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TOLERANCE EXTENSIONS IN DISTRIBUTIVE LATTICES

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It is known that for every distributive lattice D, all congruences on each its sublattice L have extensions on D. An analogous assertion does not hold for compatible tolerances. The aim of this paper is to give answers on questions of the following structure:

Let D, L denote distributive lattices, $D : L$ in the relation overlattice: sublattice, let T denote a compatible tolerance on L. Under which conditions set on **GIVEN**, has T an extension on D for arbitrary **VARIABLE**? **GIVEN** and **VARIABLE** denote all possible combinations of D, L, T. These combinations are shown in the following table:

GIVEN	DLT	DL	DT	LT	D	L	T
VARIABLE	–	T	L	D	LT	DT	DL
QUESTION NB	Q0	Q1	*)	Q2	Q3	Q4	*)

*) T cannot exist without L.

Answers to the questions Q0, Q2, Q4 are given in this paper. The question Q1 was answered in [3] and the question Q3 in [1].

1. PRELIMINARIES

Tolerance relation is a symmetric and reflexive binary relation.

Compatible tolerance on an algebra A is a tolerance relation on the support of A being a subalgebra of $A \times A$.

For a compatible tolerance T on a lattice L the following is valid:

TL 1 $[x, y] \in T \Leftrightarrow [x \wedge y, x \vee y] \in T$

TL 2 $(x \leqq y \leqq z \leqq w, [x, w] \in T) \Rightarrow [y, z] \in T$

TL 3 $([x, y] \in T, [y, z] \in T, y \vee w = x \vee y \vee z, y \wedge w = x \wedge y \wedge z) \Rightarrow$
$\Rightarrow [x \wedge y \wedge z, x \vee y \vee z] \in T$

TL 4 $([x, w] \in T, [y, z] \in T, u \wedge z = x, u \vee w = y) \Rightarrow [x, y] \in T$

An intersection of arbitrary system of compatible tolerances on an algebra A is again a compatible tolerance on A.

For L a sublattice of L' a compatible tolerance T' on L' is called an *extension* of a compatible tolerance T on L if T is the restriction of T' on L.

Let L be a lattice, $\langle a, b\rangle$ an interval in L, $e, c, d \in \langle a, b\rangle$. The element e is called a *relative bicomplement* of the elements c, d in $\langle a, b\rangle$ if $a = c \wedge d \wedge e$ and $b = c \vee d \vee e$.

Clearly, in this case c is a relative bicomplement of e, d in $\langle a, b\rangle$ and d is a relative bicomplement of e, c in $\langle a, b\rangle$.

Let L be a sublattice of a lattice L'. L is said to be *closed under relative bicomplements* in L' if for each interval $\langle a, b\rangle_L$ and each two elements $c, d \in \langle a, b\rangle_L$ there holds the following: c, d have a relative bicomplement in $\langle a, b\rangle_{L'}$ iff c, d have a relative bicomplement in $\langle a, b\rangle_L$.

2. TOLERANCES ON DISTRIBUTIVE LATTICES

The following assertion is well known.

Lemma 1. *Let D be a distributive lattice, J an ideal in D, $a \in D \setminus J$. Then there exists an ideal I which is maximal among all those ideals in D containing J and not containing a. I is prime.*

Definition. Let D be a distributive lattice, $a, b \in D$, $a < b$. Let I be a maximal ideal under all those ideals in D not containing b, let F be a maximal dual ideal among all those dual ideals in D not containing a. If $D = I \cup F$, $T = (I \times I) \cup (F \times F)$ is called a *τ-tolerance* belonging to $[a, b]$.

Note. Clearly, such a τ-tolerance is a compatible tolerance on D.

Lemma 2. [3] *Let D be a distributive lattice, T a compatible tolerance on D, $a, b \in D$, $a < b$, $[a, b] \notin T$. Then there exists a τ-tolerance on D belonging to $[a, b]$ and containing T.*

Lemma 3. [3] *Let D be a distributive lattice, T a compatible tolerance on D. Then there exists a family $\{S_i\}_{i \in I}$ of τ-tolerances on D with $T = \bigcap_{i \in I} S_i$.*

3. TOLERANCE EXTENSIONS IN DISTRIBUTIVE TLATICES

Lemma 4. *Let D be a distributive lattice, L its sublattice, T a compatible tolerance on L. If (A) is satisfied, then for every natural number n there holds (V_n), where:*

(A) $\quad (o, p, q, r \in L, x \in D, o \leqq p \leqq r \leqq q, [o, r] \in T, [p, q] \in T, x \wedge p = o,$
$\quad x \vee r = q) \Rightarrow [o, q] \in T$

(V_n) $\quad (a, b, a_1, \ldots, a_n, b_1, \ldots, b_n \in L, x_1, \ldots, x_n \in D, a \leqq a_i \leqq b_i \leqq b,$
$\quad a \leqq x_i \leqq b_i, \bigvee_{i=1}^{n} x_i = b, x_i \wedge a_i = a, [a_i, b_i] \in T) \Rightarrow [a, b] \in T.$

Proof. Suppose (A).

Ad (V_1): Let $a, b, a_1, b_1 \in L$, $x_1 \in D$, $a \leqq a_1 \leqq b_1 \leqq b$, $a \leqq x_1 \leqq b_1$, $x_1 = T$, $x_1 \wedge a_1 = a$, $[a_1, b_1] \in T$. Then $x_1 = b = b_1$, $a_1 = a$, i.e. $[a, b] \in b$.

Ad (V_n), $n \geqq 2$: Let $(V_1), \ldots, (V_{n-1})$ hold. Let $a, b, a_1, \ldots, a_n, b_1, \ldots, b_n, x_1, \ldots, x_n$ satisfy the assumptions of the left side of (V_n). Denote $I = \{1, \ldots, n\}$, $I_k = I \setminus \{k\}$, $I_{k,l} = I \setminus \{k, l\}$.

Let $k \neq l$, $k, l \in I$.

Set for $i \in I_{k,l}$

$$
\begin{array}{lll}
x_i' = a_k \vee a_l \vee x_i & x_0' = a_k \vee a_l \vee x_k \vee x_l & a' = a_k \vee a_l \\
a_i' = a_k \vee a_l \vee a_i & a_0' = a_k \vee a_l & \\
b_i' = a_k \vee a_l \vee b_i & b_0' = b_k \vee b_l & b' = b
\end{array}
$$

Denote $I' = I_{k,l} \cup \{0\}$. I' has $n - 1$ elements and the "primed" system satisfies the assumptions of (V_{n-1}), hence $[a_k \vee a_l, b] \in T$. Put $z = \bigwedge_{\substack{i,j \in I \\ i \neq j}} (a_i \vee a_j)$. Clearly, $[z, b] \in T$. Therefore $[a_k \wedge z, b_k] \in T$ for all $k \in I$. Then the elements $\bar{a}_k = a_k \wedge z$ have all properties required for a_k and $\bar{a}_k \vee \bar{a}_l = z$ for $k \neq l$. The bar may be omitted and it may be supposed $a_k \vee a_l = z$. Clearly $a_k = a_k \wedge b = a_k \wedge \bigvee_I x_i = \bigvee_{I_k} (a_k \wedge x_i)$.

Denote for $i \in I_k$

$$
\begin{array}{lll}
x_i' = a_k \wedge x_i & & \\
a_i' = a_k \wedge a_i & a' = a & \\
b_i' = a_k \wedge b_i & b' = a_k & I' = I_k.
\end{array}
$$

The obtained "primed" system satisfies all assumptions of the left side of (V_{n-1}), therefore $[a, a_k] \in T$. Hence $[a, z] \in T$. Let $\bar{b}_k = b_k \vee z$. Then $\bar{b}_k$ satisfies all assumptions on b_k. The bar may be omitted. Clearly $a_k \leqq z \wedge \bigvee_{I_k} x_i = \bigvee_{I_k} (z \wedge x_i) =$
$= \bigvee_{I_k} ((a_i \vee a_k) \wedge x_i) = \bigvee_{I_k} (a_k \wedge x_i) \leqq a_k.$

Thereby $z \wedge \bigvee_{I_k} x_i = a_k$, $b_k \vee \bigvee_{I_k} x_i = b$ and $a_k \leqq z \leqq b_k \leqq b$. $[a_k, b_k] \in T$ implies $[z, b] \in T$ (for $z = \bigvee_I a_i, b = \bigvee_I b_i$) and by (A) $[a_k, b] \in T$. Hence, because $a = \bigwedge_I a_i$, $[a, b] \in T$ q.e.d.

Theorem. *Let D be a distributive lattice, L its sublattice and T compatible tolerance on L. Then following assertions are equivalent:*

(i) *T has an extension on D*

(ii) *for each pair $a, b \in L$ with $a < b$, $[a, b] \notin T$ there exists a τ-tolerance on D belonging to $[a, b]$ and containing T*

(iii) *whenever $a, b, c, d \in L$, $a < c \leqq d < b$, $[a, d] \in T$, $[c, b] \in T$, $[a, b] \notin T$ then there exists no relative bicomplement of the elements c, d in $\langle a, b \rangle_D$.*

Proof. (i)$\Rightarrow$(iii): If there were such an element, denote it by x, then $[a, b] = [(a \vee x) \wedge c, (d \vee x) \wedge b] \in T$, but $[a, b] \notin T$.

(iii)$\Rightarrow$(ii): Clearly (iii)$\Leftrightarrow$$(A)$. Suppose (iii) holds, let $a, b \in L$, $a < b$, $[a, b] \notin T$. Denote $R = \{r \in D \mid \exists\, z,\ t \in L,\ z \leqq t,\ [t, z] \in T,\ r \leqq t,\ r \wedge z = a\}$. Let J be the ideal generated by R in D. Clearly $a \in J$. If $b \in J$, then there must exist an n-tuple $r_1, \ldots, r_n$ of elements of R with $b \leqq \bigvee_{i=1}^{n} r_i$. Denote $a_i = z_i \wedge b$, $b_i = t_i \wedge b$, $x_i = r_i \wedge b$, where z_i, t_i are the elements from the definition of R belonging to r_i. Then the assumptions of (V_n) are satisfied and from this $[a, b] \in T$. Hence $b \notin J$. There exists an ideal I containing J and not containing b, a maximal one with this property. Let $P = (D \setminus I) \cup \{d \in L \mid \exists\, c \in L \cap (D \setminus I),\ [c, d] \in T\}$. Let E be the dual ideal generated by P. If $a \in E$, then there must exist $e \in D \setminus I$, $z \in L$, $t \in L \cap (D \setminus I)$, such that $[t, z] \in T$, $a \leqq e$, $t, z \leqq b$, $z \leqq t$, $e \wedge z = a$. But for $r = e \wedge t \in D \setminus I$ there holds $r \leqq t$, $r \wedge z = a$, consequently $r \in R$. This is a contradiction, therefore $a \notin E$. Let F denote a dual ideal containing E and not containing a a maximal one with this property. Then I, F are tolerance classes of the τ-tolerance in request. (ii)$\Rightarrow$(i): The intersection of all those τ-tolerances is an extension of T on D q.e.d.

It follows a list of answers to forementioned questions.

AQ0 Let D be a distributive lattice, L its sublattice, T a compatible tolerance on L. T has an extension on D iff there exists no relative bicomplement in D of elements b, c in interval $\langle a, d \rangle_D$ whenever $a, d \in L$, $b, c \in \langle a, d \rangle_L$ satisfying $b \leqq c$, $[a, c] \in T$, $[b, d] \in T$, $[a, d] \notin T$.

AQ1 Let D be a distributive lattice, L its sublattice .T has an extension on D for arbitrary compatible tolerance T on L iff L is closed in D under relative bicomplements.

AQ2 Let L be a distributive lattice, T a compatible tolerance on L. T has an extension on D for arbitrary distributive lattice D being an overlattice of L iff T is a congruence.

AQ3 Let D be a distributive lattice. T has an extension on D for arbitrary sublattice L of the lattice D and for arbitrary compatible tolerance T on L iff D is a chain.

AQ4 Let L be a distributive lattice. T has an extension on D for arbitrary compatible tolerance T on L and for arbitrary distributive lattice D being an overlattice of L iff L is relatively complemented.

AQ2 follows from AQO, AQ4 follows from AQl,

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