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TOLERANCES AND ORDERINGS ON SEMILATTICES

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Let $\mathfrak{A} = (A, F)$ be an algebra. A binary relation R on $\mathfrak{A}$ has the *Substitution Property*, briefly *SP*, if R is a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$. We shall denote by Δ the so called diagonal $\{\langle x, x\rangle; x \in A\}$ of $\mathfrak{A}$. Clearly Δ has *SP*. By a *tolerance* we shall mean a reflexive and symmetric relation on $\mathfrak{A}$ having *SP*. Denote by $LT(\mathfrak{A})$ the lattice of all tolerances on $\mathfrak{A}$ ordered by set inclusion. Clearly $LT(\mathfrak{A})$ is an algebraic lattice, where Δ is its least and $A \times A$ its greatest element; see [3] and [5]. Hence, there exists the least tolerance $T(a, b)$ containing $\langle a, b\rangle$ for every two elements a, b of $\mathfrak{A}$.

By an *ordering* on $\mathfrak{A}$ we shall mean a reflexive, transitive and antisymmetric binary relation on $\mathfrak{A}$ having *SP*. Let $\leqq$ be a (fix) ordering on $\mathfrak{A}$. Following [1] and [7],

$$LD(\mathfrak{A}) = \{R; \Delta \subseteqq R \subseteqq \leqq \text{ and } R \text{ has } SP \text{ on } \mathfrak{A}\}$$

is the lattice of all reflexive (i.e. diagonal) binary relations having *SP* on $\mathfrak{A}$ and contained in $\leqq$. Clearly $LD(\mathfrak{A})$ is an algebraic lattice with respect to set inclusion. When a and b are two elements of $\mathfrak{A}$ such that $a \leqq b$, we denote by $D(a, b)$ the least element of $LD(\mathfrak{A})$ containing $\langle a, b\rangle$.

Let $\mathfrak{L}$ be a lattice and $\leqq$ its ordering. D. Schweigert [7] and H.-J. Bandelt [1] proved that the lattices $LT(\mathfrak{L})$ and $LD(\mathfrak{L})$ are isomorphic. We proceed to show that the situation is different for semilattices.

Theorem 1. *Let $\mathfrak{S} = (S, \vee)$ be a semilattice and $\leqq$ its induced ordering, i.e. $a \leqq b$ if and only if $a \vee b = b$. Then*

(i) *there exists a subset L of $LT(\mathfrak{S})$ which is a lattice with respect to the order on $LT(\mathfrak{S})$, and $LD(\mathfrak{S})$ is isomorphic to L;*

(ii) *the isomorphism of* (i) *is a mapping $\psi : LD(\mathfrak{S}) \to L$, where $\psi(R) = \{\langle x, y\rangle; \langle x, x \vee y\rangle \in R$ and $\langle y, x \vee y\rangle \in R\}$;*

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(iii) *L and $LT(\mathfrak{S})$ have a common least and a common greatest element i.e.* $\psi(\leqq) = S \times S$ *and* $\psi(\Delta) = \Delta$.

Proof. Let $\zeta : LT(\mathfrak{S}) \to LD(\mathfrak{S})$ be a mapping given by $\zeta(T) = T \cap \leqq$. It is clear that ζ and ψ are order-preserving and

$$\zeta\psi(R) = \zeta(\{\langle x, y\rangle; \langle x, x \vee y\rangle \in R \text{ and } \langle y, x \vee y\rangle \in R\}) = \\ = \{\langle x, y\rangle; \langle x, x \vee y\rangle \in \text{ and } \langle y, x \vee y\rangle \in R\} \cap \leqq = R.$$

Hence, ψ is an order-preserving one-to-one mapping of $LD(\mathfrak{S})$ into $LT(\mathfrak{S})$, i.e. $LD(\mathfrak{S})$ is mapped by ψ isomorphically to a lattice L which is a subset of $LT(\mathfrak{S})$. Finally,

$$\psi(\leqq) = \{\langle x, y\rangle; x \leqq x \vee y \text{ and } y \leqq x \vee y\} = S \times S$$

and

$$\psi(\Delta) = \{\langle x, y\rangle; x = x \vee y \text{ and } y = x \vee y\} = \Delta.$$ □

Remark. The lattice of Theorem 1 need not be a sublattice of $LT(\mathfrak{S})$. Indeed, if $\mathfrak{S}$ is a $\vee$-semilattice of three elements a, b and c such that $a \vee b = c$, with $R_1 = \{\langle a, c\rangle\} \cup \Delta$ and $R_2 = \{\langle b, c\rangle\} \cup \Delta$, then clearly $R_1, R_2 \in LD(\mathfrak{S})$ and $\psi(R_1 \vee R_2) = \psi(\leqq) = S \times S \neq \{\langle a, c\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\} \cup \Delta = \psi(R_1) \vee \psi(R_2)$, where the join on the left is formed in $LD(\mathfrak{S})$ and the join on the right is formed in $LT(\mathfrak{S})$.

The next theorem characterizes semilattices $\mathfrak{S}$ for which $LD(\mathfrak{S})$ and $LT(\mathfrak{S})$ are isomorphic.

Theorem 2. *Let $\mathfrak{S} = (S, \vee)$ be a semilattice and $\leqq$ its induced ordering. If $\mathfrak{S}$ is a chain, then $LD(\mathfrak{S})$ and $LT(\mathfrak{S})$ are isomorphic. If $\mathfrak{S}$ is not a chain, then $LD(\mathfrak{S})$ is isomorphic to a proper sublattice of $LT(\mathfrak{S})$.*

Proof. Let ζ and ψ be the mappings of the proof of Theorem 1. When $\mathfrak{S}$ is a chain, then $\psi\zeta(T) = T$ for every $T \in LT(\mathfrak{S})$, because $\psi(R)$ is the symmetric envelop of R. Applying now Theorem 1, we have $LD(\mathfrak{S}) \cong LT(\mathfrak{S})$.

On the contrary, suppose $\mathfrak{S}$ is not a chain, i.e. there exist elements x, y of $\mathfrak{S}$ such that $\{x, y, x \vee y\}$ constitutes a three-element subsemilattice $\mathfrak{C}$ of $\mathfrak{S}$. Now we can define two different tolerances $T_1 \in LT(\mathfrak{S})$, $T_2 \in LT(\mathfrak{S})$ such that $\mathfrak{C}$ is contained in a single block of T_2, but in T_1 it is divided into two blocks one containing $\{x, x \vee y\}$ and the other $\{y, x \vee y\}$; elsewhere $T_1 = T_2$ (see [3]). Then $\zeta(T_1) = \zeta(T_2)$. Suppose that there exist relations $R_1 \in LD(\mathfrak{S})$, $R_2 \in LD(\mathfrak{S})$ such that $T_1 = \psi(R_1)$, $T_2 = \psi(R_2)$. As $\zeta\psi(R) = R$ for each $R \in LD(\mathfrak{S})$, we have $R_1 = \zeta\psi(R_1) = \zeta(T_1) = \zeta(T_2) = \zeta\psi(R_2) = R_2$. But this implies also $\psi(R_1) = T_1 = \psi(R_2)$, which is a contradiction. Thus at least one of the relations T_1, T_2 is not an immage of a relation from $LD(\mathfrak{S})$ in the mapping ψ, and ψ maps $LD(\mathfrak{S})$ onto a proper subset of $LT(\mathfrak{S})$, not onto whole $LT(\mathfrak{S})$. □

This theorem does not exclude the case when there exists isomorphism of $LD(\mathfrak{S})$ onto $LT(\mathfrak{S})$ and onto a proper subset of $LT(\mathfrak{S})$ simultaneously. In such a case $LT(\mathfrak{S})$

would be isomorphic to its proper subset and evidently it would be infinite. We have:

Corollary 1. *Let $\mathfrak{S} = (S, \vee)$ be a finite semilattice and $\leqq$ its induced ordering. The lattices $LD(\mathfrak{S})$ and $LT(\mathfrak{S})$ are isomorphic if and only if $\mathfrak{S}$ is a chain.*

As known, the compact elements of $LT(\mathfrak{A})$ are finite joins of tolerances $T(a, b)$ for elements a, b of $\mathfrak{A}$, see [3]. Clearly the compact elements of $LD(\mathfrak{A})$ are the finite joins of $D(a, b)$ for $a \leqq b$, where $\leqq$ is the fixordering of $\mathfrak{A}$. A semilattice $\mathfrak{S}$ is called a *tree-semilattice* if the interval $[a, b]$ is a chain for every pair $a \leqq b$ of elements a, b in $\mathfrak{S}$. If $\mathfrak{S}$ is a finite tree-semilattice, its Hasse diagram is a tree in the graph theoretical sense.

Theorem 3. *Let $\mathfrak{S}$ be a semilattice and $\leqq$ its induced ordering, let $a \leqq b$ in $\mathfrak{S}$. Then:*

(1) $\psi(D(a, b)) \supseteqq T(a, b)$;

(2) $\psi(D(a, b)) = T(a, b)$ *for every pair $a \leqq b$ of $\mathfrak{S}$ if and only if $\mathfrak{S}$ is a tree-semilattice.*

Proof. If $a \leqq b$ in $\mathfrak{S}$, then $D(a, b) = \{\langle x, y\rangle; x = a \vee c, y = b \vee c \text{ for } c \in \} \cup \Delta$. Hence,

$$\psi(D(a, b)) = \{\langle x, y\rangle; \langle x, x \vee y\rangle \in D(a, b) \text{ and } \langle y, x \vee y\rangle \in D(a, b)\} =$$
$$= \{\langle x, y\rangle; x = a \vee c, y = a \vee d, x \vee y = b \vee c = b \vee d \text{ for } c, d \in \mathfrak{S}\} \cup \Delta.$$

Choosing $c = a$ and $d = b$ we obtain, $\langle a, b\rangle \in \psi(D(a, b)) \in LT(\mathfrak{S})$, and thus $T(a, b) \subseteqq \psi(D(a, b))$.

Now, let $\mathfrak{S}$ be a tree-semilattice. Then $a \leqq a \vee d \leqq b \vee d$ and $a \leqq a \vee c \leqq \leqq b \vee c = b \vee d$, whence both $a \vee c$ and $a \vee d$ lie in the interval $[a, b \vee d]$. Since $\mathfrak{S}$ is a tree-semilattice, $[a, b \vee d]$ is a chain, whence $a \vee c$ and $a \vee d$ are comparable. Then

$$\psi(D(a, b)) = \{\langle x, y\rangle; \langle x, y\rangle \in D(a, b)\} \cup \{\langle x, y\rangle; \langle y, x\rangle \in D(a, b)\} \cup \Delta = T(a, b).$$

On the contrary, if $\mathfrak{S}$ is not a tree-semilattice, there exist elements a, b, c of $\mathfrak{S}$ such that a and b are non-comparable and c is a lower bound of a and b. Thus $\{c, a, b, a \vee b\}$ constitutes a four-element subsemilattice of $\mathfrak{S}$, where we denote briefly $d = a \vee b$. Since $\langle a, d\rangle = \langle a \vee c, a \vee d\rangle \in D(c, d)$ and $\langle b, d\rangle = \langle b \vee c, b \vee d\rangle \in D(c, d)$, we have $D(c, d) = \{\langle c, d\rangle, \langle a, d\rangle, \langle b, d\rangle\} \cup \Delta$, and moreover, $\psi(D(c, d)) = \{\langle c, d\rangle, \langle d, c\rangle, \langle a, d\rangle, \langle d, a\rangle, \langle b, d\rangle, \langle d, b\rangle, \langle a, b\rangle, \langle b, a\rangle\} \cup \Delta$. The other parts but $\langle a, b\rangle \in \psi(D(c, d))$ are trivial, and $\langle a, b\rangle \in \psi(D(c, d))$ follows from $\langle a, a \vee b\rangle = \langle a, d\rangle \in D(c, d)$ and $\langle b, a \vee b\rangle = \langle b, d\rangle \in D(c, d)$. However, $T(c, d) = \{\langle c, d\rangle, \langle d, c\rangle, \langle a, d\rangle, \langle d, a\rangle, \langle b, d\rangle, \langle d, b\rangle\} \cup \Delta$ as we can easily see [6], [8]. Hence $T(c, d) \neq \psi(D(c, d))$, and the assertion follows. □

The foregoing theorem gives a characterization of tree-semilattices by means of tolerances $T(a, b)$ and diagonal relations $D(a, b)$. In the next part we proceed to give an explicite description of $D(a, b)$.

Let $\leqq$ be a (fix) ordering on an algebra $\mathfrak{A}$. We denote by $LO(\mathfrak{A})$ the set of all orderings on $\mathfrak{A}$ contained in $\leqq$. Clearly also $LO(\mathfrak{A})$ is a complete lattice. Hence, if $a \leqq b$ in $\mathfrak{A}$, there is a least element in $LO(\mathfrak{A})$ containing $\langle a, b\rangle$, and we shall denote that element by $P(a, b)$.

Theorem 4. *Let $\mathfrak{S}$ be a semilattice and $\leqq$ its induced ordering. If $D \in LD(\mathfrak{S})$, then the transitive closure $C(D)$ of D is an ordering on $\mathfrak{S}$, i.e. $C(D) \in LO(\mathfrak{S})$.*

Proof. Because $D \subseteqq \leqq$, also $C(D) \subseteqq \leqq$. Now, if $C(D)$ has SP, then it is an ordering on $\mathfrak{S}$, and thus it remains to prove SP for $C(D)$. Suppose $\langle a, b\rangle, \langle c, d\rangle \in C(D)$. Then there exist elements $x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n$ such that $a = x_0 \leqq x_1 \leqq \ldots \leqq x_m = b$ and $c = y_0 \leqq y_1 \leqq \ldots \leqq y_n = d$, where $\langle x_i, x_{i+1}\rangle \in D$ for $i = 0, 1, \ldots, m-1$ and $\langle y_j, y_{j+1}\rangle \in D$ for $j = 0, 1, \ldots, n-1$. Without loosing generality we assume that $m \leqq n$, and put $x_i = b$ for $m \leqq i \leqq n$. Let now $z_i = x_i \vee y_i$ for $i = 0, 1, \ldots, n$. Then $a \vee c = z_0 \leqq z_1 \leqq \ldots \leqq z_n = b \vee d$ and $\langle z_i, z_{i+1}\rangle \in D$ for $i = 0, 1, \ldots, n$ because of SP of D. Hence $\langle a \vee c, b \vee d\rangle \in C(D)$ and $C(D)$ has SP. □

Theorem 5. *Let $\mathfrak{S}$ be a semilattice with the induced ordering $\leqq$, a, b two elements of $\mathfrak{S}$, and $a \leqq b$. Then $D(a, b) = P(a, b)$.*

Proof. Evidently, $D(a, b) = \{\langle a \vee x, b \vee x\rangle;\ x \in \mathfrak{S}\} \cup \Delta$. We shall prove the transitivity of $D(a, b)$. Let c, d and e be elements of $\mathfrak{S}$ such that $c \leqq d \leqq e$ and $\langle c, d\rangle, \langle d, e\rangle \in D(a, b)$. If $c = d$ or $d = e$, there is nothing to prove. Suppose that $\langle c, d\rangle = \langle a \vee x, b \vee x\rangle$ and $\langle d, e\rangle = \langle a \vee y, b \vee y\rangle$ for some elements $x, y \in \mathfrak{S}$. Then $d = b \vee x = a \vee y$, and moreover, $d = d \vee d = a \vee b \vee x \vee y = b \vee x \vee y \geqq b \vee y = e$. Because $d \leqq e$ and $d \geqq e$, we have $d = e$, whence also $\langle c, e\rangle = \langle c, d\rangle \in D(a, b)$, and thus $D(a, b)$ is transitive. Then $D(a, b) \in LO(\mathfrak{S})$ and the equality $D(a, b) = P(a, b)$ is evident. □

Theorem 6. *Let $\mathfrak{S}$ be a tree-semilattice and $\leqq$ its induced ordering. If $a, b \in \mathfrak{S}$ and $a \leqq b$, then $D(a, b) = \{\langle x, b\rangle;\ a \leqq x \leqq b\} \cup \Delta$.*

Proof. By Theorem 5, $D(a, b) = P(a, b)$, and thus $D(a, b)$ is the least ordering on $\mathfrak{S}$ containing the ordered pair $a \leqq b$. Let $R = \{\langle x, b\rangle;\ a \leqq x \leqq b\} \cup \Delta$. By putting $x = a$, we obtain $\langle a, b\rangle \in R$, and according to the definition of R, $R \subseteqq D(a, b)$. It remains to prove that R has SP. Suppose $\langle y_1, z_1\rangle, \langle y_2, z_2\rangle \in R$, and if $\langle y_1, z_1\rangle, \langle y_2, z_2\rangle \in \Delta$, there is nothing to prove. If $\langle y_1, z_1\rangle \in \Delta$ and $\langle y_2, z_2\rangle \in R \setminus \Delta$, then $y_1 = z_1$, $z_2 = b$ and $a \leqq y_2 \leqq b$. These fact imply that $\langle y_1 \vee y_2, z_1 \vee z_2\rangle = \langle y_1 \vee y_2, y_1 \vee b\rangle$. If $y_1 \leqq b$, then $a \leqq y_1 \vee y_2 \leqq b$ and $z_1 \vee z_2 = y_1 \vee b = b$, and thus $\langle y_1 \vee y_2, z_1 \vee z_2\rangle \in R$. In the opposite case $y_1 \vee b > b$. On the other hand b and $y_1 \vee y_2$ belong to the interval $[y_2, y_1 \vee b]$, and because $\mathfrak{S}$ is a tree-semilattice, b and $y_1 \vee y_2$ are comparable. The inequality $y_1 \vee y_2 \leqq b$ implies that $y_1 \leqq b$, which is a contradiction. Thus $y_1 \vee y_2 > b$, and moreover, $y_1 \vee y_2 = y_1 \vee b$. Hence, $\langle y_1 \vee y_2, z_1 \vee z_2 \rangle = \langle y_1 \vee y_2, y_1 \vee b \rangle \in \Delta \subseteq R$.

If $\langle y_1, z_1 \rangle \in R \backslash \Delta$ and $\langle y_2, z_2 \rangle \in R \backslash \Delta$, then according to the proof above we have $\langle y_1 \vee y_2, z_1 \vee y_2 \rangle$, $\langle z_1 \vee y_2, z_2 \vee y_2 \rangle \in R$. Because R is trivially transitive, we obtain $\langle y_1 \vee y_2, z_1 \vee z_2 \rangle \in R$, and thus R has *SP*. □

Theorem 7. *Let $\mathfrak{S}$ be a tree-semilattice with the induced ordering $\leqq$ and P a reflexive binary relation on $\mathfrak{S}$ contained in $\leqq$. Then $P \in LD(\mathfrak{S})$ if and only if $\langle a, b \rangle \in P$ implies $\langle x, b \rangle \in P$ for any elements $a, b, x \in \mathfrak{S}$ such that $a \leqq x \leqq b$.*

Proof. If $P \in LD(\mathfrak{S})$ and $\langle a, b \rangle \in P$, then for any $x, a \leqq x \leqq b$, $\langle x, b \rangle = \langle a \vee x, b \vee x \rangle \in P$, and the first part of the proof follows.

Conversely, suppose that P has the property $\langle a, b \rangle \in P$ implies $\langle x, b \rangle \in P$ for any $a, b, x \in \mathfrak{S}$ with $a \leqq x \leqq b$. We shall prove *SP* of P. Let $\langle a, b \rangle$, $\langle c, d \rangle \in P$. If b and d are incomparable, then $a \vee c = b \vee d$, because $\mathfrak{S}$ is a tree-semilattice, and thus $\langle a \vee c, b \vee d \rangle \in \Delta \subseteq P$. If e.g. $b \leqq d$, then $b \vee d = d$ and $c \leqq a \vee c \leqq b \vee d = d$. But then $\langle c, d \rangle \in P$ implies $\langle a \vee c, b \vee d \rangle = \langle a \vee c, d \rangle \in P$ according to the property of P. The case $d \leqq b$ is analogous. □

Corollary 2. *Let $\mathfrak{S}$ be a tree-semilattice with the induced ordering $\leqq$ and P a reflexive, antisymmetric and transitive binary relation on $\mathfrak{S}$ with $P \subseteq \leqq$. Then $P \in LO(\mathfrak{S})$ if and only if $\langle a, b \rangle \in P$ implies $\langle x, b \rangle \in P$ for any elements a, b, x of $\mathfrak{S}$ with $a \leqq x \leqq b$.*

Remark. Theorem 7 and its Corollary give a possibility to describe the join operation in $LD(\mathfrak{S})$ and in $LO(\mathfrak{S})$, respectively, when $\mathfrak{S}$ is a tree-semilattice.

The join $\vee$ in $LD(\mathfrak{S})$: $P, Q \in LD(\mathfrak{S}) \Rightarrow P \vee Q = P \cup Q$.

The join $\vee$ in $LO(\mathfrak{S})$: $R, U \in LO(\mathfrak{S}) \Rightarrow R \vee U$ is the transitive closure of $R \cup U$.

The remaining part of the paper is devoted to the extension properties of relations of $LD(\mathfrak{S})$ and $LO(\mathfrak{S})$. The first attempt to study the extension property of other relations than congruences was done by Chajda in [2] for relations of $LT(\mathfrak{S})$. We recall first briefly the necessary concepts:

A class $\mathfrak{C}$ of algebras satisfies the *Tolerance Extension Property* (briefly *TEP*) if for every $\mathfrak{A} \in \mathfrak{C}$ and every subalgebra $\mathfrak{L}$ of $\mathfrak{A}$, each tolerance T on $\mathfrak{L}$ is the restriction of some tolerance T^* on $\mathfrak{A}$, i.e. $T = T^* \cap (\mathfrak{L} \times \mathfrak{L})$.

Proposition. (Theorem 2 and the Example in [2]) *Every class of tree-semilattices satisfies TEP. The variety of all semilattices does not satisfy TEP.*

We can define the extension property analogously for relations of $LD(\mathfrak{A})$ and $LO(\mathfrak{A})$:

Definition. *Let $\mathfrak{C}$ be a class of ordered algebras such that every $\mathfrak{A}$ of $\mathfrak{C}$ is ordered by a fixordering $\leqq$. $\mathfrak{C}$ satisfies the Extension Property of Orderings if for every*

$\mathfrak{A} \in \mathfrak{C}$ and for every subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, each $P \in LO(\mathfrak{B})$ is the restriction of some $P^* \in LO(\mathfrak{A})$. $\mathfrak{C}$ satisfies the D-Extension Property if for every $\mathfrak{A} \in \mathfrak{C}$ and for every subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, each $D \in LD(\mathfrak{B})$ is the restriction of some $D^* \in LD(\mathfrak{A})$.

Theorem 8. *The variety of all semilattices has the D-Extension Property.*

Proof. Let $\mathfrak{S}_0$ be subsemilattice of a semilattice $\mathfrak{S}$, $D_0 \in LD(\mathfrak{S}_0)$, and let us consider the relation $D = D_0 \cup \Delta \cup \{\langle a \vee x,\ b \vee x\rangle;\ \langle a, b\rangle \in D_0 \text{ and } x \in \mathfrak{S}\}$. Then clearly $D_0, \Delta \subseteqq D$, and we shall prove that $D \in LD(\mathfrak{S})$. If $\langle c, d\rangle, \langle e, f\rangle \in D_0$, then $\langle c \vee e,\ d \vee f\rangle \in D_0 \subseteqq D$ according to SP of D_0 and the definition of D. If $\langle c, d\rangle \in D_0$ and $\langle e, f\rangle \in \Delta$, the proof follows from the definition of D, as well as in the case $\langle c, d\rangle, \langle e, f\rangle \in \Delta$. Thus suppose $\langle c, d\rangle = \langle a \vee x, b \vee x\rangle$ for $\langle a, b\rangle \in D_0$ and $x \in \mathfrak{S}$. If $\langle e, f\rangle \in D_0$, then $\langle c \vee e, d \vee f\rangle = \langle a \vee e \vee x, b \vee f \vee x\rangle$, where $\langle a \vee e,\ b \vee f\rangle \in D_0$, whence $\langle c \vee e,\ d \vee f\rangle \in D$. If $\langle e, f\rangle \in \Delta$, the proof is trivial. Thus, let $\langle e, f\rangle = \langle a' \vee x', b' \vee x'\rangle$ for some $\langle a', b'\rangle \in D_0$ and $x' \in \mathfrak{S}$. Then $\langle c \vee c,\ d \vee f\rangle = \langle a \vee a' \vee x \vee x',\ b \vee b' \vee x \vee x'\rangle$, and on the other hand, by SP of D_0, $\langle a \vee a',\ b \vee b'\rangle \in D_0$. Therefore, $\langle c \vee e,\ d \vee f\rangle \in D$, and the SP of D follows. But then $D \in LD(\mathfrak{S})$, and so it remains to prove that $D \cap (\mathfrak{S}_0 \times \mathfrak{S}_0) = D_0$. Let $\langle a \vee x,\ b \vee x\rangle \in D$ such that $\langle a, b\rangle \in D_0$ and $x \in \mathfrak{S} \setminus \mathfrak{S}_0$. If $a \vee x \in \mathfrak{S}_0$, then $b \vee a \vee x = (b \vee a) \vee x = b \vee x$, because $a \leqq b$, and thus $b \vee x \in \mathfrak{S}_0$. Hence $\langle a \vee x,\ b \vee x\rangle \in D_0$ and $D \cap (\mathfrak{S}_0 \times \mathfrak{S}_0) \subseteqq D_0$. The converse is trivial, and the desired property follows. □

The first attempt to characterize the Extension Property of Orderings was done in [4] for a single algebra ($\mathfrak{C} = \{\mathfrak{A}\}$). The next theorem solves the problem of Extension Property of Orderings on semilattices:

Theorem 9. *The variety of all semilattices has the Extension Property of Orderings.*

Proof. Let $\mathfrak{S}_0$ be a subsemilattice of a semilattice $\mathfrak{S}$ and $P_0 \in LO(\mathfrak{S}_0)$. Let $P = P_0 \cup \Delta \cup \{\langle a \vee x,\ b \vee x\rangle;\ \langle a, b\rangle \in P_0 \text{ and } x \in \mathfrak{S}\}$ and $C(P)$ be the transitive closure of P. According to Theorems 4 and 8, $C(P) \in LO(\mathfrak{S})$ and $P_0 \subseteqq C(P) \cap (\mathfrak{S}_0 \times \mathfrak{S}_0)$. Thus it remains to prove that $C(P) \cap (\mathfrak{S}_0 \times \mathfrak{S}_0) \subseteqq P_0$. Let $c, d \in \mathfrak{S}_0$ and $\langle c, d\rangle \in C(P) \setminus P$. According to the proof of Theorem 8, $\langle c, d\rangle \notin P \setminus P_0$. Therefore, there exist elements $y_0, y_1, \ldots, y_n$ such that $c = y_0 \leqq y_1 \leqq \ldots \leqq y_n = d$, $\langle y_i, y_{i+1}\rangle \in P$ for $i = 0, 1, \ldots, n-1$ and at least one pair $\langle y_j, y_{j+1}\rangle \notin P_0$. Then by the proof of Theorem 8, $y_j \notin \mathfrak{S}_0$. Hence also $\langle y_{j-1}, y_j\rangle \notin P_0$ and $y_{j-1} \notin \mathfrak{S}_0$. By induction we conclude that $y_k \notin \mathfrak{S}_0$ for all $k \leqq j$, and thus $c \notin \mathfrak{S}_0$, which is a contradiction. Accordingly, $P_0 \supseteqq C(P) \cap (\mathfrak{S}_0 \times \mathfrak{S}_0)$ holds, and the theorem follows. □

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