on L, and aRb, as $a = a \wedge b$ and $b = b \wedge b$. Thus $N_{ab} \subseteq R$. On the other hand, if rRt, then $r = t$ implies rNt, or $kN_{ab}k$ and $aN_{ab}b$ together imply $k \wedge a = rN_{ab}t = k \wedge b$, whence $R \subseteq N_{ab}$, and accordingly, $R = N_{ab}$. By substituting rRt by sRt in 4° above, one sees that R is in fact a compatible tolerance relation on L.

In the class of distributive lattices all the normality relations can be characterized by a join of meets of two types of normality relations; these types are a modification of a specified kind of compatible tolerance relations on lattices considered by Chajda in [2] and called constructible tolerances on L. We denote by $(k]$ and $[k)$ the principal ideal and filter, respectively, of L generated by the element $k \in L$, where $(k] = \{a \mid a \leqq k, a \in L\}$ and $[k) = \{a \mid k \leqq a, a \in L\}$. At first we shall show how $(k]$ and $[k)$ generate a normality relation on L; this is done similarly as in [2].

Let L be a lattice and $c \in L$. If for each $a, b \in Lc$ satisfies the identity $(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c$, c is called a semidistributive element of L. Further, c is called a distributive element of L, it $(a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c$ holds for each $a, b \in L$.

Lemma 3. *In a modular lattice L each principal filter $[k)$, where k is a distributive* *element of L, generates a normality relation $N_{[k)}$ defined as follows: $aN_{[k)}b \Leftrightarrow a = b$, or $a \leqq b$ and there is an element $q \in L$ such that $a = q \wedge k$ and $b \in \{q \wedge j \mid j \in [k)\}$.*

Proof. The dual formulation of the proof of [2, Thm. 1] given by Chajda, shows that $\{q \wedge j \mid j \in [k)\}$ is a convex sublattice of L when L is modular.

According to the definition, $N_{[k)}$ satisfies the conditions 1° and 2° of the definition of a normality relation. Let $aN_{[k)}b$ and $cN_{[k)}d$. Then there exist two elements q and w such that $a = q \wedge k$, $b = q \wedge j_1$ for some $j_1 \in [k)$, $c = w \wedge k$ and $d = w \wedge j_2$ for some $j_2 \in [k)$, as the set $\{w \wedge j \mid j \in [k)\}$ is a convex sublattice of L. Then $a \wedge c = (q \wedge w) \wedge k$ and $b \wedge d = (q \wedge w) \wedge (j_1 \wedge j_2)$, where $j_1 \wedge j_2 \in [k)$, whence $a \wedge bN_{[k)}c \wedge d$. The proof is similar, if $a = b$ or $c = d$. Thus 3° holds for $N_{[k)}$.

4°: let $aN_{[k)}b$ and $aN_{[k)}c$, and so there are elements $q, w \in L$ such that $a = q \wedge k = w \wedge k$, $b = q \wedge j_1$ and $c = w \wedge j_2$, where $j_1, j_2 \in [k)$. $a = (q \wedge k) \vee (w \wedge k) = (q \wedge w) \wedge k$ as k is distributive in L. Further, $b \vee c = (q \wedge j_1) \vee (w \wedge j_2) \leqq (q \vee w) \wedge (j_1 \wedge j_2)$. On the other hand, $b \vee c \geqq (q \wedge k) \vee (w \wedge k) = (q \vee w) \wedge k$, and as $\{(q \vee w) \wedge j \mid j \in [k)\}$ is a convex sublattice of L, $b \vee c = (q \vee w) \wedge j$ for some $j \in [k)$, whence $aN_{[k)}b \vee c$. If $a = b$, then $b \vee c = a \vee c = c$, and $aN_{[k)}b \vee c$ holds. Thus $N_{[k)}$ is a normality relation on L.

The proof of Lemma 4 is a direct copy of that of [2, Thm. 2] given by Chajda, and hence we omit it.

Lemma 4. *Let L be a modular lattice with a least element 0. For each semidistributive element $k \in L$, $(k]$ generates a normality relation $N_{(k]}$ on L defined as follows: $aN_{(k]}b \Leftrightarrow a = b$, or $a \leqq b$ and there exists an element $q \in L$ such that $q \vee 0 = a$ and $b \in \{q \vee j \mid j \in (k]\}$.*

This lemma can be formulated in a slightly different form in the case of distributive lattices. This is done in the following corollary, the proof of which is similar to that of Lemma 4.

Corollary. *Let L be a distributive lattice and $0 \in L$. Each ideal J of L generates a normality relation N_J on L defined as follows: $aN_Jb \Leftrightarrow a \leqq b$ and there is an element $q \in L$ such that $q \vee 0 = a$ and $b \in \{q \vee j \mid j \in J\}$.*

Now we can express each N_{ab} by means of the relations $N_{(k]}$ and $N_{[k)}$ in a distributive lattice L having a least element, as shown in the following theorem.

Theorem 4. *Let L be a distributive lattice having a least element 0. Then $N_{ab} = N_{[a)} \wedge N_{(b]}$ for each two elements $a, b \in L$, $a \leqq b$.*

Proof. We show that $N_{ab} \subseteq N_{[a)} \wedge N_{(b]}$ and $N_{ab} \supseteq N_{[a)} \wedge N_{(b]}$, from which the validity of the assertion follows.

Let $a \leqq b$. Thus $b \wedge a = a$ and $b = b \wedge b$, $b \in [a)$, whence $aN_{[a)}b$. On the other hand, as $0 \in L$, $a \vee 0 = a$ and $a \vee b = b$, $a \in (b]$, whence $aN_{(b]}b$. Thus also $a(N_{[a)} \wedge N_{(b]})\, b$, from which the relation $N_{ab} \subseteq N_{[a)} \wedge N_{(b]}$ follows.

Let $r(N_{[a)} \wedge N_{(b]})\, t$, and we assume that $r < t$; the case $r = t$ is obvious. Consequently, $rN_{[a)}t$ and $rN_{(b]}t$. The definition of $N_{[a)}$ implies that there is an element $q \in L$ such that $r = q \wedge a$ and $t = q \wedge j$ for some $j \in [a)$; thus $r \leqq a \leqq b$. From $rN_{(b]}t$ we can conclude that there is an element $w \in L$ such that $w \vee 0 = r = w$ and $w \vee i = r \vee i = t$ for some $i \in (b]$. As $r, i \in (b]$, also $t \in (b]$ and $t \leqq b$. Hence, $b \wedge t = t = b \wedge (q \wedge j)$. As $j \in [a)$, $a = a \wedge j$, and moreover, $r = a \wedge q = a \wedge (q \wedge \wedge j)$. Thus $rN_{ab}t$, from which $N_{[a)} \wedge N_{(b]} \subseteq N_{ab}$ follows.

According to Theorem 3, the normality relations on a distributive lattice L with a least element 0 are joins of meets of the type $N_{[a)} \wedge N_{(b]}$. This property gives a viewpoint for determining the distributivity and modularity of the lattice $N(L)$.

In [4] Chajda and Zelinka consider the completing of a tolerance relation on an algebra $\mathfrak{A}$ by the transitivity property; they call this construction the transitive hull of a tolerance relation on $\mathfrak{A}$ and obtain numerous results on the transitive hulls of an algebra. As the normality relation N has not the general substitution property over the operation $\vee$ on L (cf. the property 4°), i.e. N is non-compatible, we shall consider the least compatible normality relation containing N on L, and this relation is called the compatible hull of N and denoted by H_N. By considering the covering generated by H_N on L, one can obtain information about the type of H_N.

Lemma 5. *Let H_N be the compatible hull of a normality relation N on a lattice L. If aNb, then xH_Ny for any two elements $x, y \in [a, b]$, $x \leqq y$.*

Proof. According to the definition of H_N, aH_Nb, and also xH_Nx, yH_Ny. As H_N is compatible, $x \vee a = xH_Nb = y \vee b$, and by applying the relation yH_Ny, we obtain the final result: $x \wedge y = xH_Ny = y \wedge b$.

The compatible hull H_N of a normality relation N on L has accordingly the properties: (i) aH_Na for each $a \in L$; (ii) $aH_Nb \Rightarrow a \leqq b$, $a, b \in L$; (iii) aH_Nb and $cH_Nd \Rightarrow a \wedge cH_Nb \wedge d$ and $a \vee cH_Nb \vee d$, $a, b, c, d \in L$. Now we are ready to define the covering given by H_N in L.

Definition 3. *Let $\mathfrak{N} = \{M_\gamma, \gamma \in \Gamma\}$ be a collection of convex sublattices of a lattice L. $\mathfrak{N}$ is called an η'-covering of L, if $\mathfrak{N}$ satisfies the conditions*

(1) $\bigcup_{\gamma \in \Gamma} M_\gamma = L$;

(2) *for each two elements $M_\alpha, M_\beta \in \mathfrak{N}$ there are indices $\delta, \sigma \in \Gamma$ such that $\{x \wedge y \mid x \in M_\alpha$ and $y \in M_\beta\} \subseteq M_\delta$ and $\{x \vee y \mid x \in M_\alpha$ and $y \in M_\beta\} \subseteq M_\sigma$.*

Theorem 5. *Let L be a lattice. Each η'-covering $\mathfrak{N}$ of L determines a binary relation $H_{\mathfrak{N}}$ satisfying the conditions of of a compatible hull on L, and conversely, each compatible hull H_N generates an η'-covering $\mathfrak{N}_H$ of L as follows: $xHy \Leftrightarrow$ there is an index $\gamma \in \Gamma$ of $\mathfrak{N}$ such that $x \leqq y$ and $x, y \in M_\gamma$.*

The proof is obvious according to the proofs of Lemma 1 and Lemma 5.

Following the terminology used in [2], an η'-covering $\mathfrak{N}$ can be called a compatible covering of L, and as each subset $M_\gamma \in \mathfrak{N}$ is a convex sublattice of L, each η'-covering of L determines a compatible tolerance relation $T_{\mathfrak{N}}$ on L as follows: $aT_{\mathfrak{N}}b \Leftrightarrow a, b \in M_\gamma$ for some $\gamma \in \Gamma$ in $\mathfrak{N}$ (see [5]). On the other hand, $aTb \Leftrightarrow a \wedge \wedge\, bTa \vee b$ for each compatible tolerance relation on a lattice L [3, Thm. 1], and hence $H_{\mathfrak{N}}$ is a restricted form of the corresponding compatible tolerance relation $T_{\mathfrak{N}}$ on L. The results obtained for compatible hulls are analogous to those obtained for transitive hulls in [4], whence they are omitted.

Let us consider the analogy of Theorem 10 in [4]; according to Theorem 7 given below, this analogy can give something new. As each congruence relation C on L is a compatible tolerance relation on L, the partition $\mathfrak{C}$ of L determined by C is an η'-covering of L and gives a relation $H_{\mathfrak{C}}$ defined similarly as the compatible hull in Theorem 5. Now we can formulate Theorem 10 of [4] given by Chajda and Zelinka in terms of normality relations:

Theorem 6. *Let L be a lattice, C a congruence relation on L, and $F(C)$ the filter of $N(L)$ consisting of all elements of $N(L)$ which are greater or equal to $H_{\mathfrak{C}}$. Then there exists a joinhomomorphism of $F(C)$ onto $N(L/C)$, where L/C is the factor-lattice of L given by C.*

Proof. Let X be the natural homomorphism of L onto L/C. Similarly as in the proof of [4, Thm. 10], one can show that there exists a surjection φ from $F(C) \subseteq \subseteq N(L)$ onto $N(L/C)$: $x\varphi(N)\, y \Leftrightarrow$ there exist elements $x', y' \in L$ such that $X(x') = = x$, $X(y') = y$ and $x'Ny'$. The proof is now a direct copy of the proof given in [4]. Thus normality relations on lattices have a property analogous to the properties of compatible tolerance relations on abstract algebras. The interesting point is, that Theorem 10 of [4] can be sharpened in the case of lattices as follows: the mapping φ of $F(C) \subseteq LT(L)$ onto $LT(L/C)$ is an homomorphism, where $LT(L)$ is the lattice of all compatible tolerance relations on L.

Theorem 7. *Let L be a lattice, C a congruence relation on L, and $F(C)$ the filter of $LT(L)$ consisting of all elements of $LT(L)$ which are greater or equal to C. Then there exists a homomorphism of $F(C)$ onto $LT(L/C)$.*

Proof. According to the proof of Theorem 10 in [4], it remains to show that $\varphi(T \wedge T') \subseteq \varphi(T) \wedge \varphi(T')$ and $\varphi(T) \wedge \varphi(T') \subseteq \varphi(T \wedge T')$.

Let $a\varphi(T \wedge T')\, b$, $a, b \in L/C$. Then $a'(T \wedge T')\, b'$ for some $a', b' \in L$, where $X(a') = a$ and $X(b') = b$. Thus $a'Tb'$ and $a'T'b'$, whence $X(a') = a\varphi(T)\, b = = X(b')$ and $X(a') = a\varphi(T')\, b = X(b)$, from which the first part of the proof follows.

Let $a(\varphi(T) \wedge \varphi(T'))\, b$, and so $a\varphi(T)\, b$, $a\varphi(T')\, b$. According to the definition of the mapping φ, $a'Tb'$ and $a''T'b''$ in L for a', b', a'', b'', where $X(a') = X(a'') = a$ and $X(b') = X(b'') = b$. On the other hand, as $xTy \Leftrightarrow x \wedge yTx \vee y$ in lattices

[3, Thm. 1], we can assume without loosing generality that $a' \leqq b'$ and $a'' \leqq b''$. As T and T' are compatible tolerance relations on L, $(a' \vee a'')\, T(b' \vee a'')$ and $(a' \vee a'')\, T'(b'' \vee a')$. The homomorphism X from L onto L/C is also an order-homomorphism, whence $q \leqq r$ in L implies $X(q) \leqq X(r)$, and thus $X(a') = a \leqq \leqq b = X(b')$. As $X(b' \vee a'') = X(b') \vee X(a'') = b \vee a = b$, $b' \vee a''$ (and $b'' \vee a'$, as well) belongs to the congruence class $C(b)$ of C, the figure of which is b in L/C under the mapping X. As $C(b)$ is a sublattice of L, $(b' \vee a'') \wedge (b' \vee a'') = b^* \in C(b)$. The element $b^* \in [a' \vee a'', b' \vee a'']$ and $b^* \in [a' \vee a'', b'' \vee a']$, whence $(a' \vee a'')\, Tb^*$ and $(a' \vee a'')\, T'b^*$. Consequently, $(a' \vee a'')\, (T \wedge T')\, b^*$, where $X(a' \vee a'') = a$ and $X(b^*) = b$, whence $a\varphi(T \wedge T')\, b$, and the relation $\varphi(T) \wedge \varphi(T') \subseteq \varphi(T \wedge T')$ follows.

Some recent observations on normality relations on finite (distributive) lattices are also given in [10].

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