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SOME REMARKS ON A TERMINAL VALUE PROBLEM

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Abstract. Some results concerning the bounded solutions of a terminal value problem for differential equations with deviating arguments and a uniqueness theorem for the retarded case of this problem are given.

Key words. Differential equations with deviating arguments, terminal value problems, uniqueness of solutions, asymptotic behavior.

The purpose of this note is to add some new results in a paper by B. A. Staikos and myself entitled "On the terminal value problem for differential equations with deviating arguments" [2].

In that paper we obtained some theorems concerning the existence and uniqueness of solutions of the differential equations with deviating arguments

$$x'(t) = f(t; x[\sigma_1(t)], \dots, x[\sigma_k(t)]), \tag{E}$$

which satisfy the "terminal" condition

$$\lim_{t \to \infty} x(t) \quad \xi, \tag{C}$$

where $\xi \in K^n$ and $K = R$ or $K = C$. The function $f : [t_0, \infty) \times (K^n)^k \to K^n$ is a locally Caratheodory one and moreover the real-valued functions σ_i, $i = 1, \dots, k$ are continuous on $[t_0, \infty)$ and, as usually, such that

$$\lim_{t \to \infty} \sigma_i(t) = \infty \qquad (i = 1, \dots, k).$$

Here, we relate the above terminal value problem $(E)-(C)$ with the well known one of the asymptotic equilibrium for the equation (E). Moreover we extend the uniqueness theorem for the problem $(E)-(C)$ in [2], to a class of equations of the form (E) with arguments not necessary of advanced type.

In [2] we introduced the function a_r by

$$a_r(t) = \max_{|z_i| \leq r} | f(t; z_1, \dots, z_k) |, \qquad t \geqq t_0, \qquad \mathrm{i} = 1, \dots, k$$

and we proved the following theorem.

Theorem 1. *If for every $r > 0$,*

$$(C_1) \qquad \int^{\infty} a_r(t)\,dt < \infty,$$

then for every $\xi \in K^n$ the terminal value problem $(E)-(C)$ has at least one solution x on an interval of the form $[T, \infty)$.

Now, we give the following definition.

Definition. *The equation (E) has "asymptotic equilibrium" if and only if there exists a $T > 0$ such that for every $\xi \in K^n$ there exists a solution x of (E) defined on $[T, \infty)$, such that $\lim_{t\to\infty} x(t) = \xi$ and for every solution x of (E) defined on $[T, \infty)$, the $\lim_{t\to\infty} x(t)$ exists in K^n.*

Thus, from this point of view, we can say that under condition (C_1) the equation (E) satisfies a part of this definition.

Results for the asymptotic equilibrium of retarded differential equations can be found in [1], [3] and [4]. Condition (C_1) is not sufficient for the asymptotic equilibrium of the equation (E). This done clear by the following example.

Example 1. Condition (C_1) is satisfied for the retarded differential equation

$$x'(t) = \frac{2}{t^2}(x(\sqrt{t}))^3 \qquad t \geqq 1,$$

but this equation has the solution $x(t) = t^2$, for which $\lim_{t\to\infty} x(t)$ is not finite.

In spite of the above remark, condition (C_1) is sufficient for the existence of the $\lim_{t\to\infty} x(t)$ for every bounded solution $x = x(t)$ of (E). That is the first of main results in this note.

Proposition 1. *If the condition (C_1) holds, then for every bounded solution $x = x(t)$ of (E) the $\lim_{t\to\infty} x(t)$ exists in K^n.*

Proof. Let $x = x(t)$, $t \geqq t_0$ be a bounded solution of (E) for which $|x(t)| < r$, $t \geqq t_0$, where $r > 0$. Then for an arbitrary $\varepsilon > 0$ by (C_1) we choose a $T \geqq t_0$ so that

$$\int_T a_r(t)\,dt < \varepsilon \qquad \text{and} \qquad \min_{\substack{t \geqq T \\ i=1,\dots,k}} \sigma_i(t) \geqq t_0 .$$

Then for any t_1, t_2 in R with $t_2 \geqq t_1 \geqq T$ we have

$$|x(t_2) - x(t_1)| \leqq \int_{t_1}^{t_2} |f(t; x[\sigma_1(t)], \dots, x[\sigma_k(t)])|\,dt \leqq \int_{t_1}^{t_2} a_r(t)\,ds \leqq \int_T^{\infty} a_r(t)\,dt < \varepsilon.$$

Therefore, Cauchy criterion gives that $\lim_{t\to\infty} x(t)$ exists in K^n.

In [2], we deal also with the uniqueness of solutions of the terminal value problem $(E)-(C)$. More precisely in [2] we proved the following result.

Theorem 2. *If the equation* (E) *is of advanced type and for every compact subset* B *of the space* $(K^n)^k$, *the function* f *satisfies*

$$(C_2) \qquad | f(t; x_1, \ldots, x_k) - f(t; y_1, \ldots, y_k) | \leqq L_B(t) \sum_{i=1}^{k} | x_i - y_i |$$

for every $t \geqq t_0$ *and* $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k)$ *in* B, *where* L_B *is a real valued function with*

$$(C_3) \qquad \int^{\infty} L_B(t) < \infty,$$

then the terminal value problem $(E)-(C)$ *has at most one solution on an interval of the form* $[T, \infty)$.

By appropriate examples in [2] we prove that the uniqueness of solutions of the terminal value problem $(E)-(C)$ fails in the case where (E) is not advanced type.

In order to extend the result of Theorem 2 to a class of retarded type (or mixed type) equations, we establish the following notations.

Without loss of generality we suppose that the arguments σ_i, $i = 1, \ldots, m$ $(1 \leqq m \leqq k)$ in (E) are of retarded type and the rest arguments are of advancend type. Also, we denote by S the set of common fixed points of the arguments σ_i, $i = 1, \ldots, m$. That is

$$S = \{t \geqq t_0 \quad \sigma_i(t) = t \text{ for all } i = 1, \ldots, m\}.$$

Moreover the functions σ_i, $i = 1, \ldots, m$ satisfy a condition of "weak monotonicity" with respect to the set S. More precisely, for any $s \in S$

$$\sigma_i(t) \geqq s \qquad \text{for every } t \geqq s \qquad (i = 1, \ldots, m).$$

Results for such retarded equations appeared in [3].

Proposition 2. *Suppose that conditions* (C_2) *and* (C_3) *of Theorem 2 are hold and the set* S *is unbounded above. Then the terminal value problem* $(E)-(C)$ *has at most one solution on an interval of the form* $[T, \infty)$.

Proof. We assume that x and y are solutions of the terminal value problem $(E)-(C)$. Then, because of $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \xi$, for any $\varepsilon > 0$ and some $T_0 \geqq t_0$ we have

$$| x(t) | \leqq | \xi | + \varepsilon \qquad \text{and} \qquad | y(t) | \leqq | \xi | + \varepsilon \qquad \text{for every } t \geqq T_0.$$

Because of the above unboundedness of S we can assume, without loss of generality, that $T_0 \in S$.

Now, we set

$$B = \{(z_1, \ldots, z_k) \in (K^n)^k : | z_i | \leqq | \xi | + \varepsilon\}.$$

Then, for $i = 1, \dots, k$ and every $t \geqq T_0$ we have

$$x(t) \in B,\ y(t) \in B,\ x[\sigma_i(t)] \in B \quad \text{and} \quad y[\sigma_i(t)] \in B,$$

since for every $t \geqq T_0$ we have $\sigma_i(t) \geqq T_0$ $(i = 1, \dots, k)$.

We consider now the function L_B and, because of (C_3), we choose a $T \in S$, $T \geqq T_0$, so that

$$\int_T^\infty L_B(s)\,\mathrm{d}s \leqq \frac{1}{2k}.$$

Thus, taking into account condition (C_2) and the fact that $\sigma_i(t) \geqq T$ $(i = 1, \dots, m)$ for all $t \geqq T$, for every $t \geqq T$ we have

$$|x(t) - y(t)| \leqq \int_t^\infty |f(s; x[\sigma_1(s)], \dots, x[\sigma_k(s]) - f(s; y[\sigma_1(s)], \dots, y[\sigma_k(s)])|\,\mathrm{d}s \leqq$$

$$\leqq \int_t^\infty L_B(s) \sum_{i=1}^k |x[\sigma_i(s)] - y[\sigma_i(s)]|\,\mathrm{d}s \leqq kp(T) \int_t^\infty L_B(s)\,\mathrm{d}s \leqq \frac{p(T)}{2},$$

where $p(T) = \sup\{|x(s) - y(s)| : s \in [T, \infty)\}$. Therefore $p(T) \leqq 1/2\, p(T)$ and so $p(T) = 0$, which proves the Proposition.

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