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HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM-LIOUVILLE FUNCTIONS

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1. INTRODUCTION AND NOTATION

In [1] there is derived a simple sufficient condition for the monotonicity of order n of the sequence of diferences of consecutive zeros of linear combination of any solution and its first derivative of the differential equation

$$y'' + q(x)\,y = 0 \qquad \text{(q)} \qquad \left(' = \frac{d}{dx}\right)$$

in the interval (a, ∞), where a is a real number.

In [4] there are given sufficient conditions for the monotonicity of the sequence of extremants (i.e. zeros of the 1-st derivative) of an arbitrary solution of the differential equation (q).

In this paper, using the first accompanying equation with regard to the basis α, β, where α, β are real numbers with the property $\alpha^2 + \beta^2 > 0$, we extend the above-mentioned results from [1] and [4] to the function

$$\alpha y + \beta g(x)\, y',$$

where $y(x)$ is a solution of the equation

$$(g(x)\, y')' + f(x)\, y = 0.$$

Finally, we introduce certain applications of the derived results for Bessel functions.

In [2] M. Laitoch introduced the first accompanying equation (Q) towards the differential equation (q) with regard to the basis α, β in the form

$$y'' + Q(x)\, y = 0, \qquad \text{(Q)}$$

where

$$Q(x) = q + \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q} + \frac{1}{2}\,\frac{\beta^2 q''}{\alpha^2 + \beta^2 q} - \frac{3}{4}\,\frac{\beta^4 q'^2}{(\alpha^2 + \beta^2 q)^2}, \qquad (1_q)$$

under the assumption that $q(x) \in C_2$, $q(x) > 0$ for $x \in (a, \infty)$, and α, β are real numbers with the property $\alpha^2 + \beta^2 > 0$.

In [2] it is proved that if $y(x)$ is a solution of (q), then the function

$$Y(x) = \frac{\alpha y + \beta y'}{\sqrt{\alpha^2 + \beta^2 q(x)}},$$

is a solution of the differential equation (Q) and conversely, if $Y(x)$ is any solution of (Q), then there exists a solution $\bar{y}(x)$ of the equation (q) such that

$$\frac{\alpha \bar{y} + \beta \bar{y}'}{\sqrt{\alpha^2 + \beta^2 q(x)}} = Y(x).$$

A function $f(x)$ is said to be n-times monotonic (or monotonic of order n) on an interval (a, ∞) if

(2) $$(-1)^i f^{(i)}(x) \geqq 0, \qquad i = 0, 1, \ldots, n, \qquad x \in (a, \infty).$$

For such a function we write $f(x) \in M_n(a, \infty)$. If strict inequality holds troughout (2), we write $f(x) \in M_n^*(a, \infty)$. We say that $f(x)$ is completely monotonic on (a, ∞) if (2) holds for $n = \infty$.

A sequence $\{x_k\}_{k=1}^{\infty}$, denoted simply by $\{x_k\}$, is said to be n-times monotonic if

(3) $$(-1)^i \Delta^i x_k \geqq 0, \qquad i = 0, 1, \ldots, n, \qquad k = 1, 2, \ldots$$

Here

$$\Delta^\circ x_k = x_k, \Delta x_k = x_{k+1} - x_k, \ldots, \Delta^n x_k = \Delta^{n-1} x_{k+1} - \Delta^{n-1} x_k.$$

For such a sequence we write $\{x_k\} \in M_n$. If strict inequality holds throughout (3), we srite $\{x_k\} \in M_n^*$. The sequence $\{x_k\}$ is called completely monotonic if (3) holds for $n = \infty$.

2. NEW BASIC RESULTS

In this section we consider the differential equation

(4) $$(g(x)\, y')' + f(x)\, y = 0,$$

with $f(x)$ and $g(x)$ continuous, $g(x) > 0$ for $x \in (a, \infty)$ and $g(x) \in M_n(a, \infty)$, $n \geqq 2$.

The change of variable

(5) $$\xi = \int_a^x [g(t)]^{-1}\, dt,$$

where the integral is assumed convergent, transforms (4) into

(6) $$\ddot{\eta} + \Phi(\xi)\, \eta = 0, \qquad \left(\dot{} = \frac{d}{d\xi}\right)$$

where $\eta(\xi) = y(x)$ and $\varphi(\xi) = f(x)\, g(x)$.

For $n \geqq 2$, $g(x)$ is non-increasing. Hence, the mapping (5) takes the x-interval (a, ∞) into the ξ-interval $(0, \infty)$.

Let $\varphi(\xi) \in C_2$, $\varphi(\xi) > 0$ on $(0, \infty)$. The first accompanying equation towards the differential equation (6) with regard to the basis α, β has the form

$$\ddot{\eta} + \Phi(\xi)\,\eta = 0, \tag{7}$$

where $\Phi(\xi)$ is given by (1φ).

Lemma 1. *Let $n \geqq 2$ be an integer. Let $f(x)$, $g(x)$, $(f(x)\,g(x))'$ in (4) be positive on (a, ∞) and let $g(x)$, $(f(x)\,g(x))'$ belong to $M_n(a, \infty)$. Then for the carrier $\varphi(\xi)$ of the differential equation (6) we have*

$$\varphi(\xi) > 0,\ \dot{\varphi}(\xi) > 0 \quad \text{on} \quad (0, \infty) \quad \text{and} \quad \dot{\varphi}(\xi) \in M_n(0, \infty).$$

Proof. Consider a carrier $\varphi(\xi)$ of the equation (6). It is obvious that $\varphi(\xi) = f(x)\,g(x)$. Therefore, by hypotheses, we have $\varphi(\xi) > 0$ on $(0, \infty)$.

The second part of the assertion is proved in [3], Theorem 3.1.

Lemma 2. *Let the assumptions of Lemma 1 hold. Let α, β be real numbers such that $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$. Then for the carrier $\Phi(\xi)$ of the first accompanying equation towards the differential equation (6) with regard to the basis α, β we have*

$$\dot{\Phi}(\xi) > 0 \quad on \quad (0, \infty), \quad \dot{\Phi}(\xi) \in M_{n-2}(0, \infty) \quad and \quad 0 < \Phi(\infty) = \varphi(\infty) \leqq \infty.$$

Proof. Consider a carrier $\Phi(\xi)$ of the equation (7). Lemma 1 implies that $\dot{\varphi}(\xi) > 0$ on $(0, \infty)$ and $\dot{\varphi}(\xi) \in M_n(0, \infty)$. Since $\alpha^2 + \beta^2 > 0$ and $\dot{\varphi}(\xi) \in M_n(0, \infty)$ we receive from ([5], Lemma 2.3), that $\dfrac{1}{\alpha^2 + \beta^2\varphi(\xi)} \in M_{n+1}(0, \infty)$.

The functions $\dfrac{\beta^2\dot{\varphi}(\xi)}{\alpha^2 + \beta^2\varphi(\xi)} \in M_n(0, \infty)$ because the sum and the product of two functions of the class $M_n(0, \infty)$ are functions belonging again to the class $M_n(0, \infty)$ [5].

Therefore, using ([5], Lemma 2.3), we have

$$\left[-\frac{3}{4}\,\frac{(\beta^2\dot{\varphi})^2}{(\alpha^2 + \beta^2\varphi)^2}\right]^{\cdot} \in M_{n-1}(0, \infty), \qquad \left[\frac{1}{2}\,\frac{\beta^2\ddot{\varphi}}{\alpha^2 + \beta^2\varphi}\right]^{\cdot} \in M_{n-2}(0, \infty)$$

and since $\alpha\beta \leqq 0$ also

$$\left[\frac{\alpha\beta\varphi}{\alpha^2 + \beta^2\varphi}\right]^{\cdot} \in M_{n-1}(0, \infty).$$

This implies $\dot{\Phi}(\xi) \in M_{n-2}(0, \infty)$ and since $\dot{\varphi}(\xi) > 0$ on $(0, \infty)$ we receive from (1_φ) that $\dot{\Phi}(\xi) > 0$ on $(0, \infty)$. From Lemma 1 and ([1], Lemma 1) we get $0 < \Phi(\infty) = \varphi(\infty) \leqq \infty$ and the proof is complete.

Let us denote, for fixed $\lambda > -1$,

$$(9) \qquad P_k = \int_{x_k}^{x_{k+1}} W(x) \frac{1}{g(x)} \left| \frac{\alpha y + \beta g(x) y'}{\sqrt{\alpha^2 + \beta^2 f(x) g(x)}} \right|^{\lambda} dx, \qquad k = 1, 2, \ldots,$$

where $y(x)$ is an arbitrary solution of (4) and $\{x_k\}$ is a sequence of consecutive zeros of the function $\alpha z(x) + \beta g(x) z'(x)$, where $z(x)$ is any solution of (4) which may or may not be linearly independent of $y(x)$. The function $W(x)$ is any sufficiently monotonic function.

Theorem 1. *Let $n \geqq 2$ be an integer and α, β be real numbers such that $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$. Let $f(x)$, $g(x)$, $(f(x) g(x))'$ in (4) be positive on (a, ∞), $g(x) \in M_n(a, \infty)$, $(f(x) g(x))' \in M_n(a, \infty)$ and let*

$$(10) \qquad W(x) > 0, \qquad W(x) \in M_{n-2}(a, \infty), \qquad x \in (a, \infty).$$

Then for P_k defined by (9) there holds

$$(11) \qquad \{P_k\} \in M^*_{n-2}.$$

Proof. Let $y(x)$, $z(x)$ be solutions of the differential equation (4) and $\eta(\xi) = y(x)$, $\zeta(\xi) = z(x)$ be solutions of the equation (6). It follows from [2] that the functions

$$H(\xi) = \frac{\alpha\eta + \beta\dot{\eta}}{\sqrt{\alpha^2 + \beta^2\varphi(\xi)}} = \frac{\alpha y + \beta g y'}{\sqrt{\alpha^2 + \beta^2 f g}},$$

$$Z(\xi) = \frac{\alpha\zeta + \beta\dot{\zeta}}{\sqrt{\alpha^2 + \beta^2\varphi(\xi)}} = \frac{\alpha z + \beta g z'}{\sqrt{\alpha^2 + \beta^2 f g}},$$

are solutions of the equation (7).

By Lemma 2, we have $0 < \Phi(\infty) \leqq \infty$. This shows that $\alpha z(x) + \beta g(x) z'(x)$ does indeed have an infinite sequence of zeros on (a, ∞).

Using the change of variable (5) we get

$$\int_{x_k}^{x_{k+1}} W(x) \frac{1}{g(x)} \left| \frac{\alpha y + \beta g(x) y'}{\sqrt{\alpha^2 + \beta^2 f(x) g(x)}} \right|^{\lambda} dx = \int_{\xi_k}^{\xi_{k+1}} W(x(\xi)) \left| \frac{\alpha\eta + \beta\eta}{\sqrt{\alpha^2 + \beta^2\varphi(\xi)}} \right|^{\lambda} d\xi,$$

where $\{\xi_k\}$ are consecutive zeros of $\alpha\zeta(\xi) + \beta\dot{\zeta}(\xi)$ corresponding, respectively, to consecutive zeros $\{x_k\}$ of $\alpha z(x) + \beta g(x) z'(x)$, here $\alpha\zeta(\xi) + \beta\dot{\zeta}(\xi) = \alpha z(x) + \beta g(x) z'(x)$.

By hypotheses, we have $W(x(\xi)) > 0$ on $(0, \infty)$. Since $W(x) \in M_{n-2}(a, \infty)$, using (8) and ([15], Lemma 2.3), we have $W(x(\xi)) \in M_{n-2}(0, \infty)$. By Lemma 2, $\dot{\Phi}(\xi) > 0$ on $(0, \infty)$ and $\dot{\Phi}(\xi) \in M_{n-2}(0, \infty)$. So, the conditions of ([3], Theorem 3.1) are fulfilled. Using this theorem we have

$$\{N_k\} \in M^*_{n-2},$$

where N_k is defined by

$$N_k = \int_{t_k}^{t_{k+1}} W(x(\xi)) \, | H(\xi) |^\lambda \, d\xi, \qquad \lambda > -1, \qquad k = 1, 2, \ldots$$

Here $H(\xi)$ is the solution of (7) and $\{t_k\}$ denotes the sequence of consecutive zeros of the solution $Z(\xi)$ of (7).

Since $Z(\xi)\sqrt{\alpha^2 + \beta^2\varphi(\xi)} = \alpha\xi(\xi) + \beta\dot{\xi}(\xi)$, we have $\{t_k\} = \{\xi_k\}$. Hence it follows that

$$N_k = \int_{\xi_k}^{\xi_{k+1}} W(x(\xi)) \left| \frac{\alpha\eta + \beta\dot{\eta}}{\sqrt{\alpha^2 + \beta^2\varphi(\xi)}} \right|^\lambda d\xi = P_k,$$

so that (11) holds, and the theorem is proved.

Corollary 1. *Let the conditions of Theorem 1 are satisfied. Then*

$$\{ \int_{x_k}^{x_{k+1}} W(x) \, | \alpha y(x) + \beta g(x, y'(x) |^\lambda \, dx \} \in M^*_{n-2},$$

for $\lambda \in (-1, 0\rangle$, $k = 1, 2, \ldots$

Proof of this corollary follows directly from Theorem 1. (11) remains valid when $W(x)$ is replaced by

$$W(x) \, g(x) \, (\alpha^2 + \beta^2 f(x) \, g(x))^{\lambda/2}, \qquad \lambda \in (-1, 0\rangle,$$

since the last function belongs to $M_{n-2}(a, \infty)$.

If we put $W(x) = 1$, we receive

Corollary 2. *Under the hypotheses of Theorem* 1 *we have*

$$\{ \int_{x_k}^{x_{k+1}} | \alpha y(x) + \beta g(x) \, y'(x) |^\lambda \, dx \} \in M^*_{n-2},$$

for $\lambda \in (-1, 0\rangle$, $k = 1, 2, \ldots$

Remark 1. If in the above considerations we choose $\alpha = 1$, $\beta = 0$, then we obtain the results from [1] concerning the monotonicity of consecutive zeros of any arbitrary solution $y(x)$ of the equation (4).

If we choose $\alpha = 0$, $\beta = 1$, then we obtain the results from [4] for the monotonicity of the sequence of extremants of an arbitrary solution of the equation (4).

3. APPLICATIONS TO BESSEL AND GENERALIZED AIRY FUNCTIONS

Throughout this section we suppose that α, β are real numbers such that $\alpha^2 + \beta^2 > 0$, $\alpha\beta \leqq 0$.

1. Let $\mathfrak{C}_n(x)$ denote any Bessel (cylinder) function of order ν, i.e. any nontrivial

solution of the Bessel equation

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0, \qquad x \in (0, \infty). \tag{12}$$

Then the function

$$y(x) = x^{1/2} \mathfrak{C}_\nu(x)$$

is a solution of the differential equation

$$y'' + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) y = 0. \tag{13}$$

Let $\{a_{\nu k}\}$ denote the sequence of consecutive positive zeros of the function

$$\alpha x^{1/2} \mathfrak{C}_\nu(x) + \beta (x^{1/2} \mathfrak{C}_\nu(x))'$$

and let $\{A_{\nu k}\}$ denote the analogous sequence of the function

$$\alpha x^{1/2} \bar{\mathfrak{C}}_\nu(x) + (\beta x^{1/2} \bar{\mathfrak{C}}_\nu(x))',$$

where $\bar{\mathfrak{C}}_\nu(x)$ denotes any Bessel function of order ν, possibly $\mathfrak{C}_\nu(x)$ again.

Theorem 2. *Let $n \geqq 2$ be an integer, $\nu > \frac{1}{2}$ be an arbitrary number and $a = \left(\nu^2 - \frac{1}{4}\right)^{1/2}$. Let*

$$W(x) > 0, \qquad W(x) \in M_{n-2}(a, \infty), \qquad x \in (a, \infty)$$

and let $R_{\nu k}$ be defined for $x \in (a, \infty)$ and $\lambda > -1$ by

$$R_{\nu k} = \int_{A_{\nu k}}^{A_{\nu, k+1}} W(x) \left| \frac{\alpha x^{1/2} \mathfrak{C}_\nu(x) + \beta (x^{1/2} \mathfrak{C}_\nu(x))'}{\sqrt{\alpha^2 + \beta^2 \left(x^2 - \nu^2 + \frac{1}{4}\right) x^{-2}}} \right|^\lambda \mathrm{d}x, \qquad k = 1, 2, \dots \tag{14}$$

Let p be the smallest integer satisfying $a \leqq A_{\nu p}$. Then

$$\{R_{\nu k}\}_{k=p}^{\infty} \in M_{n-2}^{*}. \tag{15}$$

Proof. In the case of the differential equation (13) the coefficients $f(x)$ and $g(x)$ have the form

$$f(x) = 1 - \left(\nu^2 - \frac{1}{4}\right) x^{-2} \qquad \text{and} \qquad g(x) = 1.$$

It is obvious that $f'(x) \in M_{\infty}^{*}(a, \infty)$ and $f(a) = 0$. This implies $f(x) > 0$ for $x \in (a, \infty)$.

The expression P_k defined in (9) is of the form (14). So, the assertion (15) follows immediately from Theorem 1.

Remark 2. Let $v > \frac{1}{2}$ be an arbitrary number and $a = \left(v^2 - \frac{1}{4}\right)^{1/2}$. Let

$$W(x) > 0, \qquad W(x) \in M_\infty(a, \infty), \qquad x \in (a, \infty),$$

and let R_{vk} be defined by (14). Then

$$\{R_{vk}\}_{k=p}^{\infty} \in M_\infty^*.$$

The remark is the case $n = \infty$ in Theorem 2.

Corollary 3. *Under the hypotheses of Theorem 2 we have*

$$\{\int_{A_{vk}}^{A_{v,k+1}} W(x)\,|\,\alpha x^{1/2}\mathfrak{C}_v(x) + \beta(x^{1/2}\mathfrak{C}_v(x))'\,|^\lambda\,dx\}_{k=p}^{\infty} \in M_{n-1}^*,$$

for some fixed $\lambda \in (-1, 0\rangle$.

The proof of this corollary follows from Theorem 2. The assertion (15) remains valid when $W(x)$ is replaced by

$$W(x)\left(\alpha^2 + \beta^2\left(x^2 - v^2 + \frac{1}{4}\right)x^{-2}\right)^{\lambda/2}, \qquad \lambda \in (-1, 0\rangle,$$

since the last function belongs to $M_{n-2}(a, \infty)$.

Remark 3. As a direct conclusion of Theorem 2 we get

(16) $$\{(a_{v,k+1})^\gamma - (a_{vk})^\gamma\}_{k=p}^{\infty} \in M_\infty^*, \qquad 0 < \gamma \leqq 1,$$

(17) $$\left\{\lg \frac{a_{v,k+1}}{a_{vk}}\right\}_{k=p}^{\infty} \in M_\infty^*.$$

The assertion (16) is an immediate consequence of Theorem 2 with $\bar{\mathfrak{C}}_v(x) \equiv \mathfrak{C}_v(x)$ $\lambda = 0$ and $W(x) = \gamma x^{\gamma-1}$.

The assertion (17) follows from Theorem 2 if $\bar{\mathfrak{C}}_v(x) = \mathfrak{C}_v(x)$, $\lambda = 0$ and $W(x) = x^{-1}$.

Remark 4. Let the assumptions of Theorem 2 hold and let $\gamma > 0$. Then

(18) $$\{(a_{vk})^{-\gamma}\}_{k=p}^{\infty} \in M_\infty^*,$$

(19) $$\{(\lg a_{vk})^{-\gamma}\}_{k=p}^{\infty} \in M_\infty^*, \qquad a_{vp} > 1,$$

$$\{\exp(-\gamma a_{vk})\}_{k=p}^{\infty} \in M_\infty^*.$$

The assertion (18) follows from Theorem 2 if $\bar{\mathfrak{C}}_v(x) = \mathfrak{C}_v(x)$, $\lambda = 0$ and $W(x) = -w'(x)$, where $w(x) = x^{-\gamma}$.

It is obvious that $w(x) \in M_\infty^*(a, \infty)$. Therefore we have $\Delta^\circ w(a_{vk}) > 0$, $k = p, p+1, \ldots$ Moreover,

$$-\Delta w(a_{vk}) = \int_{a_{vk}}^{a_{v,k+1}} [-w'(x)]\,dx,$$

and, since $-w'(x) \in M^*_\infty(a, \infty)$, we can see, from Theorem 2, that

$$\{-\Delta w(a_{\nu k})\}_{k=p}^{\infty} \in M^*_\infty .$$

This implies

$$\{w(a_{\nu k})\}_{k=p}^{\infty} \in M^*_\infty .$$

Thus (18) holds.

The assertions (19) and (20) follow from Theorem 2 if

$$\bar{\mathfrak{C}}_\nu(x) = \mathfrak{C}_\nu(x), \quad \lambda = 0, \quad W(x) = -[(\lg x)^{-\gamma}]' \quad \text{and} \quad W(x) = -[e^{-\gamma x}]',$$

respectively.

2. We apply Theorem 1 to certain generalized Airy functions, i.e., solutions of

$$y'' + \delta^2 x^{2\delta-2} y = 0, \tag{21}$$

where $1 < \delta \leqq \frac{3}{2}$. The solutions $y(x)$ of (21) are expressible in terms of cylinder functions:

$$y(x) = x^{1/2}\mathfrak{C}_{1/(2\delta)}(x^\delta).$$

Let $\{b_{\nu k}\}$ denote the sequence of consecutive positive zeros of the function

$$\alpha x^{1/2}\mathfrak{C}_{1/(2\delta)}(x^\delta) + \beta(x^{1/2}\mathfrak{C}_{1/(2\delta)}(x^\delta))'$$

and let $\{B_{\nu k}\}$ denote the analogous sequence of the function

$$\alpha x^{1/2}\bar{\mathfrak{C}}_{1/(2\delta)}(x^\delta) + \beta(x^{1/2}\bar{\mathfrak{C}}_{1/(2\delta)}(x^\delta))',$$

where $\bar{\mathfrak{C}}_\nu(x)$ denotes any Bessel function of order ν, possibly $\mathfrak{C}_\nu(x)$ again.

Theorem 3. *Let $n \geqq 2$ be an integer and $1 < \delta \leqq \frac{3}{2}$ be an arbitrary number. Let*

$$W(x) > 0, \quad W(x) \in M_{n-2}(a, \infty), \quad x \in (a, \infty), \quad 0 \leqq a < B_{\nu 1},$$

and let $N_{\delta k}$ be defined for $x \in (a, \infty)$ and $\lambda > -1$ by

$$N_{\delta k} = \int_{2_{\delta k}}^{2_{\delta, k+1}} W(x) \left| \frac{\alpha x^{1/2}\mathfrak{C}_{1/(2\delta)}(x^\delta) + \beta(x(^{1/2}\mathfrak{C}_{1/(2\delta)}(x^\delta))'}{\sqrt{\alpha^2 + \beta^2\delta^2 x^{2\delta-2}}} \right|^\lambda dx, \tag{22}$$

$$k = 1, 2, \ldots$$

Then

$$\{N_{\delta k}\} \in M^*_{n-2}. \tag{23}$$

Proof. The assertion (23) is an immediate consequence of Theorem 1, applied to the equation (4) with

$$f(x) = \delta^2 x^{2\delta-2} \quad \text{and} \quad g(x) = 1.$$

It is obvious that $f(x) > 0$ on (a, ∞) and $f'(x) \in M^*_\infty(a, \infty)$. The expression P_k defined in (9) is of the form (22), so that (23) holds and the theorem is proved.

Remark 5. Let $1 < \delta \leqq \frac{3}{2}$ be an arbitrary number. Let

$$W(x) > 0, \qquad W(x) \in M_\infty(a, \infty), \qquad x \in (a, \infty), \qquad 0 \leqq a < B_{\nu 1},$$

and let $N_{\delta k}$ be defined by (22). Then

$$\{N_{\delta k}\} \in M_\infty^*.$$

Rhe remark is the case $n = \infty$ in Theorem 3.

Corollary 4. *Under the hypotheses of Theorem 3 we have*

$$\{ \int_{B_{\delta k}}^{B_{\delta, k+1}} | \alpha x^{1/2} \mathfrak{E}_{1/(2\delta)}(x^\delta) + \beta(x^{1/2}\mathfrak{E}_{1/(2\delta)}(x^\delta))' |^\lambda \, dx\} \in M_{n-2}^*,$$

for some fixed $\lambda \in (-1, 0\rangle$.

Proof. In Theorem 3, we set

$$W(x) = (\alpha^2 + \beta^2\delta^2x^{2\delta-2})^{\lambda/2}, \quad \lambda \in (-1, 0\rangle.$$

Remark 6. As a direct conclusion of Theorem 3 we get

$$\{(b_{\delta,k+1})^\gamma - (b_{\delta k})^\gamma\} \in M_\infty^*, \qquad 0 < \gamma < 1, \tag{24}$$

$$\left\{\lg \frac{b_{\delta,k+1}}{b_{\delta k}}\right\} \in M_\infty^*, \tag{25}$$

$$\{(b_{\delta k})^{-\gamma}\} \in M_\infty^*, \qquad \gamma > 0, \tag{26}$$

$$\{(\lg b_{\delta k})^{-\gamma}\} \in M_\infty^*, \qquad \gamma > 0, \qquad b_{\delta k} > 1, \tag{27}$$

$$\{\exp(-\gamma b_{\delta k})\} \in M_\infty^*, \qquad \gamma > 0. \tag{28}$$

The proof is quite similar to the proof of Remark 3.

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