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A REMARK TO A MULTIPOINT BOUNDARY VALUE PROBLEM

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Abstract. For each $p = 0, 1, \ldots, n-1$, some properties of the p-th derivative $\frac{\partial^p G}{\partial t^p}$ of the Green function G for a scalar multipoint boundary value problem are established. By means of them the Φ_p-positivity of a positive linear operator A_p is proved.

Key words. Green function, multipoint boundary value problem, positive linear operator.

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In the paper some properties of the Green function $G = G(t, s)$ for the scalar De la Vallée Poussin problem are investigated which involve that a positive linear operator $A_p x(t) = \int_a^b \left| \frac{\partial^p G(t,s)}{\partial t^p} \right| x(s)\, ds$ is Φ_p-positive where $\Phi_p(t) = A_p(1)(t)$, $p = 0, 1, \ldots, n-1$. This fact has important consequences for the existence theory on nonlinear vector multipoint boundary value problem as it is shown in [4].

Consider the Green function $G = G(t, s)$, $a \leqq t, s \leqq b$, for the scalar multipoint boundary value problem

$$x^{(n)} = 0, \tag{1}$$

$$x^{(i-1)}(t_j) = 0, \qquad i = 1, \ldots, r_j,\ j = 1, 2, \ldots, m, \tag{2}$$

where $n \geqq 2$, $2 \leqq m \leqq n$, $1 \leqq r_j$ are natural numbers such that $r_1 + r_2 + \ldots + r_m = n$, and $-\infty < a = t_1 < \ldots < t_m = b < \infty$ are real numbers.

Since the Cauchy function $u = u(t, s)$ for (1) has the form $u(t,s) = \frac{(t-s)^{n-1}}{(n-1)!}$ and hence $\frac{\partial^{j-1} u(t,s)}{\partial t^{j-1}} = \frac{(t-s)^{n-j}}{(n-j)!}$, $a \leqq t, s \leqq b$, $j = 1, 2, \ldots, n$, by [1], p. 137, (compare with [3], p. 376)

$$(3)\qquad G(t,s)=\begin{cases}\sum_{l=1}^{k}\sum_{j=1}^{r_l} v_{jl}(t)\dfrac{(t_l-s)^{n-j}}{(n-j)!}, & t\geqq s,\ t_k\leqq s\leqq t_{k+1},\\ -\sum_{l=k+1}^{m}\sum_{j=1}^{r_l} v_{jl}(t)\dfrac{(t_l-s)^{n-j}}{(n-j)!}, & t\leqq s\,(k=1,\dots,m-1),\end{cases}$$

where v_{jl}, $l = 1, 2, \dots, m$, $j = 1, \dots, r_1$, is the solution of (1) satisfying the conditions

$v_{jl}^{(i-1)}(t_k) = 0$, for each $k \neq 1$, $i = 1, \dots, r_k$, $k = 1, 2, \dots, m$,

$v_{jl}^{(i-1)}(t_l) = \delta_{ji}$, $i = 1, \dots, r_l$. δ_{ji} is the delta Kronecker symbol.

Further we shall consider the Banach space $E = C([a, b], R)$ with the sup-norm $\|\,.\,\|$, partially ordered by the relation $x \leqq y$ iff $x(t) \leqq y(t)$ for all $t \in [a, b]$. Then $(E, \leqq)$ is an ordered Banach space with positive cone $P = \{x \in E : x(t) \geqq 0, a \leqq t \leqq b\}$. P is normal and generating (compare with [4]).

For each $p \in \{0, 1, \dots, n - 1\}$ let us define the operator $A_p : E \to E$ by

$$(4)\qquad A_p x(t) = \int_a^b \left|\frac{\partial^p G(t, s)}{\partial t^p}\right| x(s)\,\mathrm{d}s, \qquad a \leqq t \leqq b, x \in E.$$

A_p is a positive, completely continuous linear operator. Then the function

$$(5)\qquad \Phi_p(t) = \int_a^b \left|\frac{\partial^p G(t, s)}{\partial t^p}\right| \mathrm{d}s, \qquad a \leqq t \leqq b,$$

belongs to P and the operator A_p is Φ_p-bounded from above, since on basis of (4) for each $x \in P$

$$(6)\qquad A_p x(t) \leqq \|x\|\, \Phi_p(t), \qquad a \leqq t \leqq b.$$

The most important properties of the Green function G have been summoned up in Lemma 1, [3], p. 375. For convenience we mention here the following ones:

1. $\dfrac{\partial^{i-1}G}{\partial t^{i-1}}$, $i = 1, \dots, n - 1$, is continuous in $[a, b] \times [a, b]$.

2. $\dfrac{\partial^{n-1}G}{\partial t^{n-1}}$ is continuous in the triangles $a \leqq t \leqq s \leqq b$, $a \leqq s \leqq t \leqq b$. Further,

$$\lim_{t\to s+} \frac{\partial^{n-1}G(t, s)}{\partial t^{n-1}} - \lim_{t\to s-} \frac{\partial^{n-1}G(t, s)}{\partial t^{n-1}} = 1$$

for each s, $a < s < b$.

3. For each s, $a < s < b$, the function $G(., s)$ satisfies (1) in $[a, s) \cup (s, b]$ and the boundary conditions (2).

4. The sign of G is determined by the inequality

$$G(t, s)\,(t - t_1)^{r_1}(t - t_2)^{r_2} \dots (t - t_m)^{r_m} \geqq 0, \qquad a \leqq t, s \leqq b$$

and

$$G(t, s) \neq 0 \qquad \text{for } t_k < t < t_{k+1}, \qquad a < s < b,\ k = 1, \dots, m - 1.$$

The meaning of the Green function is given by the statement

5. If $f \in E$, then the function

$$y(t) = \int_a^b G(t, s) f(s) \, ds, \qquad a \leqq t \leqq b,$$

is the unique solution of the problem (2), $y^{(n)} = f(t)$.

Further properties of the function G are given in the following lemma.

Lemma 1. *Let $t_0 \in [a, b]$, $s_0 \in [a, b]$, $p \in \{0, 1, \dots, n - 1\}$. Then the following statements are true.*

1.

$$\frac{\partial^p G(t_0, s)}{\partial t^p} = 0 \qquad \text{for all } s \in [a, b], \tag{7}$$

iff there is an $l \in \{1, 2, \dots, m\}$ such that

$$t_0 = t_l \qquad \text{and} \qquad 0 \leqq p \leqq r_l - 1. \tag{8}$$

2. *If*

$$\frac{\partial^p G(t_0, s)}{\partial t^p} = 0 \qquad \text{for all } s \text{ in a subinterval } [a_1, b_1] \subset [a, b], \tag{9}$$

then for each k, $k = 1, 2, \dots, m - 1$, such that $(t_k, t_{k+1}) \cap (a_1, b_1) \neq \emptyset$ and $t_0 \notin (t_k, t_{k+1})$

$$\frac{\partial^p G(t_0, s)}{\partial t^p} = 0 \qquad \text{for all } s \in [t_k, t_{k+1}].$$

If $t_0 \in (t_k, t_{k+1})$, $(a_1, b_1) \cap (t_k, t_0) \neq \emptyset$ $((a_1, b_1) \cap (t_0, t_{k+1}) \neq \emptyset)$, then

$$\frac{\partial^p G(t_0, s)}{\partial t^p} = 0 \qquad \text{for all } s,\ t_k \leqq s \leqq t_0,$$

$$\left(\frac{\partial^p G(t_0, s)}{\partial t^p} = 0 \text{ for all } s,\ t_0 \leqq s \leqq t_{k+1}\right).$$

3. *If* $\dfrac{\partial^p G(t, s_0)}{\partial t^p} = 0$ *for all t from a subinterval $[a_2, b_2] \subset [a, b]$, then in case $(a_2, b_2) \cap (a, s_0) \neq \emptyset$ (in case $(a_2, b_2) \cap (s_0, b) \neq \emptyset$)*

$$\frac{\partial^p G(t, s_0)}{\partial t^p} = 0 \qquad \text{for all } t,\ a \leqq t \leqq s_0,$$

$$\left(\frac{\partial^p G(t, s_0)}{\partial t^p} = 0 \text{ for all } t,\ s_0 \leqq t \leqq b\right).$$

4. *If* $m = 2$ *and* $t_0 \in (a, b)$, *then there is no subinterval* $[a_1, b_1] \subset [a, b]$ *such that* (9) *is true.*

Proof. 1. If (8) is true, then (7) holds by the properties 3 and 1 of the Green function. Conversely, if (7) is valid, then for all functions $y \in C^n([a, b], R)$ satisfying (2) we must have $y^{(p)}(t_0) = 0$ which is only so possible when (8) is true. To show the last implication we construct in case that (8) is not true a function $y \in C^n([a, b], R)$ satisfying (2) for which $y^{(p)}(t_0) \neq 0$. To that aim consider the function

$$g_1(t) = \frac{1}{n!}(t - t_1)^{r_1}(t - t_2)^{r_2} \dots (t - t_m)^{r_m}, \qquad a \leqq t \leqq b.$$

Since $r_1 + r_2 + \dots + r_m = n$, this function satisfies the boundary value problem $y^{(n)} = 1$, (2). For each l, $l = 1, 2, \dots, m$,

$$g_1^{(r_l)}(t_l) = \sum_{s=0}^{r_l} \binom{r_l}{s} [(t - t_l)^{r_l}]^{(r_l - s)}(t_l) [(t - t_1)^{r_1} \dots (t - t_{l-1})^{r_{l-1}} \times$$
$$\times (t - t_{l+1})^{r_{l+1}} \dots (t - t_m)^{r_m}]^{(s)}(t_l) =$$
$$(10) \qquad = r_l!(t_l - t_1)^{r_1} \dots (t_l - t_{l-1})^{r_{l-1}}(t_l - t_{l+1})^{r_{l+1}} \dots (t_l - t_m)^{r_m} \neq 0.$$

Two cases may happen. Either $g_1^{(p)}(t_0) \neq 0$ and then the proof is done or $g_1^{(p)}(t_0) = 0$. In this case we multiply g_1 by the function

$$g_2(t) = g_2(t_0) + \frac{g_2'(t_0)}{1!}(t - t_0) + \dots + \frac{g_2^{(p)}(t_0)}{p!}(t - t_0)^p, \qquad a \leqq t \leqq b,$$

where $g_2(t_0), g_2'(t_0), \dots, g_2^{(p)}(t_0)$ will be suitably chosen. We have

$$(g_1 \cdot g_2)^{(p)}(t_0) = \sum_{s=0}^{p} \binom{p}{s} g_1^{(p-s)}(t_0)\, g_2^{(s)}(t_0).$$

If all values $g_1(t_0) = g_1'(t_0) = \dots = g_1^{(p)}(t_0) = 0$, then $t_0 = t_l$ for an l, $l \in \{1, 2, \dots, m\}$ and, in view of (10), $0 \leqq p \leqq r_l - 1$, which is (8) and thus, it cannot happen. Similarly we come to contradiction with our assumption, when $p = 0$. Hence we can suppose that $p \geqq 1$ and that in the sequence $g_1(t_0)$, $g_1'(t_0), \dots, g_1^{(p)}(t_0)$ at least one term is different from zero, say $g_1^{(p-k)}(t_0) \neq 0$. Then we choose $g_2(t) = \dfrac{g_2^{(k)}(t_0)}{k!}(t - t_0)^k$, with $g_2^{(k)}(t_0) \neq 0$ and hence $(g_1 \cdot g_2)^{(p)}(t_0) = \binom{p}{k} g_1^{(p-k)}(t_0)\, g_2^{(k)}(t_0) \neq 0$. We have found a polynomial y of degree $n + p$ at most satisfying (2) and such that $y^{(p)}(t_0) \neq 0$. This completes the proof that (7) and (8) are equivalent each to other.

2. and 3. By (3), $\dfrac{\partial^p G(t_0, s)}{\partial t^p}$ is an analytic function of the variable s in any interval

$[t_k, t_{k+1}]$, $k = 1, \ldots, m-1$, such that $t_0 \notin (t_k, t_{k+1})$ as well as in the intervals $[t_k, t_0]$ and $[t_0, t_{k+1}]$, when $t_0 \in (t_k, t_{k+1})$.

Similarly (3) implies that $\dfrac{\partial^p G(t, s_0)}{\partial t^p}$ is analytic in $[a, s_0]$ when $a < s_0 \leqq b$ as well as in $[s_0, b]$ if $a \leqq s_0 < b$. By the uniqueness theorem for analytic functions, the statements 2 and 3 follow.

4. If $m = 2$, then there is a $k \in \{1, \ldots, n-1\}$ such that the conditions (2) have the form

$$(2') \qquad x^{(i-1)}(a) = 0, \qquad i = 1, \ldots, k, \; x^{(i-1)}(b) = 0, \; i = 1, \ldots, n-k.$$

Suppose that (9) is true. By the statement 2 of this lemma either

$$(11) \qquad \frac{\partial^p G(t_0, s)}{\partial t^p} = 0 \qquad \text{for all } s, \; a \leqq s \leqq t_0$$

or $\dfrac{\partial^p G(t_0, s)}{\partial t^p} = 0$ for all s, $t_0 \leqq s \leqq b$. Consider only the former case. The latter would be proceeded in a similar way. By (3), (11) means the equality $\sum_{j=1}^{k} v_{j1}^{(p)}(t_0) \times \times \dfrac{(a-s)^{n-j}}{(n-j)!} \equiv 0$ in $[a, t_0]$, and hence, (11) is equivalent to

$$(12) \qquad v_{j1}^{(p)}(t_0) = 0, \qquad j = 1, \ldots, k.$$

Consider the functions

$$(13) \qquad u_j(t) = (t-a)^{j-1}(t-b)^{n-k}, \qquad a \leqq t \leqq b, \; j = 1, \ldots, k.$$

By the definition, v_{j1} are nontrivial solutions of (1) satisfying the conditions $v_{j1}^{(i-1)}(b) = 0$, $i = 1, \ldots, n-k$, $v_{j1}^{(i-1)}(a) = \delta_{ij}$, $i = 1, \ldots, k$. This implies that there exist constants $c_{k,0}, c_{k-1,1}, c_{k-1,0}, \ldots, c_{1,k-1}, c_{1,k-2}, \ldots, c_{1,0}$ such that

$$(14) \qquad \begin{aligned} v_{k1}(t) &= c_{k,0} u_k(t), \qquad c_{k,0} \neq 0 \\ v_{k-1,1}(t) &= u_{k-1}(t)\left[c_{k-1,1}(t-a) + c_{k-1,0}\right] = \\ &= c_{k-1,1} u_k(t) + c_{k-1,0} u_{k-1}(t), \qquad c_{k-1,0} \neq 0 \\ v_{k-2,1}(t) &= c_{k-2,2} u_k(t) + c_{k-2,1} u_{k-1}(t) + c_{k-2,0} u_{k-2}(t), \; c_{k-2,0} \neq 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_{1,1}(t) &= c_{1,k-1} u_k(t) + c_{1,k-2} u_{k-1}(t) + c_{1,k-3} u_{k-2}(t) + \ldots + c_{1,0} u_1(t), \\ & c_{1,0} \neq 0, \; a \leqq t \leqq b. \end{aligned}$$

By (14), (12) are equivalent to the relations

$$(15) \qquad u_j^{(p)}(t_0) = 0, \qquad j = 1, \ldots, k.$$

(13) implies that $u_k(t_0) \neq 0$, $u_k^{(n-1)}(t_0) \neq 0$ and hence $p \neq 0$, $p \neq n-1$. Similarly

the degree of $u_{k-1}(t), \dots, u_1(t)$, respectively, being $n-2, \dots, n-k$, respectively, p must be different from $n-2, \dots, n-k$. Since $u_1(t) = (t-b)^{n-k}$, $p \neq n - k - 1, \dots, p \neq 1$. We have found that there is no $t_0 \in (a, b)$ for which (15) are true and the statement is completely proved.

Theorem 1. *Let $p \in \{0, 1, \dots, n-1\}$ and let $x \in P$, $x \neq 0$ in $[a, b]$. Then*
1. *$A_p x(t) > 0$ in (a, b) if $m = 2$;*
2. *$A_p x(t) = 0$ cannot hold in any subinterval $[a_1, b_1] \subset [a, b]$ for $m > 2$.*

Proof. The first statement follows from the statement 4 of Lemma 1 and nonegativity of the functions $x(t)$, $\left|\dfrac{\partial^p G(t, s)}{\partial t^p}\right|$.

To prove the second statement let us suppose that there is an interval $[a_2, b_2] \subset \subset [a, b]$ in which $A_p x(t) = 0$. As the functions $x(s)$, $\left|\dfrac{\partial^p G(t, s)}{\partial t^p}\right|$ are nonnegative, it follows that for each $t_0 \in [a_2, b_2]$ supp $x(s) \subset S_{t_0} = \left\{s \in [a, b] : \dfrac{\partial^p G(t_0, s)}{\partial t^p} = 0\right\}$. Hence there is an interval $[a_1, b_1]$ such that $\dfrac{\partial^p G(t_0, s)}{\partial t^p} = 0$ for each $t_0 \in [a_2, b_2]$, $s \in [a_1, b_1]$. On basis of the statements 2 and 3 of Lemma 1 there exists an interval $[t_k, t_{k+1}]$, $k \in \{1, \dots, n-1\}$, such that either

$$\frac{\partial^p G(t, s)}{\partial t^p} \equiv 0 \qquad \text{for } t_k \leqq a \leqq t_{k+1}, a \leqq s \leqq t \leqq b, \tag{16}$$

or $\dfrac{\partial^p G(t, s)}{\partial t^p} \equiv 0$ for $t_k \leqq s \leqq t_{k+1}$, $a \leqq t \leqq s \leqq b$. Consider only the case (16). The other one can be investigated in a similar way.

Let us choose a sufficiently small $\varepsilon > 0$. Two cases may happen. Either 1. $k > 1$ or 2. $k = 1$. We shall consider only the first case, the second one would be proceeded in a similar way (instead of $[t_k, t_k + \varepsilon]$ we would consider $[t_{k+1} - \varepsilon, t_{k+1}]$). (16) implies that for each function $y \in C^n([a, b], R)$ satisfying the boundary conditions (2) and such that supp $y^{(n)} \subset [t_k, t_k + \varepsilon]$

$$|y^{(p)}(t)| \leqq \int_a^b \left|\frac{\partial^p G(t, s)}{\partial t^p}\right| |y^{(n)}(s)| \, ds = \int_{t_k}^{t_k+\varepsilon} \left|\frac{\partial^p G(t, s)}{\partial t^p}\right| |y^{(n)}(s)| \, ds = 0,$$

for all t, $t_k + \varepsilon \leqq t \leqq b$, and hence,

$$y^{(p)}(t) \equiv 0 \qquad \text{for } t_k + \varepsilon \leqq t \leqq b. \tag{17}$$

Now we shall construct a function $y \in C^n([a, b], R)$ with supp $y^{(n)} \subset [t_k, t_k + \varepsilon]$, satisfying (2) and for which (17) is not true. Let

$$(18)\qquad y(t) = \begin{cases} (t-t_1)^{r_1}\dots(t-t_k)^{r_k}, & t_1 \leqq t \leqq t_k, \\ y(t_k) + \dfrac{y'(t_k)}{1!}(t-t_k) + \dots + \dfrac{y^{(n-1)}(t_k)}{(n-1)!}(t-t_k)^{n-1} + \\ + \displaystyle\int_{t_k}^{t} \frac{(t-s)^{n-1}}{(n-1)!} h(s)\,\mathrm{d}s, & t_k \leqq t \leqq t_k + \varepsilon, \\ (t-t_{k+1})^{r_{k+1}}\dots(t-t_m)^{r_m} P_1(t), & t_k + \varepsilon \leqq t \leqq t_m, \end{cases}$$

where the polynomial $P_1(t)$ is such that the degree of the polynomial $(t-t_{k+1})^{r_{k+1}}\dots(t-t_m)^{r_m}. P_1(t)$ is $n-1$, otherwise $P_1(t)$ can be arbitrary and $h(t) = y^{(n)}(t)$, $t_k \leqq t \leqq t_k + \varepsilon$, is continuous in $[t_k, t_k + \varepsilon]$ and $h(t_k) = h(t_k + \varepsilon) = 0$. The function $h(t)$ will be determined in such a way that $y \in C^{n-1}([a, b], R)$. Then

$$y^{(n)}(t) = \begin{cases} 0, & t_1 \leqq t \leqq t_k \\ h(t), & t_k \leqq t \leqq t_k + \varepsilon \\ 0, & t_k + \varepsilon \leqq t \leqq t_m \end{cases}$$

and hence, $y \in C^n([a, b], R)$, $\operatorname{supp} y^{(n)} \subset [t_k, t_k + \varepsilon]$ and y satisfies the boundary conditions (2). Since $y(t)$ is a polynomial of degree $n-1$ in $[t_k + \varepsilon, t_m]$, it cannot satisfy (17) and this contradiction shows that the second statement of Theorem 1 is true.

In view of the definition (18) it suffices to check the continuity of $y(t)$ and that of its derivatives up to order $n-1$ at the points t_k, $t_k + \varepsilon$. By the uniqueness theorem for initial value problem, $y^{(j)}(t)$, $j = 0, 1, \dots, n-1$, exist and are continuous at the point t_k. The same will happen at $t_k + \varepsilon$, if h satisfies

$$\int_{t_k}^{t_k+\varepsilon} \frac{(t_k + \varepsilon - s)^{n-1-j}}{(n-1-j)!} h(s)\,\mathrm{d}s = y^{(j)}(t_k + \varepsilon) - \big[y^{(j)}(t_k) +$$

$$(19)\qquad + \frac{y^{(j+1)}(t_k)}{1!}\varepsilon + \dots + \frac{y^{(n-1)}(t_k)}{(n-1-j)!}\varepsilon^{n-1-j}\big], \qquad j = 0, 1, \dots, n-1.$$

Denote the right-hand side of (19) as a_j, $j = 0, 1, \dots, n-1$. If we put

$$(20)\qquad h(t) = (t_k + \varepsilon - t)(t - t_k)\, g(t), \qquad t_k \leqq t \leqq t_k + \varepsilon,$$

then the condition (19) has the form

$$(21)\qquad \int_{t_k}^{t_k+\varepsilon} (t_k + \varepsilon - s)^{n-j} (s - t_k)\, g(s)\,\mathrm{d}s = a_j (n-1-j)!, \qquad j = 0, 1, \dots, n-1.$$

Since the functions

$$(22)\qquad y_j(t) = (t_k + \varepsilon - t)^{n-j} (t - t_k), \qquad t_k \leqq t \leqq t_k + \varepsilon, j = 0, 1, \dots, n-1,$$

are linearly independent in $[t_k, t_k + \varepsilon]$, by *E.* Schmidt's orthogonalization process an orthonormal sequence $\{x_j(t)\}_{j=0}^{n-1}$ in the real Hilbert space $L^2([t_k, t_k + \varepsilon])$ can be constructed such that

$$(23)\qquad x_j(t) = \sum_{l=0}^{j} d_{j,l} y_l(t), \qquad t_k \leqq t \leqq t_k + \varepsilon, j = 0, 1, \dots, n-1,$$

with uniquely determined constants $d_{j,l}$, j, $l = 0, 1, \dots, n-1$ and $d_{j,j} \neq 0$, $j = 0, 1, \dots, n-1$. If $(.,.)$ is the scalar product in $L^2([t_k, t_k + \varepsilon])$, then the conditions (21) mean the relations

$$(21')\qquad (y_j, g) = a_j(n-1-j)!, \qquad j = 0, 1, \dots, n-1,$$

which on basis of (22), (23) are equivalent to

$$(24)\qquad (x_j, g) = \sum_{l=0}^{j} d_{j,l} a_l (n-1-l)! = b_j, \qquad j = 0, 1, \dots, n-1.$$

The function

$$g(t) = \sum_{j=0}^{n-1} (x_j, g)\, x_j(t) = \sum_{j=0}^{n-1} b_j x_j(t) = \sum_{j=0}^{n-1} \sum_{l=0}^{j} b_j d_{j,l} y_l(t) =$$

$$(25)\qquad = \sum_{l=0}^{n-1} \Big(\sum_{j=l}^{n-1} b_j d_{j,l} \Big) y_l(t), \qquad t_k \leqq t \leqq t_k + \varepsilon,$$

satisfies (24) as well as (21). Thus, in view of (20), (25), there is a polynomial $h(t)$ such that $h(t_k) = h(t_k + \varepsilon) = 0$ which satisfies (19) and the proof is complete.

Corollary. *If $p \in \{0, 1, \dots, n-1\}$ and the function Φ_p is given by (5), then the operator A_p is Φ_p-positive.*

Proof. By (6), A_p is Φ_p-bounded from above. According to [2], p. 59, it will be Φ_p-bounded from below if to any function $x \in P$, $x \not\equiv 0$, there exists a constant $\alpha = \alpha(x) > 0$ such that

$$(26)\qquad A_p^2 x(t) \geqq \alpha(x)\, \Phi_p(t), \qquad a \leqq t \leqq b,$$

where A_p^2 means the second iterate of A_p. By statement 2 of Theorem 1, $A_p x(t) \equiv 0$ cannot hold in any subinterval $[a_1, b_1] \subset [a, b]$ and hence, putting $A_p x(t) = y(t)$, $a \leqq t \leqq b$, we get that $y \in P$, $y(t) \not\equiv 0$ on any subinterval $[a_1, b_1] \subset [a, b]$ and (26) reduces to the inequality

$$(27)\qquad A_p y(t) \geqq \alpha(x)\, \Phi_p(t), \qquad a \leqq t \leqq b.$$

Thus the Φ_p-boundedness of A_p from below will be shown if to any function $y \in P$, $y(t) \not\equiv 0$ on any subinterval $[a_1, b_1] \subset [a, b]$ there exists a constant $\alpha > 0$ such that (27) is true. This has been proved in Lemma 8 in [4]. Then Theorem 2.2 in [2], p. 62, completes the proof of the corollary.

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