Archivum Mathematicum

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Archivum Mathematicum, Vol. 23 (1987), No. 3, 131--145

Persistent URL: http://dml.cz/dmlcz/107290

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 23, No. 3 (1987), 131–146

ANOTHER APPROACH TO THE CLASSICAL CALCULUS OF VARIATIONS III

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(Received September 9, 1985)

Abstract. The article concludes the previous investigations of the same author on formal analysis of the most fundamental concepts of the classical calculus of variations and consists of a brief overview of the general theory and several examples devoted to the interrelations between the Lagrange and the Hamiltonian approaches. Complete proofs and some other results will be published elsewhere.

Key words. Euler–Lagrange system, Hamilton system, Hamilton–Jacobi equation, regular and singular variational problem.

MS Classification. 49 B 99, 49 C 99

The primary version of the present paper was based only on the results of the preceeding parts [2] and dealt with applications of the general theory to some familiar variational problems. But the publication was delayed and in the meantime, it becames not actual. Moreover, the proceeding parts prove to be unsatisfactory from the contemporary point of view and the author decides to compile a more thorough and complete exposition for the interesting fibered case. Alas, the resulting paper [1] is excessively long for this Journal. A part of the achievements can be, however, easily understood and explained within the framework of [2] and appropriately concludes them. So we hope that a brief survey may be useful. In order to make the present paper as self-contained as possible, we recall the most important general principles of our method. We also call some attention to the terminology which being adapted to [1] slightly differs from the preceeding parts [2].

1. Introduction. Let M be a manifold, P a compact oriented n-dimensional manifold ($n \geqq 1$) with boundary Q, $\partial : Q \to P$ the natural inclusion of the boundary. Let φ (ψ) be an exterior differential n-form (($n - 1$)-form) on M, $\mathfrak{A}$ ($\mathfrak{B}$) be a given submodule of the $C^\infty(M)$-module of all n-forms (($n - 1$)-forms) on M.

We shall consider the space V of all embeddings $p : P \to M$ and the subspace $\boldsymbol{P}$ of V consisting of all such p that satisfy the requirements

$$(1) \qquad p^*\alpha \equiv 0 \quad (\alpha \in \mathfrak{A}), \qquad (p \circ \partial)^*\beta \equiv 0 \quad (\beta \in \mathfrak{B}).$$

We are interested in *critical points* of the functional

$$(2) \qquad F(p) = \int_P p^*\varphi + \int_Q (p \circ \partial)^*\psi$$

on the subset P of V. This is a very general setting of the familiar *Lagrange problem* denoted here by $\mathscr{LP}$, for brevity.

In order to simplify the exposition, we shall restrict only on the interior phenomenae completely omitting the behaviour of the mapping $p \in V$ at the boundary Q. Following this point of view, the data $Q, \partial, \psi, \mathfrak{B}$ are not longer needed and a mapping $p \in P$ is called an *extremal* (to the problem $\mathscr{LP}$) if $p^*\alpha \equiv 0$ $(\alpha \in \mathfrak{A})$ and the identity

$$(3) \qquad p^*Z \lrcorner \, \mathrm{d}(\varphi + \bar{\alpha}) \equiv 0$$

is satisfied for an appropriate $\bar{\alpha} \in \mathfrak{A}$ and all vector fields Z on M. (Note that the notation slightly differs from [2] where the difference $\varphi - \bar{\alpha}$ is used. The present choice is in better accordance with the classical formulae.) The set of all mentioned extremals will be denoted $\mathscr{EX}$. Note that the form $\varphi + \bar{\alpha}$ is a far going generalization of the famous *Cartan–Poincaré form* and (3) appears as a brief record of the *Euler–Lagrange system*.

We shall deal with the *fibered modification* of the above concepts which is as follows. The previous manifold M is replaced by the total space of a fibered manifold (E, π, B), $\pi : E \to B$ being the fibered space projection of the total space E onto the base B. The manifold P is retained but assuming $\dim P = n = \dim B$, we choose a fixed embedding $i_B^P : P \to B$ and instead of the previous space V, we shall deal with the space M of all mappings $p : P \to E$ that satisfy $\pi \circ p = i_B^P$. (In other words, we consider the space M of all cross-sections of the fibered manifold (E, π, B) over the subset $i_B^P(P)$ of the base B.) Quite analogously as before, φ is a given n-form on E and $\mathfrak{A}$ is a given $C^\infty(E)$-module of certain n-forms on E. We are interested in *extremals*, i.e., in mappings $p \in M$ satisfying $p^*\alpha \equiv 0$ $(\alpha \in \mathfrak{A})$ and (3). Note that it is quite sufficient to consider only the *π-vertical vector fields* Z in the condition (3), i.e., the vector fields Z on E satisfying $Z\pi^*g \equiv 0$ for all functions $g \in C^\infty(B)$. (See [2, part I, lemma 9].)

We shall deal only with the fibered modification of the problem $\mathscr{LP}$ from now on.

2. The Hamiltonian approach. In the particular case $\mathfrak{A} = \{0\}$, the trivial module, the extremals $p \in \mathscr{EX}$ are defined by the single identity

$$(4) \qquad p^*Z \lrcorner \, \mathrm{d}\varphi \equiv 0,$$

where Z runs over all π-vertical vectors on E. If the form $\mathrm{d}\varphi$ admits certain simple (we say *canonical*) expression in a special local coordinate system (in *canonical coordinates*) which can be achieved by performing an appropriate transformation (the *Legendre transformation*) of the original variables, then the condition (4) turns into the familiar *Hamilton system.* One can also say that every procedure which permits to reduce a general problem $\mathscr{L}\mathscr{P}$ with $\mathfrak{A} \neq \{0\}$ into an equivalent Lagrange problem with the relevant module $\mathfrak{A}$ trivial is the *Hamiltonian approach* to the variational problems.

A possible construction of the last kind may be outlined as follows. Starting with an arbitrary problem $\mathscr{L}\mathscr{P}$, we begin with introducing another Lagrange problem noted $\mathscr{L}\mathscr{P}^+$ and determined by certain new data (E^+, π^+, B^+), φ^+, $\mathfrak{A}^+$ where $B^+ = B$, $\mathfrak{A}^+ = \{0\}$ and the remaining objects E^+, π^+, φ^+ are specified below. Note besides that the original objects P and i_B^P are retained and the boundary data Q, ∂, ψ^+, $\mathfrak{B}^+$ are not important for us.

Going to the details, we assume (for simplicity) that every form $\alpha \in \mathfrak{A}$ can be uniquely represented by a sum

$$\alpha = a^1\alpha^1 + \ldots + a^c\alpha^c \qquad (a^1, \ldots, a^c \in C^\infty(E)), \tag{5}$$

where $\alpha^1, \ldots, \alpha^c \in \mathfrak{A}$ are certain fixed forms and $a^1, \ldots, a^c$ are arbitrary. Then we put $E^+ = \mathrm{R}^c \times E$ with coordinates in the first factor noted $p^1, \ldots, p^c$. At last we choose

$$\varphi^+ = (\pi_E^+)^*\varphi + p^1(\pi_E^+)^*\alpha^1 + \ldots + p^c(\pi_E^+)^*\alpha^c \tag{6}$$

for the sought form φ^+ on E^+; here appears the obvious projection $\pi_E^+ : E^+ \to E$ and we put $\pi^+ = \pi \circ \pi_E^+ : E^+ \to B$. One can verify the most important property of these objects; *every form of the type $\varphi + \alpha$ ($\alpha \in \mathfrak{A}$) possesses a unique representation by the pull-back of the type $\varphi + \alpha = \sigma^*\varphi^+$ with an appropriate cross-section* $\sigma : E \to E^+$ of the fibered manifold (E^+, π_E^+, E); compare with [2, part II, Section 4].

With these new data in mind, we are interested in critical points of the functional

$$F^+(p^+) = \int_P (p^+)^*\varphi^+ + \int_Q (\ldots)$$

on the set M^+ of all mappings $p^+ : P \to E^+$ satisfying $\pi^+ \circ p^+ = i_B^P$ (here $\pi^+ = \pi \circ \pi_E^+$) and some boundary conditions not specified above. The extremals $p^+ \in \mathscr{E}\mathscr{X}^+$ are defined by the corresponding conditions (4), of course, i.e. by the requirement

$$(p^+)^*Z^+ \,\lrcorner\, \mathrm{d}\varphi^+ \equiv 0 \tag{7}$$

where Z^+ runs over all π^+-vertical vector fields on the space E^+.

The main point of the construction consists in the fact (not proved here) that *the relation $p = \pi_E^+ \circ p^+$ maps the set of all extremals $p^+ \in \mathscr{E}\mathscr{X}^+$ onto the set of all extremals $p \in \mathscr{E}\mathscr{X}$ of the original problem.* (Cf. [2, part II, Section 5].)

The problem $\mathscr{L}\mathscr{P}^+$ already possesses the sought property $\mathfrak{A}^+ = \{0\}$. However, it currently suffers from two defects. *At first*, the extremals $p^+ \in \mathscr{E}\mathscr{X}^+$ need not fill the space E^+ up (the *shortage defect*) in the sense that all sets $p^+(P)$ ($p^+ \in \mathscr{E}\mathscr{X}^+$) may lie in a proper submanifold E' of E^+. By other words, there may exist a proper embedding $i' : E' \to E$ of a manifold E' into the manifold E^+ of the property that every extremal $p^+ \in \mathscr{E}\mathscr{X}^+$ can be decomposed as $p^+ = i' \circ p'$ where $p' : P \to E'$ is an appropriate mapping. *At second*, many extremals p^+ may lead to the same result $p = \pi_E^+ \circ p^+ \in \mathscr{E}\mathscr{X}$ (the *abundance defect*) and it follows that it is sufficient to handle only some particular extremals $p^+ \in \mathscr{E}\mathscr{X}^+$ in order to get all extremals $p \in \mathscr{E}\mathscr{X}$. At our stage of understanding the problem, it is a matter of taste which extremals p^+ are left out. (From the more advanced point of view, the Hamilton–Jacobi equation and boundary conditions should play a crucial role here, cf. [1].) For example, it may be sufficient to handle only that extremals which lie in an appropriate submanifold of E^+ narrower than the above mentioned E'.

The final conclusion is that we are led to investigating some other Lagrange problems induced on an appropriate submanifold of the total space E^+.

Having this in mind, let $i' : E' \to E^+$ be an embedding of a manifold E' into E^+. We shall assume that the composition $\pi' = \pi^+ \circ i' : E' \to B$ is a surjective submersion. Then we may introduce a new Lagrange problem noted $\mathscr{L}\mathscr{P}'$ and called the problem *induced* on E' by the problem $\mathscr{L}\mathscr{P}^+$. It is defined by the data (E', π', B'), φ', $\mathfrak{A}'$ where $B' = B$, $\varphi' = (i')^*\varphi^+$, $\mathfrak{A}' = \{0\}$, the original objects P and i_B^P are retained and the boundary data are unimportant.

One can easily see the following simple fact: If $p' : P \to E'$ and $p^+ = i' \circ p' \in \mathscr{E}\mathscr{X}^+$ then $p' \in \mathscr{E}\mathscr{X}'$ (the extremals of $\mathscr{L}\mathscr{P}'$). The converse is in general not true; the relation $p' \in \mathscr{E}\mathscr{X}'$ does not necessarily imply $p^+ = i' \circ p' \in \mathscr{E}\mathscr{X}^+$. Additional requirements ensuring the last inclusion will be formulated in terms of certain π^+-vertical vector fields on E^+. So let $\mathfrak{A}^+$ be a family of such vector fields. Assume that the family $\mathfrak{A}^+$ is *transverse* to the submanifold E' of E^+. (That means, for every $x \in E$, the set of all vectors V_y^+ ($V^+ \in \mathfrak{A}^+$, $y = i'(x)$) together with the linear subspace $\mathrm{d}i'(T_xE')$ of the tangent space T_yE^+ span the whole tangent space T_yE^+.) *Necessary and sufficient conditions ensuring the inclusion $p^+ = i' \circ p' \in \mathscr{E}\mathscr{X}^+$ for a given extremal $p' \in \mathscr{E}\mathscr{X}'$ can be expressed by*

$$(p')^* ((i')^* V^+ \rfloor \mathrm{d}\varphi^+) \equiv 0 \qquad (V^+ \in \mathfrak{A}^+). \tag{8}$$

Note at last that the embedding i' is called *regular* if $(i')^* V^+ \rfloor \mathrm{d}\varphi^+ \equiv 0$ for an appropriate family $\mathfrak{A}^+$ of the mentioned type. *In the regular case we have $p^+ = i' \circ p' \in \mathscr{E}\mathscr{X}^+$ for any $p' \in \mathscr{E}\mathscr{X}'$.*

After these preparatory considerations, we are going to more concrete topics, namely to various settings of the classical multiple integral variational problem. In all examples below, we begin with transferring the original problem $\mathscr{L}\mathscr{P}$ into the corresponding auxiliary problem $\mathscr{L}\mathscr{P}^+$ and then we try to remove the shortage and abundance defects. In all cases we obtain a nice Hamiltonian system completely equivalent to the original Euler–Lagrange system.

3. General setting of the variational problem. We introduce the space $M = \mathrm{R}^{n+m}$ with the coordinates x^i, y^j ($i = 1, \ldots, n$; $j = 1, \ldots, m$), $B = \mathrm{R}^n$ with coordinates x^i ($i = 1, \ldots, n$) and the obvious projection $\pi_B^M : M \to B$. If (M, π_B^M, B) is the relevant fibered manifold, then the accompanying space of all d-jets of cross-sections will be shortly denoted J^d. The jet coordinates on J^d are

$$x^i,\ y^j_{i_1\dots i_s} \qquad (0 \leqq s \leqq d;\ i, i_1, \dots, i_s = 1, \dots, n;\ j = 1, \dots, m).$$

Only *nondecreasing* multiindices $I = i_1 \dots i_s$, $i_1 \leq \dots \leq i_s$, are used here but in practise, in order to simplify some formulae, it is useful to introduce the convention

$$y^j_I = y^j_{I'}, \qquad (I = i_1 \dots i_s,\ I' = i'_1 \dots i'_s)$$

whenever I is a permutation of I' (which will be denoted $I \sim I'$). Owing to this convention, we may confortably remind the contact forms

$$\omega^j_I = \mathrm{d}y^j_I - \Sigma y^j_{Ii}\,\mathrm{d}x^i \qquad (Ii = i_1 \dots i_s i)$$

on the space J^d, for every multiindex $I = i_1 \dots i_s$ with the norm $|I| = s < d$. Note also that for every $e \leqq d$, there exists the obvious natural projection $\pi^d_e : J^d \to J^e$ forgetting the coordinates y^j_I $(e < |I| \leqq d)$. In particular, $J^0 = M$ so that $\pi^d_0 : J^d \to$ $\to J^0 = M$. We shall denote $\pi^d = \pi^M_B \circ \pi^d_0 : J^d \to B$.

The underlying fibered space for the variational problems will be (E, π, B), where $E = J^d$, $\pi = \pi^d$. The manifold P will be parametrized by the coordinates $t^1, \dots, t^n$ and the embedding i^P_B will be simply defined by $t^i \equiv (i^P_B)^* x^i$. We shall deal with the form

$$\varphi = (\pi^d_e)^* f\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n \qquad (\mathrm{f} \in C^\infty(J^e)),$$

where the constants e, d will be specified later on. At last, the module $\mathfrak{A}$ will always include all n-forms of the type

(9) $\quad \Sigma a^j_{Ii}\omega^j_I \wedge \mathrm{d}x_i \qquad (\mathrm{d}x_i = -(-1)^i\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \dots \wedge \mathrm{d}x^n)$

and, may be, some others. As usual, the boundary data need not be specified.

One can easily see that the first group of conditions (1) implies $p^*\omega^j_I \equiv 0$ $(|I| < d)$, hence

$$p^* y^j_I = \partial^s p^* y^j / \partial t^I \qquad (I = i_1 \dots i_s,\ s \leqq d,\ \partial t^I = \partial t^{i_1} \dots \partial t^{i_s}).$$

It follows that the functional (2) may be rewritten as

$$F(p) = \int_P f(t^1, \dots, t^n, \dots, \partial^s p^* y^j / \partial t^I, \dots)\,\mathrm{d}t^1 \wedge \dots \wedge \mathrm{d}t^n + \int_Q (\dots).$$

This is the common multiple integral of the classical calculus of variations dependent on higher order derivatives of the variable functions $p^* y^j \in C^\infty(P)$.

We shall now specify the constants d, e and the module $\mathfrak{A}$ in order to obtain some more concrete problems.

4. First example. We begin with the simplest and most economical but relatively unknown case by choosing $d = e$. At the same time, the module $\mathfrak{A}$ will consist exactly of all forms (9) where $a^j_{Ii} \in C^\infty(J^e)$, $|I| < e$, are arbitrary functions. Inserting such a form $\bar{\alpha} = \Sigma \bar{a}^j_{Ii} \omega^j_I \wedge \mathrm{d}x_i$ into the condition (3), the coefficients $\bar{a}^j_{Ii}$ can be eliminated and the familiar Euler – Lagrange system

(10) $$\Sigma(-1)^s \, \partial^s(p^* \, \partial f/\partial y^j_I)/\partial x^I \equiv 0 \qquad (j = 1, \ldots, n; \text{ sum over } I)$$

immediately appears. But this is a familiar story and we turn to the Hamiltonian theory.

Following the general lines (cf. (6)) and omitting some pull-backs, we have the form

(11) $$\varphi^+ = f \, \mathrm{d}x + \Sigma p^j_{Ii} \omega^j_I \wedge \mathrm{d}x_i \qquad (\mathrm{d}x = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)$$

where

$$p^j_{Ii} \qquad (i = 1, \ldots, n; j = 1, \ldots, m; 0 \leqq |I| < e; \; I \text{ nondecreasing})$$

are new coordinates along the fibres of the fibered manifold (E^+, π^+_E, E). The extremals $p^+ \in \mathscr{E}\mathscr{X}^+$ are defined by the condition (7). Now the choice $Z^+ = \partial/\partial p^j_{Ii}$ easily yields the requirements $(p^+)^* \omega^I_j \wedge \mathrm{d}x_i \equiv 0$, that is, $(p^+)^* \omega^j_I \equiv 0$ for every $|I| < e$. The choice $Z^+ = \partial/\partial y^j_I$ is more interesting and especially, for the case $|I| = e$, we get the conditions

(12) $$(p^+)^*(\partial f/\partial y^j_I - p^j_{\{I\}}) \equiv 0 \qquad (|I| = e)$$

with the so called *cyclic means*

$$p^j_{\{I\}} = \Sigma p^j_{I'i'} \qquad (\text{sum over } I', i' \text{ such that } I'i' \sim I).$$

It follows that the *shortage defect* is present and we are led to the submanifold E' of E^+ for which the corresponding embedding $i' : E' \to E^+$ satisfies

$$(i')^*(\partial f/\partial y^j_I - p^j_{\{I\}}) \equiv 0 \qquad (|I| = e).$$

One can see that *the regular case takes place if and only if the classical criterion*

(13) $$\det(\partial^2 f/\partial y^j_I \, \partial y^{j'}_{I'}) \neq 0 \qquad (j, j' = 1, \ldots, m; |I| = |I'| = e)$$

is satisfied. Note that (13) is nothing else than the transversality condition for the family $\mathfrak{A}^+$ of vector fields $\partial/\partial y^j_I$ $(|I| = e)$ to the submanifold E'.

The extremals $p' \in \mathscr{E}\mathscr{X}'$ can be defined in terms of the form

(14) $$\begin{aligned} \mathrm{d}\varphi' &= \mathrm{d}(i')^*\varphi^+ = \mathrm{d}(\Sigma p^j_{Ii} \, \mathrm{d}y^j_I \wedge \mathrm{d}x_i - H \, \mathrm{d}x) = \\ &= \Sigma \mathrm{d}p^j_{Ii} \wedge \mathrm{d}y^j_I \wedge \mathrm{d}x_i - \mathrm{d}H \wedge \mathrm{d}x = \\ &= \Sigma \left(\mathrm{d}p^j_{Ii} + \frac{\partial H}{\partial y^j_I} \mathrm{d}x^i \right) \wedge \left(\mathrm{d}y^j_I - \frac{\partial H}{\partial p^j_{Ii}} \mathrm{d}x^i \right) \wedge \mathrm{d}x_i, \end{aligned}$$

where $H = p^j_{\{I\}} y^j_I - f \in C^\infty(E')$ is the Hamilton function. If the regularity condition (13) is satisfied, then the functions

(15) $$x^i, y^j_I, p^j_{Ii} \qquad (0 \leqq |I| < e, I \text{ nondecreasing})$$

may serve for local coordinates on E' and the relation $p^+ = i' \circ p'$ yields a one-to-one correspondence between the sets $\mathscr{E}\mathscr{X}^+$ and $\mathscr{E}\mathscr{X}'$. But we know that the set $\mathscr{E}\mathscr{X}^+$ is projected onto the set $\mathscr{E}\mathscr{X}$ by π_E^+. So it follows that the relation $p = \pi_E^+ \circ p^+ = \pi_E^+ \circ i' \circ p'$ maps the set $\mathscr{E}\mathscr{X}'$ onto the set $\mathscr{E}\mathscr{X}$.

Let us summarize the above results. In the regular case (13), we have the coordinates (15) on the manifold E'. In these coordinates, the Hamilton system for the extremals $p' \in \mathscr{E}\mathscr{X}'$ can be explicitly expressed as

$$(p')^*\left(\mathrm{d}y_I^j - \frac{\partial H}{\partial p_{Ii}^j}\mathrm{d}x^i\right) \wedge \mathrm{d}x_i = (p')^*\left(\mathrm{d}p_{Ii}^j + \frac{\partial H}{\partial y_I^j}\mathrm{d}x^i\right) \wedge \mathrm{d}x_i \equiv 0$$

or, in a semi-classical notation,

$$\partial(p')^* y_I^j/\partial t^i \equiv (p')^* \frac{\partial H}{\partial p_{Ii}^j}, \qquad \sum_i \partial(p')^* p_{Ii}^j/\partial t^i = -(p')^* \frac{\partial H}{\partial y_I^j} \tag{16}$$

and, at last, as a Pfaffian system

$$(p')^*\left(\mathrm{d}y_I^j - \sum_i \frac{\partial H}{\partial p_{Ii}^j}\mathrm{d}x^i\right) = (p')^*(\mathrm{d}p_{Ii}^j + \sum_i h_{Iii}^j \mathrm{d}x^i) \equiv 0$$

with new variables h_{Iii}^j related by the constraints $\Sigma h_{Iii}^j \doteq \partial H/\partial y_I^j$. *Every solution p' of these systems yields a solution p of the Euler–Lagrange system* (10) *by a simple neglecting the coordinates p_{Ii}^j. Conversely, every extremal $p \in \mathscr{E}\mathscr{X}$ can be obtained in such a manner* (but in general from several extremals $p' \in \mathscr{E}\mathscr{X}'$ since the abundance defect is not yet removed).

5. A digression to the Hamilton—Jacobi equation. Let us look for such submanifolds E'' of E' for which the relevant embedding $i'' : E'' \to E'$ satisfies the condition

$$(i'')^* \mathrm{d}\varphi' = 0. \tag{17}$$

One can easily see that the latter condition is equivalent to the identity $\mathscr{E}\mathscr{X}'' = M''$, i.e., every cross-section $p'' : P \to E''$ is an extremal of the induced problem $\mathscr{L}\mathscr{P}''$. Let us look more closely at the regular case of the embedding i'. Then, according to the existence of local coordinates (15), there exists a natural projection $\pi'_{e-1} : E' \to J^{e-1}$ and it is possible to check that the choice $E'' = J^{e-1}$ with the embedding $i'' = \sigma : E'' = J^{e-1} \to E'$ satisfying $\pi'_{e-1} \circ i'' =$ identity (that means, σ is a cross-section of the fibered manifold $(E', \pi'_{e-1}, J^{e-1})$) is the most reasonable one. (Indeed, in this case E'' is a maximal regular solution of the exterior system (17), cf. [1, Section 39, point xii].) It may be also proved that in the latter case there exist functions $S^1, \dots, S^m \in C^\infty(J^{e-1})$ such that

$$(i'')^*\varphi' = \sigma^*\varphi' = \Sigma\, \mathrm{d}S^i \wedge \mathrm{d}x_i, \tag{18}$$

at least locally. Then the condition (17) is clearly equivalent to the system

$$\partial S^i/\partial y_I^j \equiv \sigma^* p_{Ii}^j, \qquad \Sigma\, \partial S^i/\partial x^i = -\sigma^* H \tag{19}$$

which appears as a variant of the *Hamilton–Jacobi equation.* The latter equation can be easily solved using the classical method of characteristic strips.

As a rule $\sigma : J^{e-1} \to E'$ is *not* a regular embedding and in order to ensure the inclusion $p' = \sigma \circ p'' \in \mathscr{E}\mathscr{X}'$, the mapping $p'' : P \to J^{e-1}$ must satisfy the corresponding additional conditions of the type (8). Choosing the vector fields $\partial/\partial p^j_{Ii}$ for the relevant transverse family, one can immediately check that the mentioned conditions are

$$(p'')^* \omega^j_I \equiv 0 \qquad (|I| < e), \tag{20}$$

i.e., they are identical with the contact conditions. Since they are plainly equivalent to the first group of the Hamilton equations (16), the second group (16) may be considered as a little intricate reformulation of the embedding relation $p' = \sigma \circ p''$ of an extremal $p' \in \mathscr{E}\mathscr{X}'$ into the submanifold $E'' = J^{e-1}$ of E'.

Given an arbitrary mapping $p' \in M'$ (in particular, an extremal $p' \in \mathscr{E}\mathscr{X}'$) the question arises whether the subset $p'(P)$ may be embedded into a submanifold E'' satisfying (17). By other words, given $p' : P \to E'$, we seek the embedding i'' satisfying (17) such that $p' = i'' \circ p'$ for certain map $p'' : P \to E''$. As usual, we shall mention only the above case $E = J^{e-1}$ and $i'' = \sigma$. Then the embedding conditions are expressed by the first group of equations (19), i.e., by the identities

$$(\pi'_{e-1} \circ p)^* \, \partial S^i/\partial y^j_I = (p')^* p^j_{Ii}. \tag{21}$$

The functions $S^1, \ldots, S^{n-1}$ may be chosen arbitrarily but satisfying (21). The remaining function S^n could be determined from the arising initial value problem

$$\partial S^n/\partial x^n = -h(\ldots, x^i, \ldots, \partial S^n/\partial x^k, \ldots, \partial S^n/\partial y^j_I, \ldots), \tag{22}$$

$$\frac{\partial S^n}{\partial y^j_I}(\ldots, x^i, \ldots, \bar{y}^j_I(\ldots, x^i, \ldots), \ldots) = \bar{p}^j_{In}(\ldots, x^i, \ldots), \tag{23}$$

following from (19). Here $\bar{y}^j_I \equiv (p')^* y^j_I$, $\bar{p}^j_{Ii} \equiv (p')^* p^j_{Ii}$ and $h = \sigma^* H + \partial S^k/\partial x^k$ (sum over $k = 1, \ldots, n-1$).

The familiar method of characteristic strips which may be employed for solving the problem (22), (23) is a little simplified since the right hand side of (22) does not contain the unknown function S^n. On the other side, however, the most important case when $p' \in \mathscr{E}\mathscr{X}'$ leads to some additional troubles since then the initial conditions (23) are given on a characteristic submanifold. This is a familiar result, of course, at least in the case $e = 1$ and we refer to [1, Section 39, points viii and ix] for a conceptual proof.

6. Second example. Let us return back to Section 4. We should like to remove the abundance defect and for this aim, let us analyse the conditions (7) more thoroughly. Remind that the choice $Z^+ = \partial/\partial p^j_{Ii}$ yields the contact conditions $(p^+)^* \omega^j_I \equiv 0$ ($|I| < e$). One can see that the choice $Z^+ = \partial/\partial y^j_I$ leads to the conditions (12),

$$(24)\qquad (p^+)^*\left((\partial f/\partial y^j_I - p^j_{\{I\}})\,\mathrm{d}x - \Sigma \mathrm{d}p^j_{Ii}\wedge \mathrm{d}x_i\right)\equiv 0 \qquad (0<|I|<e),$$
$$(p^+)^*\left(\partial f/\partial y^j\,.\,\mathrm{d}x - \Sigma \mathrm{d}p^j_i\wedge \mathrm{d}x_i\right)\equiv 0 \qquad (|I|=0).$$

If we know an extremal $p\in\mathscr{E}\mathscr{X}$ then (excepting the cases $n=1$ or $e=1$) the corresponding extremal $p^+\in\mathscr{E}\mathscr{X}^+$ over p (i.e., satisfying $p=\pi^+_E\circ p^+$) cannot be uniquely determined from these conditions since they all involve only the cyclic means $p^j_{\{I\}}$ It follows that according to the one-to-one correspondence $p^+=i'\circ p'$ between the sets $\mathscr{E}\mathscr{X}^+$ and $\mathscr{E}\mathscr{X}'$, there always exists an extremal $p'\in\mathscr{E}\mathscr{X}'$ over p but it is not uniquely determined.

In order to remove this *abundance defect*, a narrower submanifold of E^+ than the original manifold E' is needed. We may state only the final result here omitting the lengthy proof, cf. [1, Section 29].

We begin with the abbreviation $u^j_I\equiv p^j_{\{I\}}$ ($0<|I|\leqq e$, *I nondecreasing*) and let $v^j_{Ii}\in C^\infty(E^+)$ be certain functions labeled by $j=1,\dots,m$ and by all *not nondecreasing* multiindices Ii with $i=1,\dots,n$ and $I=i_1,\dots,i_s$ ($0<s<e$; $i_1,\dots,i_s=1,\dots,n$) already nondecreasing. We shall make the following assumptions. At first, every function v^j_{Ii} may depend only on the coordinates of the type

$$x^{i'},\, y^{j'}_{I'}\quad (|I'|<e),\qquad p^{j'}_{I'i'}\quad (|I|\leqq|I'|).$$

At second, the family of functions

$$x^i,\, y^j_I\quad(|I|\leqq e),\qquad u^j_I\quad(|I|\leqq e),\qquad v^j_{Ii}\quad(0<|I|<e)$$

may serve for a local coordinate system on E^+.

With these assumptions, the sought new submanifold E' of E^+ is defined by the conditions

$$(25)\qquad (i')^*\,(\partial f/\partial y^j_I - u^j_I)\equiv 0\quad(|I|=e),\qquad (i')^*\,v^j_{Ii}\equiv 0\quad(0<|I|<e),$$

so that in the regular case (13), the functions

$$(26)\qquad x^i,\, y^j_I\quad(|I|<e),\qquad u^j_I\quad(0<|I|\leqq e)$$

may serve for local coordinates on E. The form $\mathrm{d}\varphi'$ is expressed by (14) with additional relations (25) among the variables involved. Then the main result claims that *in the regular case* (13), *the relation* $p=\pi^+_E\circ i'\circ p'$ *yields a one-to-one correspondence between the sets* $\mathscr{E}\mathscr{X}$ and $\mathscr{E}\mathscr{X}'$. Note that the Hamilton system for the extremals $p'\in\mathscr{E}\mathscr{X}'$ may be effectively written down but we refer to [1, Section 27] and instead turn to some more popular subcases of the last result.

7. Third example. The above assumptions can be most easily realized by taking $v^j_{Ii}\equiv p^j_{Ii}$. Then (24) simplifies into

$$(i')^*\,p^j_{Ii}\equiv 0\qquad (Ii\text{ not nondecreasing}),$$
$$(i')^*\,(\partial f/\partial y^j_I - p^j_I)\equiv 0\qquad (|I|=e,\ I\text{ nondecreasing}).$$

The functions

$$x^i, y^j_I \quad (|I| < e), \qquad p^j_I \quad (0 < |I| \leqq e, I \text{ nondecreasing}) \tag{27}$$

may be used for coordinates and we obtain the Hamilton system

$$\partial(p')^* y^j_I / \partial t^i = (p')^* \frac{\partial H}{\partial p^j_{Ii}}, \qquad \Sigma \partial(p')^* p^j_{Ii} / \partial t^i = -(p')^* \frac{\partial H}{\partial y^j_I}$$

for the extremals $p' \in \mathscr{E}\mathscr{X}'$. Note that the sum runs over all indices i but only the nondecreasing multiindices Ii are taken into account. Unlike (16), we have a determined system of partial differential equations and *the relation* $p = \pi^+_E \circ i' \circ p'$ *between the sets* $\mathscr{E}\mathscr{X}$ *and* $\mathscr{E}\mathscr{X}'$ *is one-to-one in the regular case* (13).

8. Fourth example. Another realization of the assumptions of Section 6 is provided by the choice

$$v^j_{Ii} = p^j_{Ii} - p^j_{I'i'} \qquad (Ii \sim I'i')$$

with Ii not nondecreasing but $I'i'$ already nondecreasing. This particular case is very suggestive and popular.

If we deal with the regular case (13), the functions (27) may be taken for local coordinates on E'. As usual, it is useful to introduce the convention $p^j_I = p^j_{I'}$, whenever $I \sim I'$. Keeping this in mind, the expression of the form $d\varphi'$ is again (14) but the Hamilton system for the extremals $p' \in \mathscr{E}\mathscr{X}'$ looks quite another than (16) and is as follows:

$$\Sigma \partial(p')^* y^j_{I'} / \partial t^{i'} = (p')^* \frac{\partial H}{\partial p^j_I}, \qquad \sum_i \partial(p')^* p^j_{Ii} / \partial t^i = -(p')^* \frac{\partial H}{\partial y^j_I}.$$

The first sum is taken over i', I' with $I'i' \sim I$. In the second group of equations, the above symmetry convention must be taken into account.

Remind that *the relation* $p = \pi^+_E \circ i' \circ p'$ *yields a one-to-one correspondence between the sets* $\mathscr{E}\mathscr{X}$ *and* $\mathscr{E}\mathscr{X}'$.

9. Several remarks. At first, we shall briefly mention the non-regular variational problem. For this aim assume that (13) fails and there exist exactly $c (c \geqq 1)$ linearly independent vector fields

$$W^k = \Sigma w^{jk}_I \, \partial / \partial y^j_I \qquad (k = 1, \ldots, c; \text{ sum over } j \text{ and } I, |I| = e)$$

on the space J^e, solutions of the system

$$\Sigma w^{jk}_I \, \partial^2 f / \partial y^j_I \, \partial y^{j'}_{I'} \equiv 0 \qquad (I = |I'| = e, \text{ sum over } j \text{ and } I).$$

One can then prove that *the relation* $p = \pi^+_E \circ i' \circ p'$ *yields a one-to-one correspondence between extremals* $p \in \mathscr{E}\mathscr{X}$ *and extremals* $p' \in \mathscr{E}\mathscr{X}'$ *satisfying the additional conditions*

$$\Sigma (p')^* w^{jk}_{Ii} \, \partial(p')^* y^j_I / \partial t^i \equiv 0 \qquad (k = 1, \ldots, c; \text{ sum over } j, I, i).$$

On the other side, the unpleasant presence of c additional conditions is ballanced by the fact that the form φ' can be expressed by the coordinates

$$x^i, y^j_I \quad (0 \leqq |I| < e), \qquad p^j_I \quad (0 < |I| < e)$$

and certain parameters $u^1, \ldots, u^{C-c}$, where C is the number of coordinates $p^j_I(|I| = e)$. It follows that *the number of variables appearing in the Hamilton system for the extremals* $p' \in \mathscr{E}\mathscr{X}'$ *is diminished on* c. We refer to the paper [1] for more informations.

At second, we shall mention the Hamilton–Jacobi equation. Dealing with the example of Section 7, one can derive the corresponding Hamilton–Jacobi equation (19) quite analogously as in the Section 5. An important difference appears, however, since the condition $(i')^* p^j_{Ii} \equiv 0$ (Ii not nondecreasing) together with (19) imply the unpleasant restrictions

$$\partial S^i / \partial y^j_I \equiv 0 \qquad (Ii \text{ not nondecreasing})$$

for the sought unknown functions $S^1, \ldots, S^n$. It follows that S^1 may depend only on the variables $x^i, y^j_{1\ldots1}$, the function S^2 may depend only on $x^i, y^j_{1\ldots12\ldots2}$, and so on. It is not difficult to find a solution satisfying these constrants but if we try to embed a given extremal then the problem (22), (23) *cannot* be in general resolved. The matter get worse for the popular symmetrical variant of Section 8. In this case, the Hamilton–Jacobi equation should be resolved under the constraints

$$\partial S^i / \partial y^j_I \equiv \partial S^{i'} / \partial y^{j'}_{I'} \qquad (Ii \sim I'i')$$

and already the existence of *any* solution is not evident. Possibly some embeddability results can be derived for the case considered in Section 6 for an appropriate choice of the functions v^j_{Ii}, but nothing is known in this direction.

10. Fifth example. We shall essentially alter the choice of e, d and $\mathfrak{A}$ for the first time here. We put $d = 2e - 1$ and the module $\mathfrak{A}$ will consist exactly of all forms (9) where $a^j_{Ii} \in C^\infty(J^d)$ are arbitrary functions and $|I| < d$. One can verify that the conditions (3) lead to the same Euler–Lagrange system (10) as before. It follows that *the projections* $\pi^d_e \circ p$ *of the present extremals* $p \in \mathscr{E}\mathscr{X}$ *are identical with the extremals considered in Section 4.*

Turning to the Hamiltonian theory, we introduce the space $E^+ = R^c \times E\,(E = J^d)$ with coordinates

$$x^i, y^j_I \quad (|I| \leqq d), \qquad p^j_{Ii} \quad (0 < |I| < d)$$

and the form (11) on the space E^+. In general, the problem $\mathscr{L}\mathscr{P}^+$ suffers both from the abundance and the shortage defects. We are directly going to remove them but restricting only on the modification of the general construction of Section 6.

The final result is as follows. Let us abbreviate $u^j_I = p^j_{\{I\}}$ ($|I| \leqq d$, I nondecreasing) and let v^j_{Ii} be certain functions labeled by the same indices j, I, i as in Section 6. We moreover assume that every function v^j_{Ii} does not depend on the coordinates of the type $p^j_{I'i'}$ ($|I'| < |I|$) and the family of functions

$$x^i, y^j_I \quad (|I| \leqq d), \qquad u^j_I \quad (0 < |I| \leqq e), \qquad v^j_{Ii} \quad (0 < |I| < e),$$
$$p^j_{Ii} \quad (e \leqq |I| < d)$$

may serve for a local coordinate system on E^+. Then denoting by

$$\partial^i = \partial/\partial x^i + \Sigma y^j_{Ii}\, \partial/\partial y^j_I \qquad (i = 1, \ldots, n)$$

the so called *formal derivatives*, we introduce the submanifold E' of E^+ defined by the requirements

$$\begin{aligned}
(28) \qquad & (i')^* (\partial f/\partial y^j_I - u^j_I) \equiv 0 && (|I| = e),\\
& (i')^* (\partial f/\partial y^j_I - u^j_I) \equiv \partial^i (i')^* p^j_{Ii} && (|I| = e - k + 1,\ 1 < k \leq e),\\
& (i')^* v^j_{Ii} \equiv 0 && (0 < |I| < e),\\
& (i')^* p^j_{Ii} \equiv 0 && (e \leqq |I| < d),
\end{aligned}$$

on the relevant embedding $i' : E' \to E^+$. Note that the first two groups of the above requirements cannot be altered if one wish to remove the shortage defect, cf. the relations (12), (24). The functions v^j_{Ii} in the third group may be, however, chosen with a large degree of generality, even nonlinear, so that many recent results are included as a very particular case of our construction, cf. [3], [4]. At last, the fourth group of requirements (28) ensures that in certain favourable cases stated below the higher order derivatives y^j_I ($|I| \geqq e$) can be eliminated out of the corresponding Hamilton system for the extremals $p' \in \mathscr{E}\mathscr{X}'$ and we obtain exactly the same final result as in Section 6 above but under stronger limitations.

Let us go to more details. *At first* assume that the second requirement (28) *does not introduce any further relations among the functions* (26) *than the remaining requirements*, i.e., than the requirements (25) appearing already in Section 6. In other words, assume that the relations (28) can be locally resolved with respect to certain subfamily of the higher order jet variables y^j_I ($|I| > e$). Still in other words, assume that the corresponding Jacobian

$$(29) \qquad (\partial^2 f/\partial y^j_I\, \partial y^{j'}_{I'} + \Sigma\, \partial(\partial^i (i')^* p^j_{Ii})/\partial y^j_I)$$

of (28) with respect to the variables y^j_I ($|I| > e$) possesses the maximal possible rank equal to the number of conditions (28). (Note that the rows of (29) are labeled by $j' = 1, \ldots, m$ and all I' with $e < |I'| \leqq d$, the columns by $j = 1, \ldots, m$ and I with $1 \leqq |I| < e$.) *At second* assume the regular case (13). Under these assumptions, the functions (26) together with a maximal family of higher order varibles y^j_I ($|I| > e$) functionally independent on the manifold E' (i.e., not interrelated by (28)) constitute a coordinate system on E'. But the form $d\varphi'$ is expressed by the formula (14) with relations (25) among the variables involved. In particular, *none of the higher order variables y^j_I ($e \leqq |I| \leqq d$) is present and the Hamilton system is exactly the same as in Section 6.*

So we may conclude that *under the regularity assumption* (13) *and under the* *maximum rank assumption of* (29), *the relation* $p = \pi_E^+ \circ i' \circ p'$ *between the sets* $\mathscr{E}\mathscr{X}$ *and* $\mathscr{E}\mathscr{X}'$ *is one-to-one.* So we have the same result as in Section 6 but under *much more restrictive conditions.*

One can easily see that the coordinates u_I^j can be completely eliminated and only the jet variables

$$x^i, y_I^j \qquad (0 \leqq I \leqq d)$$

may be used for the coordinate system on E'. It follows that all investigations of the present Section can be expressed in terms of the jet variables. This is however a pure theory and it seems that nobody is able to write down *explicitly* any Hamilton system using *only* the jet variables. In any case, we *cannot* consider the approach of the present Section as an appropriate one.

11. Sixth and the last example. We enter with the same underlying fibered space $(E, \pi, B) = (J^e, \pi^e, B)$ and differential form $\varphi = f\,\mathrm{d}x(f \in C^\infty(J^e))$ as in Section 4, but the new module $\mathfrak{A}$ will consist of all *at least* 1-contact n-forms, i.e., $\mathfrak{A}$ consists of all n-forms of the type

$$\alpha = \Sigma a_{Ii}^j \omega_I^j \wedge \mathrm{d}x_i + \Sigma \omega_I^j \wedge \omega_{I'}^{j'} \wedge \vartheta_{II'}^{jj'}, \tag{30}$$

where $a_{Ii}^j \in C^\infty(J^e)$ are arbitrary functions and $\vartheta_{II'}^{jj'}$ are arbitrary $(n-2)$-forms on J^e. The first summand of (30) may be identified with the form (9) so that $p^*\omega_I^j \equiv 0$ ($|I| < e$) for every extremal $p \in \mathscr{E}\mathscr{X}$ (cf. Section 3). The last identities easily imply that all summands behind the first one in (30) are inessential for the defining conditions (3) of extremals and it follows that the Euler–Lagrange system is the same as before.

Although the Hamiltonian approach accompanied with the present setting of the variational problem is very natural from the point of view of general theory of partial differential equations, little results are available for the time being. For this reason, we restrict on several remarks in order to illustrate the arising difficulties.

First of all, in order make the representation (30) more explicit, we introduce the notation

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \ldots \mathrm{d}x^{i_s}, \qquad \omega_{\mathscr{I}}^J = \omega_{I_1}^{j_1} \wedge \ldots \wedge \omega_{I_r}^{j_r}, \qquad \mathrm{d}y_{\mathscr{I}}^J = \mathrm{d}y_{I_1}^{j_1} \wedge \ldots \wedge \mathrm{d}y_{I_r}^{j_r},$$

where $I = i_1 \ldots i_s$, $J = j_1 \ldots j_r$ and $\mathscr{I} = (I_1, \ldots, I_r)$ is a multiindex consisting of multiindices of the common nondecreasing type. Then the form α may be written more explicitly as the sum

$$\alpha = \Sigma a_{\mathscr{I}}^{J,I} \omega_{\mathscr{I}}^J \wedge \mathrm{d}x^I \qquad (a_{\mathscr{I}}^{J,I} \in C^\infty(J^e),\ 1 \leqq |J|, |I| + |J| = n).$$

According to Section 2, the Hamiltonian approach should handle the exterior form

$$\varphi^+ = f\,\mathrm{d}x + \Sigma p_{\mathscr{I}}^{J,I} \omega_{\mathscr{I}}^J \wedge \mathrm{d}x^I \tag{31}$$

with coefficients $p_{\mathscr{I}}^{J,I}$ playing the role of the variables along the fibres of the auxiliary fibered space (E^+, π_E^+, E). Following the general lines of Section 2, the Hamilton system for the extremals $p^+ \in \mathscr{E}\mathscr{X}^+$ and $p' \in \mathscr{E}\mathscr{X}'$ is determined from the corresponding forms $d\varphi^+$ and $d\varphi' = (i')^* d\varphi^+$, respectively. Here the form φ^+ must be rewritten using the Hamilton function H uniquely determined by

$$\varphi^+ = \Sigma p_{\mathscr{J}}^{J,I} \, dy_{\mathscr{J}}^J \wedge dx^I - H \, dx. \tag{32}$$

The interdependence among the functions $(i')^* p_{\mathscr{J}}^{J,I}$ and the Hamilton function H seems to be, however, very complicated in the general case and it is not easy to follow this way. A slightly better approach (suggested by Carathéodory, cf. also [2, part II, Section 19]) uses an interesting decomposition of the form φ^+.

For this aim, suppose $f \neq 0$. One can then easily verify that the formula (31) can be rewritten as

$$\varphi^+ = f\xi^1 \wedge \ldots \wedge \xi^n + \Sigma \omega_{\mathscr{J}}^J \wedge \vartheta_{\mathscr{J}}^J, \tag{33}$$

where $\xi^i \equiv dx^i + \Sigma p_{Ii}^j \omega_I^j / f$ and $\vartheta_{\mathscr{J}}^J$ ($|J| \geqq 2$) are certain $(n-2)$-forms. Then the conjecture $\vartheta_{\mathscr{J}}^J \equiv 0$ yields a quite interesting Hamilton system

$$\partial (p')^* y_I^j / \partial \zeta^i \equiv -(p')^* \frac{\partial \ln H}{\partial w_{Ii}^j}, \qquad \Sigma \partial (p')^* p_{Ii}^j / \partial \zeta^i \equiv (p')^* \frac{\partial \ln H}{\partial y_I^j}.$$

Here $H = \det(\delta_i^{i'} - \Sigma p_{Ii}^j y_{Ii'}^j / f)$ is the new Hamilton function, $w_{Ii}^j = \Sigma v^{ii'} p_{Ii'}^j / f$ are new Legendre coordinates where $v^{ii'}$ are entries of the inverse matrix

$$(\delta_i^{i'} - \Sigma p_{Ii}^j y_{Ii'}^j / f)^{-1} = (v^{ii'}),$$

the linear forms ζ^i are defined by

$$\zeta^i \equiv dx^i + \Sigma v^{ii'} p_{Ii'}^j \, dy_I^j / f$$

and, at last, $\partial / \partial \zeta^i$ are nonholonomic derivatives defined by the identity

$$\partial g / \partial \zeta^i \, . \, \zeta^1 \wedge \ldots \wedge \zeta^n = -(-1)^i \, dg \wedge \zeta^1 \wedge \ldots \wedge \zeta^{i-1} \wedge \zeta^{i+1} \wedge \ldots \wedge \zeta^n.$$

The latter Hamilton system possesses very nice invariance properties and there exists a large variety of another reasonable conjectures on the forms $\vartheta_{\mathscr{J}}^J$ for the particular case $e = 1$, cf. [2, part II, Section 20]. Nothing is however known for the general case $n > 1$, $e > 1$.

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