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ITERATION GROUPS GENERATED BY C^n FUNCTIONS

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Throughout this paper I denotes a non-empty open interval in R, the real numbers. Suppose G is a set of functions from I to I which is a group with respect to composition. The main result of this paper gives sufficient conditions on G to guarantee that it is the iteration group generated by a C^n diffeomorphism of I onto R. The concept of iteration group is recalled below.

Whenever a set of functions from I to I is referred to as a group it is understood that the group operation is composition of functions and the identity of the group is the identity function, i, defined by $i(x) = x$ for all $x \in I$. Every member of such a group must therefore be a bijection of I onto I and its inverse is its inverse function.

Given a set G of functions from I to I we define $\cup G = \{(x, f(x)): x \in I, f \in G\}$, the union of the graphs of members of G. Such a set is said to be *disjoint* provided the graphs of any two distinct members of G are disjoint; that is, if $f, g \in G$ and $f(x_0) = g(x_0)$ for some $x_0 \in I$ then $f(x) = g(x)$ for all $x \in I$. We say G is *complete* provided $\cup G = I \times I$.

Suppose $\varphi: I \to R$ is a bijection of I onto R and for each $\alpha \in R$ we define $\varphi_\alpha(x) = \varphi^{-1}(\varphi(x) + \alpha)$ for $x \in I$. Then it is easy to check that $\{\varphi_\alpha: \alpha \in R\}$ is a group of functions from I to I which we denote by $F[\varphi]$ and call the iteration group generated by φ. In fact the mapping $\alpha \to \varphi_\alpha$ is easily seen to be an isomorphism of the additive group R onto $F[\varphi]$ and hence $F[\varphi]$ is abelian. It is also clear that $F[\varphi]$ is disjoint. To see that $F[\varphi]$ is complete notice that if $x_0, y_0 \in I$ and $\alpha = \varphi(y_0) - \varphi(x_0)$ then $\varphi_\alpha(x_0) = y_0$. Clearly, if φ is a C^n diffeomorphism of I onto R then every member of $F[\varphi]$ is a C^n function. To summarize, given a C^n diffeomorphism φ of I onto R, $F[\varphi]$ is a complete, disjoint group of C^n functions from I to I. Our main result, which we now state, is a converse to the last assertion and answers a question raised by O. Borůvka and F. Neuman.

Theorem 1. If $n \geqq 0$ and F is a complete, disjoint group of C^n functions from I to I then $F = F[\varphi]$ for some C^n diffeomorphism φ of I onto R.

The case $n = 0$ is essentially a result of Aczél [1] for which we give a different proof.

Borůvka and Neuman have also raised the following problems. Given a disjoint group G of continuous functions from I to I such that $\cup G$ is dense in $I \times I$, can G be embedded in a complete, disjoint group F of continuous functions from I to I? If so, and if each member of G is a C^n function, is each member of F a C^n function? We answer the first question in the *affirmative*. The answer to the second question is no if $n \geqq 1$ and is the subject of a forthcoming paper of the first author.

The main tools used in this paper are a theorem of O. Hölder concerning fully ordered Archimedean groups and theorems of N. G. de Bruijn concerning the so-called difference property.

Lemma 2. If G is a disjoint group of continuous functions from I to I then each member of G is a strictly increasing bijection of I onto I.

Proof. As noted earlier, each member of G is a continuous bijection of I onto I and hence strictly monotonic. If some f in G were decreasing then there would exist an $x_0 \in I$ such that $f(x_0) = x_0 = i(x_0)$. But this is impossible since $f \neq i$ and G is disjoint. Hence each member of G is strictly increasing.

If G is a disjoint set of continuous functions from I to I and if $f, g \in G$ with $f \neq g$ then either $f(x) < g(x)$ for all $x \in I$ or $g(x) < f(x)$ for all $x \in I$. Thus we define a total (full or linear) ordering on G by letting $f < g$ in case $f, g \in G$ and $f(x) < g(x)$ for all (or some) $x \in I$. We refer the reader to [3] for terminology concerning partially ordered sets and groups.

Proposition 3. If G is a disjoint group of continuous functions from I to I then G is a fully ordered Archimedean group.

Proof. As just observed, our ordering is full. If $f, g \in G$ and $f < g$ then clearly $f \circ h < g \circ h$ and $h \circ f < h \circ g$ for all $h \in G$ since every h in G is strictly increasing. Thus G is a fully ordered group.

To see that G is Archimedean, let $f, g \in G$ with $i \leqq f$, $i \leqq g$ and suppose $f^n < g$ for all $n = 1, 2, \ldots$ If $x_0 \in I$ then $x_0 \leqq f^n(x_0) \leqq f(f^n(x_0)) = f^{n+1}(x_0) < g(x_0)$ and so, if $y_0 = \lim_{n \to +\infty} f^n(x_0)$ then $y_0 \in I$ and $f(y_0) = f(\lim_{n \to +\infty} f^n(x_0)) = \lim_{n \to +\infty} f(f^n(x_0)) = \lim_{n \to +\infty} f^{n+1}(x_0) = y_0 = i(y_0)$. It follows that $f = i$. Hence G is Archimedean.

Our proofs of the embedding result and the case $n = 0$ of Theorem 1 depend heavily on the following theorem of O. Hölder (1901). A proof, due to H. Cartan can be found on pages 45–46 of the book [3] of L. Fuchs.

Theorem 4. Every fully ordered Archimedean group is isomorphic, as an ordered group, to a subgroup of the additive group R with the natural ordering and is therefore abelian.

Corollary 5. Suppose J is a dense subset of I and J is a fully ordered Archimedean group with respect to a binary operation $*$ on J and the natural ordering inherited from R. Then there exists a strictly increasing bijection φ of I onto R such that $\varphi(x * y) = \varphi(x) + \varphi(y)$ for all $x, y \in J$.

Proof. By Theorem 4 there exists a strictly increasing $\psi : J \to R$ such that $\psi(x * y) = \psi(x) + \psi(y)$ for all $x, y \in J$. Since J is dense in I, between any two members of J we can find another member of J. Since ψ is strictly increasing, between any two members of $\psi(J)$ we can find another member of $\psi(J)$. Thus in $\psi(J)$ we can find a bounded, strictly monotonic sequence which is therefore a Cauchy sequence in R. That is, given $\varepsilon > 0$, there exists $u, v \in \psi(J)$ such that $u < v < u + \varepsilon$. Now $\psi(J)$ is a subgroup of R so $v - u \in \psi(J)$ and $0 < v - u < \varepsilon$. Thus $\psi(J)$ contains arbitrarily small positive members. It follows that $\psi(J)$ is dense in R.

Now we show that ψ is continuous. Given $\varepsilon > 0$ choose $\delta \in J$ such that $0 < \psi(\delta) < \varepsilon$. Then $e < \delta$ where e is the identity of J. If $x_0 \in J$ then $x_0\delta^{-1} < x_0 < x_0\delta$. If $x_0\delta^{-1} < x < x_0\delta$ and $x \in J$ then $\delta^{-1} < x_0^{-1}x < \delta$ and so $-\varepsilon < -\psi(\delta) = \psi(\delta^{-1}) < \psi(x_0^{-1}x) = \psi(x) - \psi(x_0) < \psi(\delta) < \varepsilon$. Now $U = \{x \in J \mid x_0\delta^{-1} < x < x_0\delta\}$ is a neighborhood of x_0 in J and we have shown that $|\psi(x) - \psi(x_0)| < \varepsilon$ if $x \in U$. Hence ψ is continuous.

Since J is dense in I there exists a continuous $\varphi : I \to R$ such that $\varphi(x) = \psi(x)$ for all $x \in J$. But ψ is strictly increasing and hence so is φ. Moreover, $\psi(J)$ is dense in R so $\varphi(I) = R$.

Theorem 6. If G is a disjoint group of continuous functions from I to I such that, for some $\Theta \in I$, $\{f(\Theta) : f \in G\}$ is dense in I then G is a subgroup of $F[\varphi]$ for some strictly increasing continuous bijection φ of I onto R. Moreover, if $\{f(\Theta) : f \in G\} = I$ then $G = F[\varphi]$.

Proof. Suppose $J = \{f(\Theta) : f \in G\}$ is dense in I for some $\Theta \in I$ and define $\Phi : G \to J$ by letting $\Phi(f) = f(\Theta)$ for all $f \in G$. Then Φ is a strictly increasing bijection of G onto J where J has the natural ordering. For $x, y \in J$ define $x * y = \Phi(\Phi^{-1}(x) \circ \Phi^{-1}(y))$. Then $(J, *)$ is a fully ordered Archimedean group with respect to the natural ordering and in fact Φ is an order isomorphism of G onto J. Notice that $\Theta = \Phi(i)$ is the identity of J. By Corollary 5 there exists a strictly increasing continuous bijection φ of I onto R such that $\varphi(x * y) = \varphi(x) + \varphi(y)$ for all $x, y \in J$. We claim that G is a subgroup of $F[\varphi]$.

Let $f \in G$ and put $\alpha = \varphi(f(\Theta))$. Notice $f(\Theta) \in J$. Then, for any $x \in J$,

$$\begin{aligned}\varphi_\alpha(x) &= \varphi^{-1}(\varphi(x) + \alpha) = \varphi^{-1}(\varphi(x) + \varphi(f(\Theta))) = \\ &= \varphi^{-1}(\varphi(f(\Theta)) + \varphi(x)) = f(\Theta) * x = \Phi(\Phi^{-1}(f(\Theta)) \circ \Phi^{-1}(x)) = \\ &= \Phi(f \circ \Phi^{-1}(x)) = (f \circ \Phi^{-1}(x))(\Theta) = f(\Phi^{-1}(x)(\Theta)) = f(x)\end{aligned}$$

since $\Phi^{-1}(x)(\Theta) = x$. Since φ_α and f are continuous and J is dense in I, $\varphi_\alpha(x) = f(x)$ for all $x \in I$. Thus G is a subgroup of $F[\varphi]$.

If $J = I$ we claim $G = F[\varphi]$. For, if $\beta \in R$, then there exists $f \in G$ such that

$f(\Theta) = \varphi_\beta(\Theta)$. But there exists $\alpha \in R$ such that $f = \varphi_\alpha$ so $\varphi_\alpha(\Theta) = \varphi_\beta(\Theta)$. Hence $\varphi_\beta = \varphi_\alpha$ since $F[\varphi]$ is disjoint and so $\varphi_\beta = f \in G$.

Notice that we have proved Theorem 1 for $n = 0$ because, if G is complete, then $\{f(\Theta) : f \in G\} = I$ for all $\Theta \in I$.

To settle the first embedding problem it is convenient to consider first the case $I = (0, 1)$. We will use

Proposition 7. Suppose G is a disjoint group of continuous functions from $(0, 1)$ to $(0, 1)$ such that $\cup\, G$ is dense in $(0, 1) \times (0, 1)$. Then

(i) for every $\varepsilon > 0$ there exists $g \in G$ such that $x < g(x) < x + \varepsilon$ for all $x \in (0, 1)$ and

(ii) for every $\Theta \in I$, $\{f(\Theta) : f \in G\}$ is dense in I.

Proof. (i) Let $0 < \varepsilon < 1/2$. For $x \in I$, $\cup\, G$ must intersect the open set $\{(t, u) : x < t < x + \varepsilon, t < u < x + \varepsilon\}$. Hence for each $x \in I$ there exists $f_x \in G$ and $t \in (0, 1)$ such that $x < t < x + \varepsilon$ and $t < f_x(t) < x + \varepsilon$. Since $t < f_x(t)$ we must have $i < f_x$ and so $x < f_x(x)$. Since $x < t$ and f_x is increasing, $f_x(x) < f_x(t) < x + \varepsilon$. Thus $x < f_x(x) < x + \varepsilon$. Since f_x is continuous, there is a neighborhood V_x of x in $(0, 1)$ such that $t < f_x(t) < t + \varepsilon$ for all $t \in V_x$. Now $[\varepsilon, 1 - \varepsilon]$ is compact so we can choose $x_1, \ldots, x_n \in [\varepsilon, 1 - \varepsilon]$ such that $[\varepsilon, 1 - \varepsilon] \subseteq \bigcup_{k=1}^{n} V_{x_k}$. Let g be the smallest of $f_{x_1}, \ldots, f_{x_n}$. It follows that $x < g(x) < x + \varepsilon$ for all $x \in [\varepsilon, 1 - \varepsilon]$. If $0 < x < \varepsilon$ then $x < g(x) < g(\varepsilon) < \varepsilon + \varepsilon < x + 2\varepsilon$. If $1 - \varepsilon < x < 1$ then $x < g(x) < 1 < x + \varepsilon$. Thus we have $x < g(x) < x + 2\varepsilon$ for all $x \in (0, 1)$.

Similarly we could show that for every $\varepsilon > 0$ there exists $h \in G$ such that $x - \varepsilon < h(x) < x$ for all $x \in (0, 1)$.

(ii) Let $\Theta \in I$ and $0 < \varepsilon$. Choose $g \in G$ such that $x < g(x) < x + \varepsilon$ for all $x \in (0, 1)$. Then $g^n(\Theta) < g(g^n(\Theta)) = g^{n+1}(\Theta) < g^n(\Theta) + \varepsilon$ for all $n = 0, 1, 2, \ldots$ Moreover $\lim_{n \to +\infty} g^n(\Theta) = 1$ because if $x_0 = \lim_{n \to +\infty} g^n(\Theta)$ and $0 < x_0 < 1$ then $g(x_0) = x_0 = i(x_0)$ which is impossible since $i < g$. It follows that $\{f(\Theta) : f \in G, i \leqq f\}$ is dense in $[\Theta, 1)$. Similarly $\{f(\Theta) : f \in G, f \leqq i\}$ is dense in $(0, \Theta]$.

Theorem 8. Suppose G is a disjoint group of continuous functions from I to I and $\cup\, G$ is dense in $I \times I$. Then there exists a strictly increasing bijection φ of I onto R such that G is a subgroup of $F[\varphi]$.

Proof. If $I = (0, 1)$ this follows from (ii) of Proposition 7 and Theorem 6. In general choose a strictly increasing homeomorphism τ of I onto $(0, 1)$ and let $\tilde{G} = \{\tau \circ f \circ \tau^{-1} : f \in G\}$. Then the mapping $f \to \tau \circ f \circ \tau^{-1}$ of G onto $\tilde{G}$ is an order preserving group isomorphism. Since the mapping $(x, y) \to (\tau(x), \tau(y))$ is a homeomorphism of $I \times I$ onto $(0, 1) \times (0, 1)$ it follows that $\cup\, \tilde{G}$ is dense in $(0, 1) \times (0, 1)$. Hence there exists a strictly increasing homeomorphism ψ of $(0, 1)$

onto R such that $\tilde{G}$ is a subgroup of $F[\psi]$. It follows that G is a subgroup of $F[\varphi]$ if $\varphi = \psi \circ \tau$.

To complete the proof of Theorem 1 we will use the following result of N. G. de Bruijn [2].

Theorem 9. Suppose $f: R \to R$ is such that for every $\alpha \in R$ the mapping $x \to f(x + \alpha) - f(x)$ is continuous on R. Then there exists a continuous $g: R \to R$ and an additive $A: R \to R$ such that $f(x) = g(x) + A(x)$ for all $x \in R$. Moreover, if $n \geqq 1$ and for every $\alpha \in R$ the mapping $x \to f(x + \alpha) - f(x)$ is n times (continuously) differentiable on R, then g is n times (continuously) differentiable.

Theorem 10. Suppose $n \geqq 1$ and F is a complete, disjoint group of n times (continuously) differentiable functions from I to I. Then there exists a strictly increasing bijection φ of I onto R which is n times (continuously) differentiable on I and such that $F = F[\varphi]$ and $\varphi'(x) > 0$ for all $x \in I$.

Proof. According to Theorem 6 there exists a strictly increasing homeomorphism φ of I onto R such that $F = F[\varphi]$.

Since φ is increasing it is differentiable almost everywhere on I (see [4], p. 264). Hence there exists $x_0 \in I$ such that $\varphi'(x_0)$ exists. Now let x_1 be an arbitrary member of I. Since F is complete there exists $f \in F$ such that $f(x_0) = x_1$. Choose $\alpha \in R$ such that $f = \varphi_\alpha$ so that

$$\varphi(f(x)) = \varphi(x) + \alpha \qquad \text{for all } x \in I.$$

Hence, for sufficiently small real $\delta \neq 0$,

$$\varphi(f(x_0 + \delta)) - \varphi(f(x_0)) = \varphi(x_0 + \delta) - \varphi(x_0)$$

and so

$$\frac{\varphi(f(x_0 + \delta)) - \varphi(f(x_0))}{f(x_0 + \delta) - f(x_0)} \cdot \frac{f(x_0 + \delta) - f(x_0)}{\delta} = \frac{\varphi(x_0 + \delta - \varphi(x_0)}{\delta}.$$

Now $\varphi'(x_0)$ exists and f is differentiable on I. Since f^{-1} is also differentiable it follows that $f'(x) > 0$ for all $x \in I$ and we concluded that $\varphi'(f(x_0))$ exists. But $x_1 = f(x_0)$ and x_1 was chosen arbitrarily from I. Hence φ is differentiable and $\varphi'(f(x))\, f'(x) = \varphi'(x)$ for all $x \in I$ and all $f \in F$.

If $\varphi'(x_0) = 0$ for some $x_0 \in I$ then $\varphi'(f(x_0)) = 0$ for all $f \in F$. This implies $\varphi'(x) = 0$ for all $x \in I$ which is impossible since φ is strictly increasing. Hence $\varphi'(x) > 0$ for all $x \in I$. Therefore φ^{-1} is differentiable as well and $(\varphi^{-1})'(x) > 0$ for all $x \in R$.

Now suppose f' is continuous for every $f \in F$. This means that, for every $\alpha \in R$, the mapping $x \to \varphi^{-1}(\varphi(x) + \alpha)$ is continuously differentiable on I. Let $\psi = \varphi^{-1}$ so that, for every $\alpha \in R$, the mapping $x \to \psi(\varphi(x) + \alpha)$ is continuously differentiable on I. Hence, for every $\alpha \in R$, the mapping $x \to \psi'(\varphi(x) + \alpha)\, \varphi'(x)$ is continuous on I. Since $\psi = \varphi^{-1}$ is a homeomorphism of R onto I, for every $\alpha \in R$ the mapping $x \to \psi'(x + \alpha)\, \varphi'(\psi(x))$ is continuous on R. Let $\varrho(x) = \log \psi'(x)$ and $\sigma(x) =$

$= \log \varphi'(\psi(x))$ for $x \in R$. Then, for every $\alpha \in R$, the mapping $x \to \varrho(x + \alpha) + \sigma(x)$ is continuous on R. In particular the mapping $x \to \varrho(x) + \sigma(x)$ is continuous. Hence, for every $\alpha \in R$, the mapping $x \to \varrho(x + \alpha) - \varrho(x)$ is continuous on R. By Theorem 9 there exists a continuous $\mu : R \to R$ and an additive $A : R \to R$ such that $\varrho(x) = \mu(x) + A(x)$ for all $x \in R$. Since ψ is differentiable, ψ' is measurable and so ϱ is measurable. Hence A is measurable and therefore there exists $c \in R$ such that $A(x) = cx$ for all $x \in R$ (see, for example, Ostrowski [5]). It follows that ϱ is continuous and hence ψ' is continuous. Therefore φ' is continuous and the proof is complete in case $n = 1$.

Next suppose $n = 2$. Then, for every $\alpha \in R$, the mapping $x \to \psi'(\varphi(x) + \alpha)\, \varphi'(x)$ is (continuously) differentiable on I. We have seen that φ' and ψ' are continuous and so for every $\alpha \in R$ the mapping $x \to \psi'(x + \alpha)\, \varphi'(\psi(x))$ is (continuously) differentiable on R. It follows from Theorem 9 that μ is (continuously) differentiable on R. But so is A and hence ϱ is (continuously) differentiable on R. Thus ψ' and φ' are (continuously) differentiable on R. This proves our result in case $n = 2$.

The proof can be completed by induction.

Our results can be reformulated to give the following analogue of the result of de Bruijn.

Theorem 11. Suppose $n \geqq 0$ and ψ is a bijection of I onto R such that for every $\alpha \in R$ the mapping $x \to \psi^{-1}(\psi(x) + \alpha)$ is continuous from I to R. Then there exists a strictly increasing, continuous bijection φ of I onto R and an additive bijection A of R onto R such that $\psi(x) = A(\varphi(x))$ for all $x \in I$. Moreover, if for every $\alpha \in R$ the mapping $x \to \psi^{-1}(\psi(x) + \alpha)$ is n times (continuously) differentiable on I, then φ is n times (continuously) differentiable on I.

Proof. The iteration group $F[\psi]$ is complete and disjoint and we are assuming that each of its members is continuous. Hence there exists a strictly increasing, continuous bijection ϱ of I onto R such that $F[\psi] = F[\varrho]$, according to Theorem 6.

Thus, for every $\alpha \in R$ there exists $B(\alpha) \in R$ such that $\psi_\alpha = \varrho_{B(\alpha)}$. Clearly B is a bijection of R onto R. Moreover, if $\alpha, \beta \in R$, then

$$\varrho_{B(\alpha+\beta)} = \psi_{\alpha+\beta} = \psi_\alpha \circ \psi_\beta = \varrho_{B(\alpha)} \circ \varrho_{B(\beta)} = \varrho_{B(\alpha)+B(\beta)}$$

so that B is additive. Choose $x_0 \in I$ such that $\psi(x_0) = 0$ and let $k = \varrho(x_0)$. Then

$$\psi^{-1}(\alpha) = \psi^{-1}(\psi(x_0) + \alpha) = \psi_\alpha(x_0) = \varrho_{B(\alpha)}(x_0) = \\ = \varrho^{-1}(\varrho(x_0) + B(\alpha)) = \varrho^{-1}(k + B(\alpha)) \qquad \text{for all } \alpha \in R.$$

Thus $x = \psi^{-1}(\psi(x)) = \varrho^{-1}(k + B(\psi(x)))$ for all $x \in I$. Hence $\varrho(x) = k + B(\psi(x))$ for all $x \in I$. Let $\varphi(x) = \varrho(x) - k$ for all $x \in I$ and let $A = B^{-1}$. Then $\psi(x) = A(\varphi(x))$ for all $x \in I$.

If $n \geqq 1$ and, for every $\alpha \in R$, the mapping $x \to \varphi^{-1}(\psi(x) + \alpha)$ is n times (continuously) differentiable on I, then from Theorem 10 it follows that ϱ (and hence φ) is n times (continuously) differentiable.

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