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NOTE ON A CONJECTURE OF P. D. T. A. ELLIOTT

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Abstract. In this paper we prove some partial results on a conjecture of P.D.T.A. Elliott' and on a restricted form of it. Elliott's conjecture is related with shifted-prime factorizations and if established it would provide the best possible answer to a recently question of Rosen [11].

Key words. Elliott's conjecture, prime numbers, shifted-prime factorizations.

1. INTRODUCTION

A completely additive arithmetic function f is a mapping of the natural numbers to the complex numbers such that $f(ab) = f(a) + f(b)$ for all natural numbers a and b. In the present paper a natural number is a positive integer.

I. Kátai [7] conjectured and P. D. T. A. Elliott [2] proved that, if f is a completely additive arithmetic function such that $f(p + 1) = 0$ for all primes p, then f must be identically zero.

An immediate consequence of this fact, established by D. Wolke [12] is the surprising result that every positive integer may be expressed in the form

$$n = \prod_{i=1}^{m} (p_i + 1)^{r_i},$$

where the p_i are primes and the exponents r_i are rationals.

We note that such expansions for an integer are not unique; for example we have $2 = (3 + 1)^{1/2} = (31 + 1)^{1/5}$.

K. H. Rosen [11] called these representations *shifted-prime factorizations* and stated some questions and conjectures related to shifted-prime factorizations. One of these questions is the following ([11, Question 2]):

Rosen's question. Give an upper bound for the minimum number of primes required to represent all positive integers not exceeding n by shifted-prime factorizations involving only these primes.

If established, the following conjecture of P. D. T. A. Elliott (see [11]) would provide the best possible answer to Rosen's question:

Elliott's conjecture. Every nonzero rational number can be expressed in the form $(p + 1)/(q + 1)$ where p and q are primes (a negative prime being allowed for negative rationals).

In fact it is clear that for Rosen's question only a restricted from of Elliott's conjecture has importance:

A restricted form of Elliott's conjecture. Every positive integer can be expressed in the form $(p + 1)/(q + 1)$, where p and q are primes.

Another restricted form of Elliott's conjecture follows from the (unproved) hypothesis H of A. Schinzel (see [8] and for progress towards this hypothesis see [4] and [5]).

The aim of this paper is to prove some partial results on Elliott's conjecture and on the above restricted form of Elliott's conjecture.

2. A PARTIAL RESULT ON ELLIOTT'S CONJECTURE

Firstly we shall prove the following

Lemma. *If a, b, c are natural numbers such that $(a, b) = (b, c) = (c, a) = 1$ and $2 \mid abc$ then there is a constant $K = K(a, b, c)$ and an infinitely many natural numbers p and q with at most K prime factors such that $ap - bq = c$.*

The above Lemma is a partial result on a conjecture of G. H. Hardy and J. E. Littlewood [6, Conjecture D, p. 45]: "If a, b, c are natural numbers with $(a, b) = (b, c) = (c, a) = 1$ and $2 \mid abc$ then there are infinitely many primes p and q such that $ap - bq = c$.

For $a = b = 1$, $c = 2$ a partial result on Hardy-Littlewood's conjecture (which become, in this case, the long standing conjecture of existence of infinitely many twin primes) was proved by A. Rényi (see [1]) and for $a = 1$ another partial result was proved by H. E. Richert (see [3] and [5]).

Note that these results are better than our Lemma in these particular cases but, as far as we know, our Lemma is the best partial result until now on the whole conjecture of Hardy and Littlewood.

Proof of Lemma. Because $(a, b) = (b, c) = (c, a) = 1$ and $2 \mid abc$, there are two natural numbers r and s such that $ar - bs = c$.

Let us consider the polynomials $f_1(x) = bx + r$, $f_2(x) = ax + s$. Then $f_1(x) f_2(x) = abx^2 + (ar + bs)\, x + rs$.

We claim that the polynomials f_1 and f_2 verifies the conditions of Theorem 1 from the classic paper of G. Ricci [10]. We shall prove this claim by *reductio ad absurdum.*

Let us suppose that there is a prime p such that p is a divisor of $f_1(x) f_2(x)$ for every positive and negative integer x. Then for $x = 0$ we obtain $p \mid rs$ and for

$x = \pm 1$ we obtain $p \mid [ab \pm (ar + bs)]$. Thus we have $p \mid 2ab$ and $p \mid [2(ar + bs)]$.

We have two situations:

i) $p = 2$

ii) $p > 2$.

Firstly we analyse the case i).

From $2 \mid rs$ we get that at least one of the natural numbers r and s is even. But r and s cannot be simultaneously even. Indeed, if contrary, r and s are even then $2 \mid (ar \pm bs)$, so $2 \mid ab$ and $2 \mid c$; contradiction with $(a, b) = (b, c) = (c, a) = 1$.

Thus, without loss of generality, we shall assume that r is even and s is odd. From $p = 2$ and $p \mid [ab + (ar + bs)] = [b(a + 1) + ar + b(s - 1)]$ it follows that $2 \mid b(a + 1)$. Hence a is odd or b is even. If b is even then, keeping in mind that $ar - bs = c$, we find that c is also even which contradicts the relation $(b, c) = 1$.

Thus b is odd and a is odd, too. In this case also $c = ar - bs$ is an odd number, contradiction with $2 \mid abc$.

Finally, we investigate the case ii), i.e. the case when $p > 2$.

From $p \mid 2ab$ and $p \mid 2(ar + bs)$ we conclude that $p \mid ab$ and $p \mid (ar + bs)$. But $p \mid rs$ and $p \mid (ar^2 + brs)$, so we have $p \mid ar^2$. Consequently we find that $p \mid ar$. In a similar way we can obtain that $p \mid bs$.

From these two results we get $p \mid (ar - bs) = c$ which contradicts $(ab, c) = 1$.

Thus we can apply Ricci's theorem to obtain that there is a constant $K = K(a, b)$ and an infinitely many natural numbers p and q with at most K prime factors such that $p = f_1(x)$ and $q = f_2(x)$. From $p = bx + r$ and $q = ax + s$ we deduce that $ap - bq = ar - bs = c$ and the proof of Lemma is finished.

Now we are ready to prove the following partial result on Elliott's conjecture:

Theorem 1. *For every nonzero rational number r there is a constant $K = K(r)$ and there are an infinitely many natural numbers p and q with at most K prime factors such that $r = (p + 1)/(q + 1)$.*

Before we start the proof of Theorem 1 we note, as in Elliott's conjecture, that a negative prime is allowed for negative rationals.

Proof of Theorem 1. In view of the above remark it is sufficient to prove Theorem 1 only for positive rationals.

Let $r > 1$ be a fixed rational number. Then there are two relatively prime numbers $a < b$ such that $r = b/a$. From $b > a$ and $(a, b) = 1$ we find that $(a, b - a) = 1$. Because $2 \mid ab(b - a)$ we can apply the above Lemma. Thus, there is a constant $K = K(a, b) = K(r)$ and an infinitely many natural numbers p and q with at most K prime factors such that $ap - bq = b - a$, so $(p + 1)/(q + 1) = b/a = r$.

Because $1 = (p + 1)/(p + 1)$ for every integer $p \neq -1$ it remains to prove the assertion of Theorem 1 for $o < r < 1$.

In this last case we have $r = a/b$ with $a < b$. Then b/a is rational and $b/a > 1$. Hence b/a can be expressed in the form $b/a = (p + 1)/(q + 1)$, so $r = a/b = (q + 1)/(p + 1)$.

Now the proof is complete.

From the proof of Theorem 1 we see that Hardy-Littlewood's conjecture implies Elliott's conjecture.

In the end of this paragraph we make the remark that Theorem 1 has also variants for the following representations:

$r = (p - 1)/(q - 1)$, using the equation $ap - bq = a - b$ $(a > b)$ for a proof;
$r = (p + 1)/(q - 1)$, using the equation $bq - ap = a + b$ $(b > a)$ for a proof;
$r = (p - 1)/(q + 1)$, using the equation $ap - bq = a + b$ $(b > a)$ for a proof.

3. A PARTIAL RESULT ON THE RESTRICTED FORM OF ELLIOTT'S CONJECTURE

As we mentioned in the first section of the present paper, for Rosen's question only a restricted form of Elliott's conjecture has importance.

In this paragraph we shall prove a partial result on this restricted form of Elliott's conjecture. Namely we shall prove the following

Theorem 2. *Let $n > 1$ be a natural number.*

i) *There are infinitely many pairs (p, q) with p a number of at most three prime factors and q a prime number such that $n = (p + 1)/(q + 1)$.*

ii) *If n is sufficiently large then there is a prime q and a number $p \leqq n^{357/200}$ with at most three prime factors such that $n = (p + 1)/(q + 1)$.*

Because we have $1 = (q + 1)/(q + 1)$ for every prime q the condition $n > 1$ from the above Theorem 2 is natural.

Proof of Theorem 2.

i) It follows from a theorem of H. E. Richert (see [3] and [5]) that there are infinitely many primes q such that $nq + n - 1$ have at most three prime factors. We denote by p_q the number $nq + n - 1$ which have at most three prime factors, for every prime q, via Richert's theorem.

From $p_q = nq + n - 1$ we deduce that $p_q + 1 = n(q + 1)$. Hence the relation $n = (p_q + 1)/(q + 1)$ holds and the first part of Theorem 2 is proved.

ii) According to a recent deep result of W. Fluch [3, Korollar] if n is sufficiently large, there is a prime number q and a number $p_q \leqq n^{357/200}$ of the form $p_q = nq + n - 1$ with at most three prime factors.

Keeping in mind the last equality from the proof of the first part of Theorem 2 we have the desired result.

Now the proof of Theorem 2 is complete.

The following proposition is another partial result of the whole Elliott's conjecture.

Corollary. *Let $r > 0$ be a rational number. There exist a natural number $k = k(r)$ and an infinitely many $2k$-tuples of integers $(p_1, \ldots, p_k, q_1, \ldots, q_k)$ such that*

$$r = \sum_{i=1}^{k} \frac{p_i + 1}{q_i + 1},$$

where p_i are primes and q_i have at most three prime factors, $i = 1, \ldots, k$.

Proof. From Theorem 2, i) we find that for every natural number $n > 1$, the rational number $1/n$ can be expressed at $1/n = (p + 1)/(q + 1)$, where p is a prime number and q has at most three prime factors. Because $r = \sum_{i=1}^{k} n_i^{-1}$ for some n_i and k (see e.g. [9]) we have the desired result.

REFERENCES

[1] E. Bombieri, *Le grand crible dans la théorie analitique des nombres*, Paris, 1974.

[2] P. D. T. A. Elliott, *A conjecture of Kátai*, Acta Arith. 26 (1974), 11–20.

[3] W. Fluch, *Über ein Theorem der linearen Progressionen*, Nederl. Akad. Wetensch. Proc. Ser. A 84 = Indag. Math. 43, (1981), 173–180.

[4] H. Halberstam, *Progress towards hypothesis H*, Journées Arithmetique Francaises (Univ, Provence, Marseille, 1971), Exp. No. 3, 8 pp. Univ. Provence, Marseille, 1971.

[5] H. Halberstam and H. E. Richert, *Sieve Methods*, London, 1974.

[6] G. H. Hardy and J. E. Littlewood, *Some problems of partitio numerorum III*, Acta Math. 44 (1923), 1–70.

[7] I. Kátai, *On sets characterizing number-theoretical functions*, Acta Arith. 13 (1968), 315–320.

[8] A. Schinzel et W. Sierpinski, *Sur certaines hypothèses concernant les nombres premiers*, Acta Arith. 4 (1958), 185–208.

[9] W. Sierpinski, *Sur les décompositions de nombres rationnels en fractions primaires*, Mathesis, 65 (1956), 16–32.

[10] G. Ricci, *Sur la congettura di Goldbach e la constante di Schnirelman*, Ann. Scuola Norm. Sup. Pisa, 6 (1937), 71–116.

[11] K. H. Rosen, *Shifted-prime factorizations*, Amer. Math. Monthly, 88 (1981), 276–278.

[12] D. Wolke, *Bemerkungen über Eindeutigkseitsmengen additiver functionen*, Elem. Math. 33 (1978), 14–16.

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